RINGS ALL OF WHOSE FINITELY GENERATED MODULES ARE INJECTIVE

B. L. OSOFSKY

The main purpose of this paper is to prove that a ring all of whose finitely generated modules are injective must be semi-simple Artin.[1]

We begin with the following information about the class of rings under consideration:

LEMMA 1. *Let R be a ring with identity, and assume each cyclic right R-module is injective. Then R is regular in the sense of von Neumann and R is right self injective.*

Proof. For any ring R with identity, it is easy to see that a right ideal I of R is generated by an idempotent if and only if I is a direct summand of the right R-module R_R. If I is an injective right ideal of R, then I is a direct summand of R_R, and therefore is generated by an idempotent. Thus if every cyclic right R-module is injective, each principal right ideal aR generated by $a \in R$ is generated by an idempotent, that is $aR = eR$ for some $e = e^2 \in R$. Then there exist $x, y \in R$ such that $e = ax$, and $a = ey$. It follows that $ea = e(ey) = ey = a$ and $a = ea = axa$. Thus R is a regular ring, and since R_R is generated by the identity, R_R is injective.

Let M_R denote a right module over a ring R. If P, N are submodules of M, let $P \,'\!\supseteqq N$ signify that P is an essential extension of N. (See Eckmann and Schopf [2].) Then N is an essential submodule of P.

For each $x \in M$, let $x^R = \{r \in R \mid xr = 0\}$. The singular submodule $Z(M)$ is defined by:

$$Z(M) = \{x \in M \mid R_R \,'\!\supseteqq x^R\}$$

$Z(R_R)$ is actually a two sided ideal of R.

If $e = e^2 \in R$, and if $x \in e^R \cap eR$, then $x = ex = 0$ and so $e^R \cap eR = 0$. Thus $Z(R_R)$ contains no idempotents $\neq 0$. In particular, if R is a

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[1] After the author obtained this characterization of rings whose cyclic modules are injective, a translation of a recent paper [5] by L. A. Skornjakov was published and brought to the author's attention. Although Skornjakov states the major portion of the author's main theorem, his proof is incorrect. In the proof of his Lemma 9, which is crucial to his proof of the theorem, Skornjakov assumes that the injective hull of a submodule in an injective module must be a unique submodule, whereas in general it is unique only up to isomorphism.

Added in proof April 13, 1964. This lemma is actually false. See the author's dissertation for a counter-example.

regular ring, $Z(R_R) = 0$. (Cf. R. E. Johnson [4]).

We need the following important (and known [3]) lemma:

LEMMA 2. *Let M_R be a module such that $Z(M) = 0$. Then each submodule N of M has a unique maximal essential extension N^* in M.*

Proof. Let $\{M_i \mid i \in I\}$ be the set of all submodules of M such that $M_i \,{}'\!\supseteqq N$. Set $N^* = \sum_{i \in I} M_i$. Then if $N^* \,{}'\!\supseteqq N$, N^* must be the unique maximal essential extension of N in M since it contains every essential extension of N in M.

For each $y \in M$, let

$$(N : y) = \{r \in R \mid yr \in N\} .$$

If $M \supseteqq Q \,{}'\!\supseteqq N$, then $R_R \,{}'\!\supseteqq (N : y)$ for all $y \in Q$. (This follows, since any non-zero right ideal I of R which satisfies $I \cap (N : y) = 0$, also satisfies $yI \neq 0$ and $yI \cap N = 0$, a contradiction.)

Now let $0 \neq x = x_{i_1} + \cdots + x_{i_n}$, $0 \neq x_{i_j} \in M_{i_j}$, $j = 1, \cdots, n$, be any element of N^*. Then

$$(N : x) \supseteqq \bigcap_{j=1}^{n} (N : x_{i_j}) .$$

Now $M_{i_j} \,{}'\!\supseteqq N$, so $R_R \,{}'\!\supseteqq (N : x_{i_j})$, and therefore $R_R \,{}'\!\supseteqq \bigcap_{j=1}^{n} (N : x_{i_j})$, hence $R_R \,{}'\!\supseteqq (N : x)$. Since $Z(M) = 0$, $x(N : x) \neq 0$, and so $x(N : x)$ is a nonzero submodule of $xR \cap N$. This proves $N^* \,{}'\!\supseteqq N$ as asserted.

We next consider certain properties of idempotents in a right self injective regular ring. Let $N \sim A$ denote the set theoretic complement of a set A in a set N.

LEMMA 3. *Let $\{e_n \mid n \in N\}$ be a set of orthogonal idempotents in a right self injective regular ring. Then for every subset A of N, there exists an idempotent $E_A \in R$ such that*

$$\begin{aligned} &E_A e_n = e_n && \textit{for all } n \in A \\ &e_{n'} E_A = E_A e_{n'} = 0 && \textit{for all } n' \in N \sim A \\ &E_A + E_{N \sim A} = E_N . \end{aligned}$$

Proof. Since R is regular, $Z(R_R) = 0$. Then, by Lemma 2, each right ideal I of R has a unique maximal essential extension I^* in R. Since R_R is injective, by [2] I has an injective hull in R_R which is a maximal essential extension of I in R_R. Thus each right ideal I has a unique injective hull I^* in R_R. Then as remarked in the proof of Lemma 1, there exists $e = e^2 \in R$ such that $I^* = eR$.

Hence for any subset A of N, there exists an idempotent $e_A \in R$ such that

$$e_A R = \Big(\sum_{n\in A} e_n R\Big)^* .$$

Since $\{e_n \mid n \in N\}$ are orthogonal,

$$\Big(\sum_{n\in A} e_n R\Big) \cap \Big(\sum_{n'\in N\sim A} e_{n'} R\Big) = 0 .$$

Then $e_A R \cap e_{N\sim A} R = 0$ (for $x \neq 0 \in e_A R \cap e_{N\sim A} R$ implies $xR \cap \sum_{n\in A} e_n R \cap \sum_{n'\in N\sim A} e_{n'} R \neq 0$, a contradiction.) Thus the sum $e_A A + e_{N\sim A} R$ is direct, and since each summand is injective, $e_A R \oplus e_{N\sim A} R$ is injective. Since injective hulls of right ideals of R are unique,

$$\Big(\sum_{n\in N} e_n R\Big)^* \supseteqq \Big(\sum_{n\in A} e_n R\Big)^* ,$$
$$\Big(\sum_{n\in N} e_n R\Big)^* \supseteqq \Big(\sum_{n'\in N\sim A} e_{n'} R\Big)^*$$

so $(\sum_{n\in N} e_n R)^* \supseteqq e_A R \oplus e_{N\sim A} R \supseteqq \sum_{n\in N} e_n R$. Then it follows $(\sum_{n\in N} e_n R)^* = e_A R \oplus e_{N\sim A} R$. Set $E_N = e_N$, where $e_N R = (\sum_{n\in N} e_n R)^*$. Let E_A (respectively $E_{N\sim A}$) be the projection of the idempotent E_N on $e_A R$ (respectively $e_{N\sim A} R$). We note that $E_A + E_{N\sim A} = E_N$ by definition, and E_A is simply the projection of the identity element of R on $e_A R$ with respect to the direct decomposition $R = (1 - E_N)R \oplus e_A R \oplus e_{N\sim A} R$. Thus E_A and $E_{N\sim A}$ are orthogonal. Furthermore

$$E_A e_n = e_n \qquad \forall n \in A$$
$$E_A e_{n'} = E_A E_{N\sim A} e_{n'} = 0 \qquad \forall n' \in N \sim A .$$

Since $e_A R' \supseteqq \sum_{n\in A} e_n R$, $(\sum_{n\in A} e_n R : E_A)$ is an essential right ideal of R. But

$$e_{n'} E_A\Big(\sum_{n\in A} e_n R : E_A\Big) \subseteqq e_{n'}\Big(\sum_{n\in A} e_n R\Big) = 0 \qquad \forall n' \in N \sim A .$$

Thus $e_{n'} E_A \in Z(R_R) = 0$, and we conclude $e_{n'} E_A = 0 \; \forall n' \in N \sim A$.

LEMMA 4. *Let R be a right self injective regular ring which contains an infinite set of orthogonal idempotents $\{e_n \mid n \in N\}$. Let $I = \sum \oplus e_n R$. For $A \subseteqq N$, let E_A be defined as in Lemma 3. Then a set $S_{\mathfrak{A}} = \{E_A \mid A \in \mathfrak{A}\}$, where each A is infinite, is independent modulo I, that is, $\sum_{A\in\mathfrak{A}} (E_A R + I)$ is direct in $R - I$, if and only if for any finite set $\{A_i \mid i = 1, \cdots, n\} \subseteqq \mathfrak{A}$, $A_i \cap \bigcup_{j\neq i} A_j$ is a finite subset of N, $1 \leqq i \leqq n$.*

Proof. Assume $S_{\mathfrak{A}}$ is independent modulo I, and let $\{A_i \mid i = 1, \cdots, n\} \subseteqq \mathfrak{A}$. Set $C = C_{ij} = A_i \cap A_j$. For all i and $j \neq i$, $E_{A_i} R \supseteqq \sum_{n\in C} e_n R$ and $E_{A_j} R \supseteqq \sum_{n\in C} e_n R$. Thus $E_{A_i} R \supseteqq (\sum_{n\in C} e_n R)^* = E_C R$ and

$E_{A_j}R \supseteqq (\sum_{n\in C} e_nR)^* = E_CR$. Since $0 = E_C - E_C = E_{A_i}E_C - E_{A_j}E_C$, $E_C \in I$. Then for all but a finite number of $n \in C$, $e_nE_C = 0$. Since this implies $e_n = e_nE_Ce_n = 0$ which is true for no n, C must be finite. Then $A_i \cap \bigcup_{j\neq i} A_j$ is a finite union of finite sets, and thus finite.

Now assume $A_i \cap \bigcup_{j\neq i} A_j$ is finite, and let $\sum_{j=1}^n E_{A_j}r_j \in I$. If $m \notin A_i$, $e_mE_{A_i}r_i = 0$ by Lemma 3. If $m \in A_i \sim \bigcup_{j\neq i} A_j$, $e_m \sum_{j=1}^n E_{A_j}r_j = e_mE_{A_i}r_i$. Since $\sum_{j=1}^n E_{A_j}r_j \in I$, there are at most a finite number of $m \in A_i \sim \bigcup_{j\neq i} A_j$ such that $e_mE_{A_i}r_i \neq 0$. Since $A_i \cap \bigcup_{j\neq i} A_j$ is also finite, the set

$$B = \{m \in N \mid e_mE_{A_i}r_i \neq 0\}$$

must be finite.

Now for all $n' \in N \sim B$,

$$\begin{aligned} 0 = e_{n'}E_{A_i}r_i &= e_{n'}([1 - E_N] + E_{N\sim B} + E_B)E_{A_i}r_i \\ &= e_{n'}E_{N\sim B}E_{A_i}r_i \ . \end{aligned}$$

Assume $E_{N\sim B}E_{A_i}r_i \neq 0$. Then, since $E_{N\sim B}R' \supseteqq \sum_{n'\in N\sim B} e_{n'}R$, there is an $s \in R$ such that $E_{N\sim B}E_{A_i}r_is \neq 0 \in \sum_{n'\in N\sim B} e_{n'}R$, so for some $n' \in N \sim B$, $e_{n'}E_{N\sim B}E_{A_i}r_is \neq 0$, a contradiction.

Then

$$E_{A_i}r_i = ([1 - E_N] + E_{N\sim B} + E_B)E_{A_i}r_i = E_BE_{A_i}r_i \ .$$

Since a finite direct sum of injective modules is injective, $\sum_{i\in B} e_iR = (\sum_{i\in B} e_iR)^* = E_BR$ and $E_BR \subseteqq I$. It follows that $E_{A_i}r_i \in I$.

LEMMA 5. *Let R be a right self injective regular ring which contains an infinite set of orthogonal idempotents $\{e_n \mid n \in N\}$. If $I = \sum_{n\in N} e_nR$, then $R - I$ is not an injective R-module.*

Proof. Let $\{A_i \mid i = 1, 2, \cdots\}$ be a countable family of subsets $A_i \subseteqq N$ such that $\{E_{A_i} \mid i = 1, 2, \cdots\}$ are independent in $R - I$. For example, the A_i may be disjoint countable subsets of N.

Let $\mathscr{P}$ denote the family of sets $S_{\mathfrak{A}} = \{E_{B_\alpha} \mid B_\alpha \subseteqq N, \alpha \in \mathfrak{A}\}$ where $\mathfrak{A}$ is some index set, such that $S_{\mathfrak{A}} \supseteqq \{E_{A_i} \mid i = 1, 2, \cdots\}$ and $S_{\mathfrak{A}}$ is independent modulo I. Partially order $\mathscr{P}$ by inclusion. Since independence modulo I depends only on finite sets of idempotents, $\mathscr{P}$ is inductive. By Zorn's lemma, select a maximal element $S \in \mathscr{P}$.

Let $J = \sum_{E_B\in S} E_BR$. Define $\varphi: J \to R - I$ by

$$\begin{aligned} \varphi(E_{A_i}) &= E_{A_i} + I \qquad \forall i = 1, 2, \cdots \\ \varphi(E_B) &= 0 + I \qquad \forall E_B \in S \sim \{E_{A_i}\} \\ \varphi\Big(\sum_{k=1}^n E_{B_k}r_k\Big) &= \sum_{k=1}^n \varphi(E_{B_k})r_k \qquad E_{B_k} \in S,\ r_k \in R \ . \end{aligned}$$

$\sum_{k=1}^n E_{B_k}r_k = 0$ implies $E_{B_k}r_k \in I$ since the idempotents of S are independent modulo I. Hence $\varphi(E_{B_k})r_k = 0 + I$, and $\varphi(\sum_{k=1}^{n} E_{B_k} r_k) = 0 + I$. Thus φ is a map which is clearly an R homomorphism.

Assume φ is induced by left multiplication by $m + I$ in $R - I$, $m \in R$. Then

$$(1) \qquad mE_{A_i} - E_{A_i} \in I \qquad \forall i = 1, 2, \cdots$$

and

$$(2) \qquad mE_B \in I \qquad \forall E_B \in S \sim \{E_{A_i}\} .$$

From (1) we conclude that for all but a finite number of $n \in A_i$, $e_n(mE_{A_i} - E_{A_i}) = 0$ and $e_n mE_{A_i} e_n = e_n E_{A_i} e_n$. Thus $e_n m e_n = e_n$ by Lemma 3.

From (2) we conclude that for all but a finite number of $n' \in B$, $e_{n'} mE_B = 0$, and $e_{n'} mE_B e_{n'} = e_{n'} m e_{n'} = 0$.

Let $j_1 \in A_1$, $e_{j_1} m e_{j_1} = e_{j_1}$. Select $j_{n+1} \in A_{n+1}$ such that

$$e_{j_{n+1}} m e_{j_{n+1}} = e_{j_{n+1}}$$

and

$$j_{n+1} \notin A_k \qquad \text{for all } k < n + 1 .$$

This is possible since $\{j \in A_{n+1} \mid e_j m e_j = e_j\}$ is infinite and Lemma 4 implies $A_{n+1} \cap \bigcup_{k=1}^{n} A_k$ is finite.

Since S is maximal in $\mathscr{P}$, by Lemma 4 $\{j_n \mid n = 1, 2, \cdots\}$ thus defined must have an infinite number of elements in common with some $B \subseteqq N$ such that $E_B \in S$. $B \neq A_i$ $i = 1, 2, \cdots$ since $j_n \notin A_i$ for all $n > i$. Therefore $\varphi(E_B) = 0$, and $e_{j'} = e_{j'} m e_{j'} = 0$ for all but a finite number of $j' \in B \cap \{j_n \mid n = 1, 2, \cdots\}$. This contradicts the assumption that $B \cap \{j_n \mid n = 1, 2, \cdots\}$ is an infinite set. Thus φ is not induced by left multiplication by $m + I$ in $R - I$. Hence $R - I$ is not injective. (See [1], p. 8.)

THEOREM. *Let R be a ring with* 1. *Then the following conditions are equivalent*:

(a) *R is semi-simple Artin.*

(b) *Every finitely generated right R-module is injective.*

(c) *Every cyclic right R-module is injective.*

Proof. $(a) \Rightarrow (b)$. By ([1], p. 11, Theorem 4.2), every right module over a semi-simple Artin ring R is injective, and so every finitely generated right R-module is injective.

(b) $\Rightarrow$ (c). Since every cyclic R-module is finitely generated by one element, (c) is a special case of (b).

(c) $\Rightarrow$ (a). If every cyclic R-module is injective, by Lemma 1, R

is right self injective and regular. By Lemma 5, R cannot contain an infinite set of orthogonal idempotents. It is well known that this condition in any regular ring R implies that R satisfies the minimum condition and hence is semi-simple Artin.

COROLLARY. *Let R be a right self injective, hereditary ring with identity. Then R is semi-simple Artin.*

Proof. R hereditary is equivalent to the condition that every quotient of an injective R-module is injective. (See [1], p. 14.) Since every cyclic module is isomorphic to a quotient of the injective module R_R, every cyclic R-module is injective. Therefore by the theorem R is semi-simple Artin.

REFERENCES

1. H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton University Press, 1956.
2. B. Eckmann and A. Schopf, *Über injektive Moduln*, Archiv der Mathematik, **4** (1953), 75–78.
3. Carl Faith, *Theory of Rings*, Lecture Notes (dittoed), Rutgers University, 1963.
4. R. E. Johnson, *The extended centralizer of a ring over a module*, Proc. Amer. Math. Soc., **2** (1951), 891–895.
5. L. A. Skornjakov, *Rings with injective cyclic modules*, Soviet Mathematics, **4** (1963), 36–38.

RUTGERS, THE STATE UNIVERSITY