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# MEASURE-THEORETIC PROPERTIES OF NON-MEASURABLE SETS

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**This article discusses the interior and exterior measures of two disjoint point sets $S_1, S_2$ and their union set $S_1 \cup S_2$. Besides well-known inequalities on the six quantities $m_i(S)$ and $m_e(S)$ for $S = S_1, S_2$, and $S_1 \cup S_2$, further inequalities are obtained. Indeed, a complete colleciton of inequalities on these six quantities is obtained, which are both necessary and sufficient conditions. The complete collection of inequalities are expressible as: there are a certain six linear combinations of the six quantities which are each $\geq 0$, and these six linear combinations can be independently assigned any nonnegative real value or $\infty$, subject to their sum being $\leq m(X)$, where $X$ is the entire space or a measurable set containing $S_1$ and $S_2$.**

**1. Introduction.** Consider any point set $S$ on the real number line or in Euclidean $n$-dimensional space. (That the space is Euclidean is unessential; general measure spaces, subject to a limitation, will be taken up in a separate article.) The set $S$ has an interior Lebesgue measure $m_i(S)$ and an exterior Lebesgue measure $m_e(S)$, which are non-negative real numbers or $\infty$ satisfying

$$(1.1) \qquad 0 \leq m_i(S) \leq m_e(S),$$

$$(1.2) \qquad m_i(S) \leq m_i(T), \quad m_e(S) \leq m_e(T) \quad \text{for } S \subset T,$$

where $S$ and $T$ are two sets with $S$ contained in $T$. A bounded set is Lebesgue measurable if $m_i(S) = m_e(S)$, and the common value is its measure $m(S)$; an unbounded set $S$ is Lebesgue measurable if the intersection of $S$ with every bounded interval is Lebesgue measurable (then $m_i(S) = m_e(S)$). For two disjoint sets $S_1$ and $S_2$, i.e. $S_1 \cap S_2 = \varnothing$ where $\varnothing$ is the symbol for the empty or null set, it is standard that if $S_1$ and $S_2$ are measurable, then $S_1 \cup S_2$ is also measurable and $m(S_1 \cup S_2) = m(S_1) + m(S_2)$. The present article considers any two disjoint sets $S_1$ and $S_2$, whether measurable or not, and obtains a *complete* collection of independent inequalities on the six quantities $m_i(S)$ and $m_e(S)$ for $S = S_1, S_2$, and $S_1 \cup S_2$.

A set $S$ is non-measurable if $m_i(S) < m_e(S)$, or if $m_i(S) = m_e(S) = \infty$ and the intersection of $S$ with some bounded interval is non-measurable. There are non-measurable sets, and indeed there are a large number of them. This is well known **[1]** and shown in books on measure theory, and also incidentally shown briefly in §7 here. For two disjoint sets $S_1$ and $S_2$, where $S_1 \cap S_2 = \varnothing$, it is known that

$$\tag{1.3} \begin{cases} m_i(S_1 \cup S_2) \geq m_i(S_1) + m_i(S_2), \\ m_e(S_1 \cup S_2) \leq m_e(S_1) + m_e(S_2). \end{cases}$$

In words, interior measure is superadditive and exterior measure is subadditive. Place

$$\tag{1.4} d_i(S_1, S_2) = m_i(S_1 \cup S_2) - m_i(S_1) - m_i(S_2) \geq 0,$$

$$\tag{1.5} d_e(S_1, S_2) = m_e(S_1) + m_e(S_2) - m_e(S_1 \cup S_2) \geq 0,$$

for any pair of disjoint sets $S_1, S_2, S_1 \cap S_2 = \varnothing$, having finite exterior (and therefore interior) measures. The definition of $d_i(S_1, S_2)$ and $d_e(S_1, S_2)$ when $m_e(S_1)$ or $m_e(S_2)$ or both are infinite will be given in §3, formulas (3.1), (3.2), and (3.3). The quantities $d_i(S_1, S_2)$ and $d_e(S_1, S_2)$ may be called the "differences" or "deficiencies" associated with the pair of disjoint sets $S_1, S_2$ and describe numerically how far the interior and exterior measures differ from the exact additivity property for measurable sets. In this article, the quantities $d_i$ and $d_e$ will be studied and a simple relation between them found, namely

$$\tag{1.6} d_i(S_1, S_2) \leq d_e(S_1, S_2).$$

Also, other inequalities for $m_i$ and $m_e$ of $S_1, S_2, S_1 \cup S_2$ will be obtained. Indeed, a *complete* collection of independent inequalities on the six quantities $m_i(S)$ and $m_e(S)$ for $S = S_1, S_2$, and $S_1 \cup S_2$ will be found. These are necessary conditions, and if six real numbers satisfy this complete collection of inequalities, there are pairs of disjoint sets $S_1, S_2$ with these values of the six quantities. Additional set functions of pairs of disjoint sets $S_1, S_2$ are introduced. These results are stated in Theorems 5 and 10, or in the paragraph containing formulas (8.14) and (8.15).

Define the *average* measure $m_a(S)$ of any set $S$ by

$$\tag{1.7} m_a(S) = \tfrac{1}{2}(m_i(S) + m_e(S)).$$

The inequality (1.6) can be rephrased, by inserting equations (1.4), (1.5) into (1.6), transposing some terms and dividing by 2, becoming

$$\tag{1.8} m_a(S_1 \cup S_2) \leq m_a(S_1) + m_a(S_2).$$

This states that the average measure $m_a$ is *subadditive* (like exterior measure and not interior measure). Indeed, it will be shown that if one has a countable number of mutually disjoint sets $S_\nu$, $\nu = 1, 2, \ldots$ to $N$ or to $\infty$, then

$$m_a\left(\bigcup_\nu S_\nu\right) \leq \sum_\nu m_a(S_\nu), \tag{1.9}$$

and so average measure is *countably* subadditive. It is known that $m_e(S)$ is countably subadditive, and $m_i(S)$ is countably superadditive, and $m(S)$ for measurable sets $S$ is countably additive.

There are non-measurable sets, and indeed a large number of them. It should be said that a non-measurable set $S$ is actually partially measurable, having an interior measure $m_i(S)$ and an exterior measure $m_e(S)$ with $0 \leq m_i(S) < m_e(S)$. A measurable set $S$, of finite measure, is just a set with $m_i(S) = m_e(S)$. One could say that a non-measurable set $S$ has as measure an undetermined value between $m_i(S)$ and $m_e(S)$, for instance $m_a(S)$, or that it has a range of values between $m_i(S)$ and $m_e(S)$. A non-measurable set may be more appropriately called a partially measurable set, having an interior measure and an exterior measure satisfying (1.1), (1.2).

In a broad sense of measure, if one is thinking of applications, not necessarily mathematical, an exact measurement might not be available for some process or subject. But a lower value and an upper value might be available, like interior measure and exterior measure. Finding properties of the lower and upper values would be of interest. Or an estimate (such as average measure) could be considered.

**2. Some lemmas.** If $S$ is any set, which may be non-measurable, it is well known that if $L$ is any measurable set $\subset S$ then $m(L) \leq m_i(S)$ and there is a measurable set $B \subset S$ with $m(B) = m_i(S)$; and if $L$ is any measurable set $\supset S$ then $m(L) \geq m_e(S)$ and there is a measurable set $K \supset S$ with $m(K) = m_e(S)$. Some lemmas concerning any sets $S$, whether measurable or not, will first be found.

LEMMA 1. *If a countable number of sets $S_\nu$, $\nu = 1, 2, \ldots$ to $N$ or to $\infty$, are contained in mutually disjoint measurable sets $L_\nu$, $S_\nu \subset L_\nu$ for all $\nu = 1, 2, \ldots$, and $L_\mu \cap L_\nu = \varnothing$ for all $\mu, \nu$ with $\mu \neq \nu$, then*

$$m_i\left(\bigcup_\nu S_\nu\right) = \sum_\nu m_i(S_\nu), \quad m_e\left(\bigcup_\nu S_\nu\right) = \sum_\nu m_e(S_\nu).$$

*Proof.* Select a measurable set $B \subset (\bigcup_\nu S_\nu)$ with $m(B) = m_i(\bigcup_\nu S_\nu)$. Then $(B \cap L_\mu) \subset ((\bigcup_\nu S_\nu) \cap L_\mu) = \bigcup_\nu (S_\nu \cap L_\mu) = S_\mu$ since $(S_\nu \cap L_\mu) \subset (L_\nu \cap L_\mu) = \varnothing$ for all $\nu \neq \mu$, and $S_\mu \cap L_\mu = S_\mu$. Therefore, $m(B \cap L_\mu) \leq m_i(S_\mu)$ for every $\mu = 1, 2, \ldots$. Now, $B \subset (\bigcup_\nu S_\nu) \subset (\bigcup_\nu L_\nu)$, so that

$$m_i\left(\bigcup_\nu S_\nu\right) = m(B) = m\left(B \cap \left(\bigcup_\nu L_\nu\right)\right) = m\left(\bigcup_\nu (B \cap L_\nu)\right)$$
$$= \sum_\nu m(B \cap L_\nu) \leq \sum_\nu m_i(S_\nu).$$

But interior measure is countably superadditive, $m_i(\bigcup_\nu S_\nu) \geq \sum_\nu m_i(S_\nu)$, and the equality $m_i(\bigcup_\nu S_\nu) = \sum_\nu m_i(S_\nu)$ is established.

Let $K$ be a measurable set $\supset (\bigcup_\nu S_\nu)$ with $m(K) = m_e(\bigcup_\nu S_\nu)$. Then $K \supset (K \cap (\bigcup_\nu L_\nu)) \supset (\bigcup_\nu S_\nu)$, so that $m(K \cap (\bigcup_\nu L_\nu)) = m_e(\bigcup_\nu S_\nu)$ also. Now, $(K \cap L_\nu) \supset S_\nu$, so that $m(K \cap L_\nu) \geq m_e(S_\nu)$, and therefore

$$m_e\left(\bigcup_\nu S_\nu\right) = m\left(K \cap \left(\bigcup_\nu L_\nu\right)\right) = m\left(\bigcup_\nu (K \cap L_\nu)\right)$$
$$= \sum_\nu m(K \cap L_\nu) \geq \sum_\nu m_e(S_\nu).$$

But exterior measure is countably subadditive, $m_e(\bigcup_\nu S_\nu) \leq \sum_\nu m_e(S_\nu)$, and the equality $m_e(\bigcup_\nu S_\nu) = \sum_\nu m_e(S_\nu)$ is established. Lemma 1 is proved.

A consequence of Lemma 1 is the following: if $L$ and $S$ are disjoint sets, where $L$ is a measurable set, then

$$(2.1) \qquad \begin{cases} m_i(L \cup S) = m(L) + m_i(S), \\ \\ m_e(L \cup S) = m(L) + m_e(S). \end{cases} \qquad L \cap S = \varnothing,$$

This is by Lemma 1, since $S$ is contained in the measurable set (entire space $-L$).

LEMMA 2. *Suppose that $S_1 \cup S_2$ is measurable, where $S_1$ and $S_2$ are disjoint sets, $S_1 \cap S_2 = \varnothing$. Then*

$$m(S_1 \cup S_2) = m_i(S_1) + m_e(S_2) = m_e(S_1) + m_i(S_2).$$

*Proof.* Let $B_1$ be a measurable set $\subset S_1$ with $m(B_1) = m_i(S_1)$. Then $(B_1 \cup S_2) \subset (S_1 \cup S_2)$, and $B_1$ is disjoint from $S_2$, so that $m(S_1 \cup S_2) \geq m_e(B_1 \cup S_2) = m(B_1) + m_e(S_2)$ by (2.1). Thus,

$$(2.2) \qquad m(S_1 \cup S_2) \geq m_i(S_1) + m_e(S_2).$$

Let $K_2$ be a measurable set $\supset S_2$ with $m(K_2) = m_e(S_2)$. Then $K_2 \supset (K_2 \cap (S_1 \cup S_2)) \supset S_2$ and $m(K_2) \geq m(K_2 \cap (S_1 \cup S_2)) \geq m_e(S_2)$, so that $m(K_2 \cap (S_1 \cup S_2)) = m_e(S_2)$ also. Now for finite $m_e(S_2)$, one obtains from $((S_1 \cup S_2) - (K_2 \cap (S_1 \cup S_2)) \subset S_1$ that

$$\begin{aligned} m_i(S_1) &\geq m[(S_1 \cup S_2) - (K_2 \cap (S_1 \cup S_2))] \\ &= m(S_1 \cup S_2) - m(K_2 \cap (S_1 \cup S_2)) = m(S_1 \cup S_2) - m_e(S_2). \end{aligned}$$

Adding the finite quantity $m_e(S_2)$ to both sides of this inequality gives $m(S_1 \cup S_2) \leq m_i(S_1) + m_e(S_2)$. If $m_e(S_2)$ is infinite, this last inequality is still true, and together with (2.2), the first equation of Lemma 2 for $m(S_1 \cup S_2)$ is obtained. Interchanging the roles of $S_1$ and $S_2$ establishes the second equation of Lemma 2 for $m(S_1 \cup S_2)$. Lemma 2 is proved.

Another formulation of Lemma 2 is as follows. Suppose that $S \subset L$ where $L$ is measurable. Then, if $m_e(S)$ is finite,

$$\begin{aligned} m_i(L - S) &= m(L) - m_e(S), \\ m_e(L - S) &= m(L) - m_i(S), \quad S \subset L. \end{aligned} \tag{2.3}$$

This is from Lemma 2 for $S_1 = S$, $S_2 = L - S$. If $m_e(S) = \infty$, replace (2.3) by Lemma 2 for $S_1 = S$, $S_2 = L - S$, which reduces to just $m_e(L - S) + m_i(S) = m(L) = \infty$. Formula (2.3), or Lemma 2, states a *complementation* property of interior and exterior measures.

Incidentally, above and subsequently, $\infty$ is a possible value of an interior measure, exterior measure, or measure, and has the properties: $\infty + \text{finite} = \infty$, $\text{finite} + \infty = \infty$, $\infty + \infty = \infty$, $\infty > \text{finite}$, $\text{finite} < \infty$, $\infty - \text{finite} = \infty$; $\infty - \infty$ is undetermined and has no meaning, and finite $-\infty$ has no meaning as a measure, interior or exterior, since these have non-negative values. The relations $<$, $>$, and $=$ are mutually exclusive; and the commutative and associative laws of addition hold.

LEMMA 3. *Suppose that $S \subset M$, where $M$ is measurable, and that $m_i(M - S) = 0$ (for finite $m(M)$, $m_i(M - S) = 0$ is equivalent to $m_e(S) = m(M)$). If $L$ is a measurable set $\subset M$, then*

$$m_e(S \cap L) = m(L) \quad \textit{and} \quad m_i(L - (S \cap L)) = 0.$$

*Proof.* $L - (S \cap L) = (M \cap L) - (S \cap L) = ((M - S) \cap L) \subset (M - S)$. Therefore $m_i(L - (S \cap L)) \leq m_i(M - S) = 0$ and so $m_i(L - (S \cap L)) = 0$. By Lemma 2, $m(L) = m_e(S \cap L) + m_i(L - (S \cap L)) = m_e(S \cap L)$. The statement in parentheses in Lemma 3 follows from Lemma 2: $m(M) = m_e(S) + m_i(M - S)$. Lemma 3 is proved.

The next lemma refers to $d_i(S_1, S_2)$ and $d_e(S_1, S_2)$ for two disjoint sets $S_1$ and $S_2$, when the pair $(S_1, S_2)$ is expressible as a countable union of pairs $(S_1^\nu, S_2^\nu)$ which are contained in mutually disjoint measurable sets $L^\nu$.

LEMMA 4. *Suppose that $S_1$ and $S_2$ are disjoint sets, $S_1 \cap S_2 = \varnothing$, and $S_1 = \bigcup_\nu S_1^\nu, S_2 = \bigcup_\nu S_2^\nu$ for a countable number of $\nu = 1, 2, \ldots$ (to $N$ or to $\infty$). And suppose that $(S_1^\nu \cup S_2^\nu) \subset L^\nu$ where $L^\nu$ is a measurable set for every $\nu = 1, 2, \ldots$, and that the measurable sets $L^\nu$ are mutually disjoint, i.e. $L^\mu \cap L^\nu = \varnothing$ for all $\mu, \nu$ with $\mu \neq \nu$. Then*

$$(2.4) \qquad d_i(S_1, S_2) = \sum_\nu d_i(S_1^\nu, S_2^\nu), \quad d_e(S_1, S_2) = \sum_\nu d_e(S_1^\nu, S_2^\nu),$$

$$(2.5) \qquad m_a(S_1) = \sum_\nu m_a(S_1^\nu).$$

*Proof.* The equality of (2.5) is true by (1.7) and Lemma 1. Incidentally, (2.5) holds for any single set $S$ by taking $S_1 = S$, $S_1^\nu = S^\nu$ for all $\nu$, and $S_2 = \varnothing$, $S_2^\nu = \varnothing$.

The equations for $d_i$ and $d_e$ in (2.4) are true when $m_e(S_1 \cup S_2)$ is finite, by using the homogeneous linear formulas (1.4) and (1.5) for $d_i$ and $d_e$, and Lemma 1. For then all the quantities $m_i(S)$ and $m_e(S)$ when $S = S_1, S_2, S_1 \cup S_2, S_1^\nu, S_2^\nu, S_1^\nu \cup S_2^\mu$ are finite, and $d_i(S_1^\nu, S_2^\nu) = m_i(S_1^\nu \cup S_2^\nu) - m_i(S_1^\nu) - m_i(S_2^\nu)$, and likewise for $d_e(S_1^\nu, S_2^\nu)$. Summing over all $\nu = 1, 2, \ldots$ (to $N$ or to $\infty$) gives

$$\sum_\nu d_i(S_1^\nu, S_2^\nu) = \sum_\nu m_i(S_1^\nu \cup S_2^\nu) - \sum_\nu m_i(S_1^\nu) - \sum_\nu m_i(S_2^\nu),$$

since

$$\sum_\nu m_i(S_1^\nu) = m_i(S_1), \quad \sum_\nu m_i(S_2^\nu) = m_i(S_2), \quad \text{and}$$

$$\sum_\nu m_i(S_1^\nu \cup S_2^\nu) = m_i(S_1 \cup S_2)$$

by Lemma 1, which are finite amounts; and so

$$\sum_\nu d_i(S_1^\nu, S_2^\nu) = m_i(S_1 \cup S_2) - m_i(S_1) - m_i(S_2) = d_i(S_1, S_2).$$

Likewise for $d_e(S_1, S_2)$, and Lemma 4 is proved when $m_e(S_1 \cup S_2)$ is finite, and in particular when $S_1$ and $S_2$ are bounded sets. The case of Lemma 4 when $m_e(S_1 \cup S_2) = \infty$ will be taken up in the §3.

The formulas (1.4) and (1.5) for $d_i$ and for $d_e$ cannot be used if a subtractive term in the formula is $\infty$. But $d_i$ and $d_e$ can still be defined, and the formulas (1.4) and (1.5) rewritten as

$$
\begin{cases} m_i(S_1 \cup S_2) = m_i(S_1) + m_i(S_2) + d_i(S_1, S_2), \\ m_e(S_1) + m_e(S_2) = m_e(S_1 \cup S_2) + d_e(S_1, S_2), \end{cases} \tag{2.6}
$$

$$
d_i(S_1, S_2) \geq 0, \quad d_e(S_1, S_2) \geq 0. \tag{2.7}
$$

The definitions of $d_i(S_1, S_2)$ and $d_e(S_1, S_2)$ are in the next section.

**3. Definitions when an exterior measure is infinite.** The entire Euclidean $n$-dimensional space of all points $(x_1, x_2, \ldots, x_n)$, for all real values of $x_1, x_2, \ldots,$ and $x_n$, can be written as the union $\bigcup_{\nu=1}^{\infty} X^{\nu}$ of a countably infinite number of mutually disjoint bounded measurable sets $X^{\nu}$, $\nu = 1, 2, \ldots$ to $\infty$; for example, as $\bigcup_{k_1} \bigcup_{k_2} \cdots \bigcup_{k_m} I^{(k_1, k_2, \ldots, k_n)}$ where $I^{(k_1, k_2, \ldots, k_n)}$ is the half-open unit interval of measure 1 consisting of all points $(x_1, x_2, \ldots, x_n)$ for which

$$
k_1 \leq x_1 < k_1 + 1, \quad k_2 \leq x_2 < k_2 + 1, \ldots, k_n \leq x_n < k_n + 1,
$$

and the $k_1, k_2, \ldots, k_n$ are integers which range over all integer values from $-\infty$ to $+\infty$ independently. All the intervals $I^{(k_1, k_2, \ldots, k_n)}$ are countably infinite in number, and can be arranged in some order as $X^{\nu}$ with $\nu = 1, 2, \ldots$ to $\infty$, so that

$$
\text{entire space} = \bigcup_{\nu} X^{\nu}, \quad X^{\mu} \cap X^{\nu} = \varnothing \text{ for all } \mu, \nu \text{ with } \mu \neq \nu, \tag{3.1}
$$

where $X^{\nu}$ are mutually disjoint bounded measurable sets. Then, for any two disjoint sets $S_1$ and $S_2$,

$$
S_1 = \bigcup_{\nu=1}^{\infty} (S_1 \cap X^{\nu}),
$$

$$
S_2 = \bigcup_{\nu=1}^{\infty} (S_2 \cap X^{\nu}).
$$

The two sets $S_1 \cap X^{\nu}$, $S_2 \cap X^{\nu}$ for any $\nu$ are disjoint bounded sets, and $d_i(S_1 \cap X^{\nu}, S_2 \cap X^{\nu})$ and $d_e(S_1 \cap X^{\nu}, S_2 \cap X^{\nu})$ can be defined as in (1.4) and (1.5), and they satisfy (2.6) and (2.7) above. Then define $d_i(S_1, S_2)$ and $d_e(S_1, S_2)$ by

$$
d_i(S_1, S_2) = \sum_{\nu=1}^{\infty} d_i(S_1 \cap X^{\nu}, S_2 \cap X^{\nu}), \tag{3.2}
$$

$$d_e(S_1, S_2) = \sum_{\nu=1}^{\infty} d_e(S_1 \cap X^\nu, S_2 \cap X^\nu). \tag{3.3}$$

These being sums of non-negative numbers, note that $\infty$ is a possible value of $d_i(S_1, S_2)$ and $d_e(S_1, S_2)$ when the corresponding infinite series diverges to $\infty$, as well as a non-negative real number.

The formulas (2.6) and (2.7) are satisfied for the pair of disjoint sets $S_1 \cap X^\nu$, $S_2 \cap X^\nu$ for each $\nu$, and (2.6) is

$$\begin{aligned} m_i((S_1 \cap X^\nu) \cup (S_2 \cap X^\nu)) = m_i(S_1 \cap X^\nu) + m_i(s_2 \cap X^\nu) \\ + d_i(S_1 \cap X^\nu, S_2 \cap X^\nu), \end{aligned}$$

and similarly for (2.7). Summing for $\nu$ from 1 to $k$ gives

$$\begin{aligned} &\sum_{\nu=1}^{k} m_i((S_1 \cup S_2) \cap X^\nu) \\ &\qquad = \sum_{\nu=1}^{k} m_i(S_1 \cap X^\nu) + \sum_{\nu=1}^{k} m_i(S_2 \cap X^\nu) \\ &\qquad\quad + \sum_{\nu=1}^{k} d_i(S_1 \cap X^\nu, S_2 \cap X^\nu), \end{aligned}$$

and letting $k \to \infty$, using Lemma 1 and the definition (3.2) gives

$$m_i(S_1 \cup S_2) = m_i(S_1) + m_i(S_2) + d_i(S_1, S_2),$$

all the quantities being $\geq 0$ and only additions being involved. Similarly for the second line of (2.6) and for (2.7). Thus, the formula (2.6) and (2.7) are established.

If $S_1$ and $S_2$ are both bounded sets, then (2.6) and (2.7) imply (1.4) and (1.5) since all the terms in (2.6) involving $m_i$ and $m_e$ are finite, and so $d_i(S_1, S_2)$ and $d_e(S_1, S_2)$ are finite by (2.6), and transpositions give (1.4) and (1.5). Thus, the definitions of $d_i(S_1, S_2)$ and $d_e(S_1, S_2)$ given in (3.2) and (3.3) agree with their definitions in (1.4) and (1.5) when $S_1$ and $S_2$ are bounded sets. A further statement concerning the definitions of $d_i(S_1, S_2)$ and $d_e(S_1, S_2)$ will be made in the paragraph following the next paragraph.

Returning to the proof of Lemma 4 in §2, it has already been proved when $S_1$ and $S_2$ are bounded sets, in §2. Now let $S_1$ and $S_2$ be any pair of disjoint sets, and $S_1 = \bigcup_\nu S_1^\nu$, $S_2 = \bigcup_\nu S_2^\nu$, and $L^\nu$ as in the hypotheses of Lemma 4. One has, by the definition (3.2), $d_i(S_1^\nu, S_2^\nu) = \sum_{\mu=1}^\infty d_i(S_1^\nu \cap X^\mu, S_2^\nu \cap X^\mu)$ and likewise for $d_e$. Therefore,

$$\tag{3.4} \begin{aligned} \sum_\nu d_i(S_1^\nu, S_2^\nu) &= \sum_\nu \left( \sum_\mu d_i(S_1^\nu \cap X^\mu, S_2^\nu \cap X^\mu) \right) \\ &= \sum_\mu \left( \sum_\nu d_i(S_1^\nu \cap X^\mu, S_2^\nu \cap X^\mu) \right), \end{aligned}$$

this interchange of the order of summation being valid since all the terms are non-negative. Likewise for $d_e$. Since $S_1 = \bigcup_\nu S_1^\nu$ and $S_2 = \bigcup_\nu S_2^\nu$, one has $S_1 \cap X^\mu = \bigcup_\nu (S_1^\nu \cap X^\mu)$ and $S_2 \cap X^\mu = \bigcup_\nu (S_2^\nu \cap X^\mu)$, and for each $\mu$ the pair $S_1 \cap X^\mu$ and $S_2 \cap X^\mu$ are two bounded disjoint sets, so that Lemma 4 is applicable and therefore

$$\tag{3.5} d_i(S_1 \cap X^\mu, S_2 \cap X^\mu) = \sum_\mu d_i(S_1^\nu \cap X^\mu, S_2^\nu \cap X^\mu),$$

from (2.4) with $S_1 \cap X^\mu, S_2 \cap X^\mu$ replacing $S_1, S_2$ in (2.4). From (3.4),

$$\begin{aligned} \sum_\nu d_i(S_1^\nu, S_2^\nu) &= \sum_\mu \left( \sum_\nu d_i(S_1^\nu \cap X^\mu, S_2^\nu \cap X^\mu) \right) \\ &= \sum_\mu d_i(S_1 \cap X^\mu, S_2 \cap X^\mu) = d_i(S_1, S_2), \end{aligned}$$

by (3.5) and (3.2). This is the first equation in (2.4). Likewise for $d_e$ in place of $d_i$ in this paragraph, which gives the second equation in (2.4). Lemma 4 is proved.

If $S_1$ and $S_2$ are disjoint sets, and the entire space is expressed as $\bigcup_{\nu=1}^\infty Y^\nu$ of another countably infinite number of mutually disjoint bounded measurable sets, as in (3.1) with $Y^\nu$ replacing $X^\nu$, then $S_1 = \bigcup_\nu (S_1 \cap Y^\nu)$ and $S_2 = \bigcup_\nu (S_2 \cap Y^\nu)$, and Lemma 4 shows that $d_i(S_1, S_2) = \sum_{\nu=1}^\infty d_i(S_1 \cap Y^\nu, S_2 \cap Y^\nu)$, $d_e(S_1, S_2) = \sum_{\nu=1}^\infty (S_1 \cap Y^\nu, S_2 \cap Y^\nu)$. Therefore, using $Y^\nu$, $\nu = 1, 2, \ldots$ to $\infty$, in place of $X^\nu$, $\nu = 1, 2, \ldots$ to $\infty$, for the definitions of $d_i(S_1, S_2)$ and $d_e(S_1, S_2)$ in (3.2) and (3.3) gives the same values of $d_i(S_1, S_2)$ and $d_e(S_1, S_2)$ respectively.

Incidentally, note that $d_i(S_1, S_2)$ and $d_e(S_1, S_2)$ depend on the pair of disjoint sets $S_1, S_2$ and not on their order, so that $d_i(S_2, S_1) = d_i(S_1, S_2)$, $d_e(S_2, S_1) = d_e(S_1, S_2)$. And, if one of the sets $S_1, S_2$ is the empty set $\varnothing$, say $S_2 = \varnothing$, then

$$d_i(S,\varnothing) = d_i(\varnothing,S) = 0, \quad d_e(S,\varnothing) = d_e(\varnothing,S) = 0. \tag{3.6}$$

This is by (1.4) and (1.5) if $S$ is a bounded set, and by (3.2) and (3.3) for any $S$. Also,

$$d_i(S_1,S_2) = d_e(S_1,S_2) = 0 \tag{3.7}$$

if $S_1$ and $S_2$ are both measurable sets, by (1.4) and (1.5) when $S_1$ and $S_2$ are bounded sets, and then by (3.2) and (3.3) for any measurable sets $S_1, S_2$. ((3.7) holds if one of $S_1, S_2$ is measurable, by (2.1).)

This §3 will be completed by the following lemma.

LEMMA 5. *For any set $S$ there is a measurable set $B \subset S$ for which $m(B) = m_i(S)$ and $m_i(S-B) = 0$, and there is a measurable set $L \supset S$ for which $m(L) = m_e(S)$ and $m_i(L-S) = 0$.*

*Proof.* It is well known that there is a measurable set $B \subset S$ for which $m(B) = m_i(S)$, and a measurable set $L \supset S$ for which $m(L) = m_e(S)$. If $m_i(S)$ is finite, then $m_i(S-B) = 0$ follows from (2.1) applied to the disjoint sets $B$ and $S-B$. If $m_e(S)$ is finite, then $m_i(L-S) = 0$ follows from (2.3). So Lemma 5 is proved for sets $S$ with finite $m_i(S)$ or $m_e(S)$, and in particular for bounded sets $S$. For $m_i(S)$ or $m_e(S)$ infinite, write the entire space as $\bigcup_\nu X^\nu$ as in (3.1). For each $\nu$ there are, by Lemma 5 for bounded sets, measurable sets $B^\nu \subset (S \cap X^\nu)$ and $L^\nu \supset (S \cap X^\nu)$ for which $m_i((S \cap X^\nu) - B^\nu) = 0$ and $m_i(L^\nu - (S \cap X^\nu)) = 0$. Place $B = \bigcup_\nu B^\nu$ and $L = \bigcup_\nu (L^\nu \cap X^\nu)$. Then

$$S - B = \left(\bigcup_\nu (S \cap X^\nu)\right) - \left(\bigcup_\nu B^\nu\right) = \bigcup_\nu ((S \cap X^\nu) - B^\nu)$$

and by Lemma 1,

$$m_i(S-B) = \sum_\nu m_i((S \cap X^\nu) - B^\nu) = \sum_{\nu=1}^{\infty} 0 = 0;$$

$m(B) = m_i(S)$ follows from (2.1) applied to the disjoint sets $B$ and $S-B$. Also,

$$L - S = \left(\bigcup_\nu (L^\nu \cap X^\nu)\right) - \left(\bigcup_\nu (S \cap X^\nu)\right) = \bigcup_\nu ((L^\nu \cap X^\nu) - (S \cap X^\nu))$$

and by Lemma 1,

$$m_i(L-S) = \sum_{\nu} m_i((L^\nu \cap X^\nu) - (S \cap X^\nu)) = \sum_{\nu=1}^{\infty} 0 = 0$$

since

$$((L^\nu \cap X^\nu) - (S \cap X^\nu)) \subset (L^\nu - (S \cap X^\nu)) \quad \text{and}$$
$$m_i(L^\nu - (S \cap X^\nu)) = 0.$$

By Lemma 2 applied to the disjoint sets $L - S$ and $S$, $m(L) = m_i(L-S) + m_e(S) = m_e(S)$. Lemma 5 is proved.

Concerning Lemma 5, it suffices to state merely $m_i(S-B) = 0$ and $m_i(L-S) = 0$ as the properties of $B \subset S$ and $L \supset S$. (Note that it is $m_i$ that appears in both $= 0$ statements.) For, $m(B) = m_i(S)$ follows from $m_i(S-B) = 0$ by (2.1) applied to the disjoint sets $B$ and $S - B$; and $m(L) = m_e(S)$ follows from $m_i(L-S) = 0$ by Lemma 2 applied to the disjoint sets $L - S$ and $S$. Incidentally, in Lemma 5, the sets $B$ and $L$ may be chosen as Borel sets.

**4. An inequality for the differences.** The first main theorem of this article is

THEOREM 1. *Suppose that $S_1$ and $S_2$ are two disjoint sets, $S_1 \cap S_2 = \varnothing$. Then*

$$0 \le d_i(S_1, S_2) \le d_e(S_1, S_2).$$

*Proof.* Select measurable sets $B_1 \subset S_1$ and $B_2 \subset S_2$ as in Lemma 5, for which $m_i(S_1 - B_1) = 0$ and $m_i(S_2 - B_2) = 0$.

Now, $S_1 = B_1 \cup (S_1 - B_1)$ and $S_2 = B_2 \cup (S_2 - B_2)$, and $(S_1 - B_1) \subset$ (the entire space $-B_1$), which is a measurable set disjoint from $B_1$, and $(S_2 - B_2) \subset$ (the entire space $-B_2$), which is a measurable set disjoint from $B_2$. By Lemma 4 for $N = 2$,

$$\begin{aligned} d_i(S_1, S_2) &= d_i(B_1, B_2) + d_i(S_1 - B_1, S_2 - B_2) \\ &= d_i(S_1 - B_1, S_2 - B_2) \quad \text{and} \end{aligned}$$

$$\begin{aligned} d_e(S_1, S_2) &= d_e(B_1, B_2) + d_e(S_1 - B_1, S_2 - B_2) \\ &= d_e(S_1 - B_1, S_2 - B_2) \end{aligned}$$

since $d_i(B_1, B_2) = d_e(B_1, B_2) = 0$, by (3.7). The sets $S_1 - B_1$ and $S_2 - B_2$ both have interior measure 0, so to prove Theorem 1 it suffices to prove Theorem 1 when both sets $S_1, S_2$ of the theorem have interior measure 0.

Suppose that $Z_1$ and $Z_2$ are disjoint sets, $Z_1 \cap Z_2 = \varnothing$, and both have interior measure 0, $m_i(Z_1) = 0$ and $m_i(Z_2) = 0$. Select measurable sets $L_1 \supset Z_1$ and $L_2 \supset Z_2$ as in Lemma 5, with $m_i(L_1 - Z_1) = 0$ and $m_i(L_2 - Z_2) = 0$. Now,

$$L_1 \cup L_2 = (L_1 \cap L_2) \cup (L_1 - (L_1 \cap L_2)) \cap (L_2 - (L_1 \cap L_2)), \quad \text{and}$$

$$Z_j \subset (L_1 \cup L_2) \quad \text{for } j = 1 \text{ and } 2,$$

so that

$$(4.1) \qquad \begin{aligned} Z_j = (Z_j \cap (L_1 \cap L_2)) &\cup [Z_j \cap (L_1 - (L_1 \cap L_2))] \\ &\cup [Z_j \cap (L_2 - (L_1 \cap L_2))] \quad \text{for } j = 1, 2. \end{aligned}$$

The three measurable sets $L_1 \cap L_2$, $L_1 - (L_1 \cap L_2)$, $L_2 - (L_1 \cap L_2)$ are mutually disjoint, and the two disjoint sets $Z_j$, for $j = 1$ and 2, are each expressed in (4.1) as a union of three sets, one in each of these three mutually disjoint measurable sets. By Lemma 4, (2.4) expresses $d_i(Z_1, Z_2)$ and $d_e(Z_1, Z_2)$ as sums of three $d_i$'s and three $d_e$'s respectively, corresponding to the three terms of the right-hand sides of (4.1). The last two terms in both these sums are zero. For, $Z_2 \cap (L_1 - (L_1 \cap L_2)) = \varnothing$ since $Z_2 \subset L_2$ and $L_2 \cap (L_1 - (L_1 \cap L_2)) = \varnothing$. The pair of sets in the second expression on the right-hand sides of (4.1) for $j = 1$ and $j = 2$ is $Z_1 \cap (L_1 - (L_1 \cap L_2))$ and $\varnothing$, and

$$\begin{aligned} &d_i(Z_1 \cap (L_1 - (L_1 \cap L_2)), \varnothing) \\ &\quad = d_e(Z_1 \cap (L_1 - (L_1 \cap L_2)), \varnothing) = 0 \quad \text{by (3.6)}. \end{aligned}$$

Likewise, the pair of sets in the third expression on the right-hand sides of (4.1), for $j = 1$ and $j = 2$, is $\varnothing$ and $Z_2 \cap (L_2 - (L_1 \cap L_2))$, so that

$$d_i(\varnothing, Z_2 \cap (L_2 - (L_1 \cap L_2))) = d_e(\varnothing, Z_2 \cap (L_2 - (L_1 \cap L_2))) = 0.$$

There remains the terms in the first expression on the right-hand sides of (4.1), for $j = 1$ and $j = 2$. The result is

$$(4.2) \qquad \begin{cases} d_i(Z_1, Z_2) = d_i(Z_1 \cap (L_1 \cap L_2), Z_2 \cap (L_1 \cap L_2)), \\ d_e(Z_1, Z_2) = d_e(Z_1 \cap (L_1 \cap L_2), Z_2 \cap (L_1 \cap L_2)). \end{cases}$$

Now, $m_i(Z_1 \cap (L_1 \cap L_2)) = 0$ and $m_i(Z_2 \cap (L_1 \cap L_2)) = 0$ since both these sets are contained in $Z_1$ and $Z_2$ respectively, and $m_i(Z_1) = 0$, $m_i(Z_2) = 0$. The first formula of (4.2) and of (2.6) give

$$(4.3) \qquad d_i(Z_1, Z_2) = m_i((Z_1 \cup Z_2) \cap (L_1 \cap L_2)).$$

The second formula of (4.2) and of (2.6) give

$$(4.4)\qquad \begin{aligned} &d_e(Z_1, Z_2) + m_e((Z_1 \cup Z_2) \cap (L_1 \cap L_2)) \\ &\quad = m_e(Z_1 \cap (L_1 \cap L_2)) + m_e(Z_2 \cap (L_1 \cap L_2)). \end{aligned}$$

But $m_i(L_1 - Z_1) = 0$ by the selection of $L_1 \supset Z_1$, and $(L_1 \cap L_2) \subset L_1$, so that Lemma 3 with $M = L_1$, and $S = Z_1$, and $L = L_1 \cap L_2$ gives $m_e(Z_1 \cap (L_1 \cap L_2)) = m(L_1 \cap L_2)$. Likewise, $m_e(Z_2 \cap (L_1 \cap L_2)) = m(L_1 \cap L_2)$. Also,

$$Z_1 \cap (L_1 \cap L_2) \subset ((Z_1 \cup Z_2) \cap (L_1 \cap L_2)) \subset (L_1 \cap L_2)$$

and $m_e(Z_1 \cap (L_1 \cap L_2)) = m(L_1 \cap L_2)$ shows that

$$m_e((Z_1 \cup Z_2) \cap (L_1 \cap L_2)) = m(L_1 \cap L_2).$$

Placing these three equal values $m(L_1 \cap L_2)$ into (4.4) gives

$$(4.5)\qquad d_e(Z_1, Z_2) + m(L_1 \cap L_2) = m(L_1 \cap L_2) + m(L_1 \cap L_2).$$

If $Z_1$ and $Z_2$ are bounded sets, then $m(L_1 \cap L_2)$ is finite, and (4.5) establishes that

$$(4.6)\qquad d_e(Z_1, Z_2) = m(L_1 \cap L_2).$$

This and (4.3) yield

$$(4.7)\qquad d_i(Z_1, Z_2) \leq d_e(Z_1, Z_2).$$

This inequality (4.7) is established when $Z_1$ and $Z_2$ are bounded sets. For any disjoint sets $Z_1, Z_2$ for which $m_i(Z_1) = m_i(Z_2) = 0$, use the definitions of $d_i(Z_1, Z_2)$ and $d_e(Z_1, Z_2)$ in (3.2) and (3.3). For each $\mu$, the sets $Z_1 \cap X^\mu$ and $Z_2 \cap X^\mu$ are bounded disjoint sets for which $m_i(Z_1 \cap X^\mu) = 0$ and $m_i(Z_2 \cap X^\mu) = 0$ so that by (4.7)

$$0 \leq d_i(Z_1 \cap X^\mu, Z_2 \cap X^\mu) \leq d_e(Z_1 \cap X^\mu, Z_2 \cap X^\mu).$$

Summing over all positive integers $\mu$ from 1 to $\infty$, and using (3.2) and (3.3) for the pair $Z_1, Z_2$ gives (4.7).

Continuing with the proof of Theorem 1, it was shown above in the first paragraph of the proof that it suffices to prove Theorem 1 for the pair of sets $S_1 - B_1$, $S_2 - B_2$, both of which have interior measure 0. Theorem 1 is proved.

A consequence of Theorem 1 is: if $d_e(S_1, S_2) = 0$ then $d_i(S_1, S_2) = 0$. That is, if $m_e(S_1 \cup S_2) = m_e(S_1) + m_e(S_2)$, then $m_i(S_1 \cup S_2) = m_i(S_1) + m_i(S_2)$. But $d_i(S_1, S_2) = 0$ does not necessarily imply that $d_e(S_1, S_2) = 0$.

**5. Average measure.** Another form of Theorem 1, and a generalization, is stated in Theorem 2 immediately below, using the *average measure* $m_a(S)$ of a set $S$, defined in (1.7).

THEOREM 2. *The average measure $m_a(S)$ of a set $S$, defined by $m_a(S) = \frac{1}{2}(m_i(S) + m_e(S))$, is subadditive, i.e., $m_a(S_1 \cup S_2) \le m_a(S_1) + m_a(S_2)$ for any two disjoint sets $S_1, S_2$. More generally, suppose that $S_\nu$, $\nu = 1, 2, \ldots$ to $N$ or to infinity, are a finite or countably infinite number of mutually disjoint sets, and consider the union $\bigcup_\nu S_\nu$ of these sets. Then*

$$m_a\left(\bigcup_\nu S_\nu\right) \le \sum_\nu m_a(S_\nu).$$

*Proof.* If $S_1$ and $S_2$ are disjoint bounded sets, insert the definitions (1.4) and (1.5) of $d_i(S_1, S_2)$ and $d_e(S_1, S_2)$ into the inequality $d_i(S_1, S_2) \le d_e(S_1, S_2)$ of Theorem 1, transpose suitable terms and divide by 2. The result is the first inequality of Theorem 2. For any disjoint sets $S_1$ and $S_2$, using (3.1),

$$\begin{aligned} m_a((S_1 \cup S_2) \cap X^\nu) &= m_a((S_1 \cap X^\nu) \cup (S_2 \cap X^\nu)) \\ &\le m_a(S_1 \cap X^\nu) + m_a(S_2 \cap X^\nu), \end{aligned}$$

and summing for all $\nu$ from 1 to $\infty$ gives $m_a(S_1 \cup S_2) \le m_a(S_1) + m_a(S_2)$ by (1.7) and Lemma 1, and the terms in the infinite series are all $\ge 0$. For any two sets $S_1$ and $S_2$, one has $S_1 \cup S_2 = S_1 \cup (S_2 - (S_2 \cap S_1))$ so that

$$m_a(S_1 \cup S_2) \le m_a(S_1) + m_a(S_2 - (S_2 \cap S_1)) \le m_a(S_1) + m_a(S_2)$$

since $(S_2 - (S_2 \cap S_1)) \subset S_2$. The first sentence of Theorem 2 is proved.

For the second sentence of Theorem 2, if the number of sets $S_\nu$ is finite, the theorem is proved by mathematical induction, as follows: Supposing the theorem true for $k$ sets, $m_a(\bigcup_{\nu=1}^k S_\nu) \le \sum_{\nu=1}^k m_a(S_\nu)$, then for $k+1$ sets

$$\begin{aligned} m_a\left(\bigcup_{\nu=1}^{k+1} S_\nu\right) &= m_a\left(\left(\bigcup_{\nu=1}^{k} S_\nu\right) \cup S_{k+1}\right) \le m_a\left(\bigcup_{\nu=1}^{k} S_\nu\right) + m_a(S_{k+1}) \\ &\le \sum_{\nu=1}^{k} m_a(S_\nu) + m_a(S_{k+1}) = \sum_{\nu=1}^{k+1} m_a(S_\nu), \end{aligned}$$

which is the theorem for $k+1$ sets. Since the theorem is true for 2 sets, the theorem is true for any finite number $N$ of sets, by the principle of mathematical induction.

If the number of sets $S_\nu$ is infinite, $\nu = 1, 2, \dots$ to infinity, note first that Theorem 2 is true if $\sum_{\nu=1}^{\infty} m_a(S_\nu)$ is divergent, or if any $m_a(S_\nu) = \infty$, since then $\sum_{\nu=1}^{\infty} m_a(S_\nu) = \infty$. Suppose that $\sum_{\nu=1}^{\infty} m_a(S_\nu)$ is convergent. Since $0 \le m_i(S) \le m_e(S)$, one has from (1.7) that $0 \le \frac{1}{2} m_e(S) \le m_a(S) \le m_e(S)$, and the convergence of $\sum_{\nu=1}^{\infty} m_a(S_\nu)$ implies (and is implied by) the convergence of $\sum_{\nu=1}^{\infty} m_e(S_\nu)$. Let $K_\nu$ be a measurable set $\supset S_\nu$ with $m(K_\nu) = m_e(S_\nu)$. Then $\sum_{\nu=1}^{\infty} m(K_\nu)$ is convergent, and given any positive number $\varepsilon$ there is an integer $k = k(\varepsilon)$ such that $\sum_{\nu=k+1}^{\infty} m(K\nu) < \varepsilon$. Now, for measurable sets it is known that $m(\bigcup_{\nu=k+1}^{\infty} K_\nu) \le \sum_{\nu=k+1}^{\infty} m(K_\nu) < \varepsilon$ (indeed $=$ holds if the $K_\nu$ are mutually disjoint), one has $m_e(\bigcup_{\nu=k+1}^{\infty} S_\nu) \le m(\bigcup_{\nu=k+1}^{\infty} K_\nu) < \varepsilon$. Then $\bigcup_{\nu=1}^{\infty} S_\nu = (\bigcup_{\nu=1}^{k} S_\nu) \cup (\bigcup_{\nu=k+1}^{\infty} S_\nu)$, so that $m_a(\bigcup_{\nu=1}^{\infty} S_\nu) \le m_a(\bigcup_{\nu=1}^{k} S_\nu) + \varepsilon$. But by the preceding paragraph, $m_a(\bigcup_{\nu=1}^{k} S_\nu) \le \sum_{\nu=1}^{k} m_a(S_\nu)$, so that

$$(5.1) \qquad m_a\left(\bigcup_{\nu=1}^{\infty} S_\nu\right) \le \sum_{\nu=1}^{k} m_a(S_\nu) + \varepsilon \le \bigcup_{\nu=1}^{\infty} m_a(S_\nu) + \varepsilon.$$

The terms involving $m_a(\ )$ on the two extreme sides of (5.1) are certain amounts, and (5.1) being true for any positive $\varepsilon$, letting $\varepsilon \to 0$ gives $m_a(\bigcup_{\nu=1}^{\infty} S_\nu) \le \sum_{\nu=1}^{\infty} m_a(S_\nu)$. Theorem 2 is proved. (Incidentally, note that the inequality $m_a(\bigcup_\nu S_\nu) \le \sum_\nu m_a(S_\nu)$ holds for any sets $S_\nu$, not necessarily mutually disjoint.)

In words, Theorem 2 states that the average measure $m_a(S)$ is a *sub*additive set funciton and indeed is a countably subadditive set function, just as the exterior measure $m_e(S)$ is. It is interesting to note that while the interior measure $m_i(S)$ is *super*-additive for disjoint sets, the average measure $m_a(S)$, which is $\frac{1}{2}$ the sum of $m_i(S)$ and $m_e(S)$, is *sub*additive like the exterior measure $m_e(S)$. For the sum $m_i(S) + m_e(S)$, the subadditivity of $m_e(S)$ overcomes the superadditivity of $m_i(S)$ resulting in the subadditivity of $m_a(S)$.

THEOREM 3. *If $S \subset L$ where $L$ is measurable, then*

$$m_a(S) + m_a(L - S) = m(L) = m_a(L).$$

*Proof.* This is a consequence of Lemma 2 with $S_1 = S$ and $S_2 = L - S$. Add the two resulting equalities of Lemma 2, and divide by 2, obtaining the equation of Theorem 3.

Theorem 3 is a *complementation* property of average measure, referring to the average measures of a set $S$ and its complement $L-S$ in a containing measurable set $L$ (such as an interval). Average measure is also non-negative and monotone increasing, $0 \leq m_a(S) \leq m_a(T)$ for $S \subset T$; and $m_a(S) = m(S)$ for measurable sets $S$, and $m_a(S) = 0$ only for sets of measure 0. Note that exterior measure $m_e(S)$ does not have this complementation property.

Also, note that $m_a(S) \geq 0$ and if $S$ is measurable, then $m_a(S) = m(S)$. And, if $m_a(S) = 0$, then $S$ is a measurable set of measure 0. For, since $m_i(S) \leq m_a(S)$, so $m_i(S) = 0$ and then $m_e(S) = 0$ from $m_a(S) = \frac{1}{2}m_i(S) + \frac{1}{2}m_e(S)$.

**6. More inequalities.** Another main theorem concerning two disjoint sets $S_1, S_2$ is

THEOREM 4. *If $S_1$ and $S_2$ are disjoint sets, $S_1 \cap S_2 = \emptyset$, then*

$$m_i(S_1 \cup S_2) \leq m_i(S_1) + m_e(S_2) \leq m_e(S_1 \cup S_2),$$
$$m_i(S_1 \cup S_2) \leq m_e(S_1) + m_i(S_2) \leq m_e(S_1 \cup S_2).$$

*Proof.* Pick a measurable set $L \supset (S_1 \cup S_2)$ with $m(L) = m_e(S_1 \cup S_2)$, and a measurable set $B_1 \subset S_1$ with $m(B_1) = m_i(S_1)$. The set $(L-B_1) \supset S_2$ since $S_2$ is disjoint from $S_1$, so that $((L-B_1) \cap (S_1 \cup S_2)) \supset S_2$, and $m_e((L-B_1) \cap (S_1 \cup S_2)) \geq m_e(S_2)$. Now,

$$\begin{aligned} S_1 \cup S_2 &= (B_1 \cap (S_1 \cup S_2)) \cup ((L-B_1) \cap (S_1 \cup S_2)) \\ &= B_1 \cup ((L-B_1) \cap (S_1 \cup S_2)). \end{aligned}$$

By Lemma 1,

$$\begin{aligned} m_e(S_1 \cup S_2) &= m(B_1) + m_e((L-B_1) \cap (S_1 \cup S_2)) \\ &\geq m_i(S_1) + m_e(S_2), \end{aligned}$$

and this is one of the inequalities of Theorem 4 for $m_e(S_1 \cup S_2)$. Interchanging the roles of $S_1$ and $S_2$ gives the other inequality of Theorem 4 for $m_e(S_1 \cup S_2)$.

Now, if $m_e(S_1 \cup S_2)$ is finite, placing (1.4) and (1.5) into the inequality of Theorem 1 gives

$$m_i(S_1 \cup S_2) - m_i(S_1) - m_i(S_2) \leq m_e(S_1) + m_e(S_2) - m_e(S_1 \cup S_2),$$

which can be written as

$$m_e(S_1 \cup S_2) - m_i(S_1) - m_e(S_2) \leq m_e(S_1) + m_i(S_2) - m_i(S_1 \cup S_2).$$

The left-hand side of this inequality is $\geq 0$ by the proved inequality in Theorem 4 for $m_e(S_1 \cup S_2)$, and therefore the right-hand side of this inequality is $\geq 0$. This is one of the inequalities of Theorem 4 for $m_i(S_1 \cup S_2)$. The other inequality of Theorem 4 for $m_i(S_1 \cup S_2)$ is obtained by interchanging the roles of $S_1$ and $S_2$. Theorem 4 is proved when $m_e(S_1 \cup S_2)$ is finite.

Suppose that $m_e(S_1 \cup S_2) = \infty$, and decompose the entire space as in (3.1). By what has just been proved

$$\begin{aligned} m_i((S_1 \cup S_2) \cap X^\nu) &= m_i((S_1 \cap X^\nu) \cup (S_2 \cap X^\nu)) \\ &\leq m_i(S_1 \cap X^\nu) + m_e(S_2 \cap X^\nu). \end{aligned}$$

Summing for all $\nu = 1, 2, \ldots$ to $\infty$, Lemma 1 gives

$$m_i(S_1 \cup S_2) \leq m_i(S_1) + m_e(S_2).$$

This is one of the inequalities of Theorem 4 for $m_i(S_1 \cup S_2)$, and interchanging the roles of $S_1$ and $S_2$ gives the other inequality for $m_i(S_1 \cup S_2)$. Theorem 4 is proved.

Besides the inequalities of Theorem 1 and Theorem 4, there are the well-known inequalities (1.3), and (1.1) for $S = S_1$ and $S_2$ and $S_1 \cup S_2$, and (1.2) for $T = S_1 \cup S_2$, $S = S_1$ and $S_2$. It will be shown that all these form a *complete* set of inequalities for $m_i(S)$ and $m_e(S)$ for $S = S_1$ and $S_2$ and $S_1 \cup S_2$, for every pair of disjoint sets $S_1$ and $S_2$. But first, the large number of these inequalities will be written in fewer and more manageable form.

For two disjoint sets $S_1$ and $S_2$, introduce the quantities $g_1(S_1, S_2)$ and $g_2(S_1, S_2)$ defined by

$$\text{(6.1)} \qquad \begin{cases} g_1(S_1, S_2) = m_e(S_1 \cup S_2) - m_i(S_1) - m_e(S_2) \geq 0, \\ g_2(S_1, S_2) = m_e(S_1 \cup S_2) - m_i(S_2) - m_e(S_1) \geq 0, \\ \hfill S_1 \cap S_2 = \varnothing, \end{cases}$$

which are non-negative by Theorem 4. And introduce the quantity $h(S_1, S_2)$ defined by

$$\text{(6.2)} \qquad h(S_1, S_2) = d_e(S_1, S_2) - d_i(S_1, S_2) \geq 0, \qquad S_1 \cap S_2 = \varnothing,$$

which is non-negative by Theorem 1. These are the definitions when $S_1$ and $S_2$ are bounded sets (and when $m_e(S_1 \cup S_2)$ is finite). Their definitions for any disjoint sets $S_1$ and $S_2$ are given the same way as for $d_i(S_1, S_2)$ and $d_e(S_1, S_2)$ in §3, by (3.1) and (3.2), (3.3), with the quantity $g_1(\ ,\ )$ replacing $d_i(\ ,\ )$ or $d_e(\ ,\ )$ in (3.2), (3.3), and likewise for $g_2(\ ,\ )$ and $h(\ ,\ )$. Replace the formula (6.1) and (6.2) by the transposed equations

$$(6.3)\quad \begin{cases} g_1(S_1,S_2)+m_i(S_1)+m_e(S_2) = g_2(S_1,S_2)+m_i(S_2)+m_e(S_1) \\ \qquad\qquad = m_e(S_1 \cup S_2), \\ h(S_1,S_2)+d_i(S_1,S_2) = d_e(S_1,S_2), \\ g_1(S_1,S_2) \geq 0, \quad g_2(S_1,S_2) \geq 0, \quad h(S_1,S_2) \geq 0, \end{cases}$$

as (1.4) and (1.5) were replaced by (2.6) and (2.7) in §2. The definitions of $g_1(S_1,S_2)$, $g_2(S_1,S_2)$, and $h(S_1,S_2)$, by (3.1) and analogously to (3.2) or (3.3), are shown to be consistent with (6.1) and (6.2), just as was done in §3 for $d_i(S_1,S_2)$ and $d_e(S_1,S_2)$. This is by using (6.3) for bounded sets and Lemma 1, thereby establishing (6.3) for any pair of disjoint sets $S_1,S_2$; and when $S_1$ and $S_2$ are bounded sets (or when $m_e(S_1 \cup S_2)$ is finite), transpositions in (6.3) give (6.1) and (6.2). Also, Lemma 4 with $g_1(\ ,\ )$ or $g_2(\ ,\ )$ or $h(\ ,\ )$ replacing $d_i(\ ,\ )$ or $d_e(\ ,\ )$ in (2.4) is proved just as in §§2 and 3. And the definitions of $g_1(S_1,S_2)$, $g_2(S_1,S_2)$, and $h(S_1,S_2)$ are independent of which decomposition (3.1) is used, as in §3.

Incidentally, the expression for $h(S_1,S_2)$ in (6.2) can also be written as

$$h(S_1,S_2) = m_i(S_1)+m_e(S_1)+m_i(S_2)+m_e(S_2) \\ - m_i(S_1 \cup S_2) - m_e(S_1 \cup S_2) \geq 0,$$

and also as

$$\tfrac{1}{2}h(S_1,S_2) = m_a(S_1)+m_a(S_2)-m_a(S_1 \cup S_2) = d_a(S_1,S_2) \geq 0,$$

where $d_a(S_1,S_2)$ is the average "difference" for the disjoint sets $S_1,S_2$.

Suppose $S_1$ and $S_2$ are any disjoint sets,

$$(6.4)\qquad S_1 \cap S_2 = \varnothing,$$

and consider their union $S_1 \cup S_2$. Place

$$(6.5)\quad \begin{cases} a_1 = m_i(S_1), & a_2 = m_i(S_2), & a = m_i(S_1 \cup S_2), \\ b_1 = m_e(S_1), & b_2 = m_e(S_2), & b = m_e(S_1 \cup S_2). \end{cases}$$

Concerning the six real numbers $a_1,b_1,a_2,b_2,a,b$, the superadditivity of interior measure and subadditivity of exterior measure is, as in (1.3),

$$(6.6)\qquad d_i = a - a_1 - a_2 \geq 0, \quad d_e = b_1 + b_2 - b \geq 0$$

where $d_i$ and $d_e$ are defined in (6.6), as in (1.4) and (1.5). The relations (6.6) may be written

$$(6.6)' \qquad d_i + a_1 + a_2 = a, \quad d_e + b = b_1 + b_2, \quad d_i \geq 0, \quad d_e \geq 0.$$

The non-negativeness and connection between interior and exterior measures (1.1), and the monotone increasing property (1.2), give

$$(6.7) \qquad \begin{cases} 0 \leq a_1 \leq b_1, & 0 \leq a_2 \leq b_2, \quad 0 \leq a \leq b, \\ a_1 \leq a, \quad b_1 \leq b, & a_2 \leq a, \quad b_2 \leq b. \end{cases}$$

In addition to these, Theorems 1 and 4 state

$$(6.8) \qquad d_i \leq d_e, \quad \text{and}$$

$$(6.9) \qquad a \leq a_1 + b_2 \leq b, \quad a \leq a_2 + b_1 \leq b.$$

The above large number of inqualities for $a_1$, $b_1$, $a_2$, $b_2$, $a$, $b$ in (6.6), (6.7), (6.8) and (6.9), 17 in all, will first be reduced in number and form. Introduce $d_i$ and $d_e$ as in (6.6) and $g_1$, $g_2$, and $h$ as in (6.1) and (6.2):

$$(6.10) \quad g_1 = b - a_1 - b_2 \geq 0, \quad g_2 = b - a_2 - b_1 \geq 0, \quad h = d_e - d_i \geq 0.$$

The relations in (6.10) can also be written as

$$(6.11) \quad g_1 + a_1 + b_2 = b, \quad g_2 + a_2 + b_1 = b, \quad g_1 \geq 0, \quad g_2 \geq 0, \quad \text{and}$$

$$(6.12) \qquad h + d_i = d_e, \quad h \geq 0.$$

From (6.6), $b_1 = (b - b_2) + d_e = g_1 + a_1 + d_e$ by (6.11), and (6.12) gives (and likewise for $b_2$)

$$(6.13) \qquad b_1 = a_1 + d_i + h + g_1, \quad b_2 = a_2 + d_i + h + g_2.$$

From (6.6) one has

$$(6.14) \qquad a = a_1 + a_2 + d_i \quad \text{and} \quad b = a_1 + a_2 + d_i + h + g_1 + g_2.$$

The second equation in (6.14) comes from (6.6), (6.13) and (6.12) :

$$\begin{aligned} b &= b_1 + b_2 - d_e \\ &= (a_1 + d_i + h + g_1) + (a_2 + d_i + h + g_2) - (d_i + h) \\ &= a_1 + a_2 + d_i + h + g_1 + g_2. \end{aligned}$$

Also, if the disjoint sets $S_1, S_2$ are both contained in a measurable set $X$, which might be an interval or the entire space, then

$$(6.15) \qquad b \leq m(X).$$

The relations (6.11), (6.12), (6.13), and (6.14) have been obtained above when all the quantities involved are finite, or $m_e(S_1 \cup S_2)$ is finite. In particular, they hold when $S_1$ and $S_2$ are bounded sets. If this is not the case, they still hold. Write the entire space as in (3.1), and they hold for the pair of sets $S_1 \cap X^\nu$, $S_2 \cap X^\nu$. Then summing these relations over all $\nu = 1, 2, \dots$ to $\infty$, and using Lemmas 1 and 4, and Lemma 4 with $g_1(\ ,\ )$ or $g_2(\ ,\ )$, or $h(\ ,\ )$ replacing $d_i(\ ,\ )$ or $d_e(\ ,\ )$ in (2.4), these relations are established just as in the establishment of (6.3) as in §3. Thus, (6.11), (6.12), (6.13) and (6.14) are proved in general, so that the quantities $a_1$, $b_1$, $a_2$, $b_2$, $a$, $b$ can be expressed in terms of $a_1$ $a_2$, $d_i$, $h$, $g_1$, $g_2$ by (6.13) and (6.14), leading to the next main theorem.

THEOREM 5. *For two disjoint sets $S_1$ and $S_2$ which are both contained in a measurable set $X$, all the relations* (6.6), (6.6)′, (6.7), (6.8), (6.9), ... *through* (6.15), *for the quantities in* (6.5), *can be written as*

$$a_1, a_2, d_i, h, g_1, g_2 \text{ are each } \geq 0, \quad \text{and} \tag{6.16}$$

$$a_1 + a_2 + d_i + h + g_1 + g_2 \leq m(X), \tag{6.17}$$

*and the quantities $b_1$, $b_2$, $a$, $b$ expressed in terms of the non-negative quantities $a_1$, $a_2$, $d_i$, $h$, $g_1$, $g_2$ by* (6.13) *and* (6.14), *and $d_e$ from* (6.12).

*Proof.* The inequalities (6.16) and (6.17), and the relations (6.12), (6.13), and (6.14) have already been obtained. Conversely, given six quantities (real numbers or $\infty$) $a_1$, $a_2$, $d_i$, $h$, $g_1$, $g_2$ satisfying (6.16) and (6.17), and obtaining $b_1$, $b_2$, $a$, $b$ from (6.13) and (6.14), and $d_e$ from (6.12), all the relations (6.6), (6.6)′, (6.7), (6.8), (6.9), ... through (6.15) are satisfied. For, $a_1 \geq 0$ and $a_2 \geq 0$ are stated in (6.16), and $a \geq 0$ comes from (6.14); and $b_1 \geq a_1$, $b_2 \geq a_2$ are evident from (6.13), and $b \geq a$ from (6.14); these are the first line of (6.7). The second line of (6.7) is evident from (6.14) and (6.13). And (6.6)′ comes from (6.14) and (6.12), (6.13), the second equation of (6.6)′ from a calculation of $d_e + b$ and $b_1 + b_2$; and (6.6) from (6.6)′ when $a_1, a_2$, and $b$ are finite. And (6.8) comes from (6.12). Concerning (6.9), $a_1 + b_2 = a_1 + a_2 + d_i + h + g_2$ from (6.13), which is evidently $\geq a$ and $\leq b$ from (6.14); and likewise for $a_2 + b_1 = a_2 + a_1 + d_i + h + g_1$; so that (6.9) is satisfied. And also (6.11) is satisfied by (6.13) and (6.14); and (6.10) from (6.11) and (6.12) when $a_1, b_2$ and $a_2, b_1$ and $d_i$ are finite. The formulas (6.12), (6.13), and (6.14) hold as stated in the theorem. And (6.15) comes from (6.14) and (6.17). Theorem 5 is proved. Incidentally, in Theorem 5, $d_e$ in (6.12) gives $d_e + b = b_1 + b_2$ and $d_e \geq 0$ in (6.6)$'$, so that (6.12) can be considered as the definition of $d_e$.

It will be shown in §8 that given *any* six non-negative numbers $a_1$, $a_2$, $d_i$, $h$, $g_1$, $g_2$, finite or $\infty$, satisfying (6.17), and obtaining $b_1$, $b_2$, $a$, $b$ from (6.13) and (6.14), there are disjoint sets $S_1$, $S_2$ both $\subset X$ such that (6.5) holds. Thus, the inequalities (6.16) and (6.17) are sufficient as well as necessary conditions.

**7. Some interesting non-measurable sets.** To prove the statements made in the preceding paragraph, some non-measurable sets will be needed. These will be obtained in this section, and are interesting in themselves. Consider first the real number line, and more particularly, the half-open unit interval $I$ of real numbers $x$, where $0 \leq x < 1$, in which addition is taken modulo 1. Or, consider the circumference $I$ of a circle of radius $1/2\pi$ in the plane, whose length is 1; addition of points on the circumference is defined by rotation of the cirumference. In either description of $I$, Lebesgue interior and exterior measure, and measurability, are defined, and are invariant under rotation of the circumference, or translation modulo 1 of the unit interval.

There is a standard construction of a non-measurable set $Z$ in the unit interval $I$, with $m_i(Z) = 0$ and $m_e(Z) > 0$. This is obtained by considering two real numbers $x$ and $y$, modulo 1, as equivalent if $x - y$ is a rational number, $x \sim y$ if $x - y = r$ where $r$ is a rational number, and forming the equivalence classes of real numbers. An equivalence class is a set $K$ of real numbers in $I$ of the form $K = \{x + r, \text{ for all } r\}$, where $x$ is a real number in $0 \leq x < 1$ and $r$ is a rational number, and addition $+$ is taken modulo 1. Two equivalence classes $K_j = \{x_j + r, \text{ for all } r\}$, $j = 1, 2$, are different, and are also disjoint, if $x_2 - x_1 \neq$ a rational number, and are identical if $x_2 - x_1 =$ a rational number. Form a set $Z$ by selecting one real number from each equivalence class, for all the different equivalence classes, using the axiom of choice. Define the set $Z + r$ in the unit interval $I$ as the set of all real numbers $z + r$, modulo 1, for all $z \in Z$ (or, on the unit circumference $I$, by rotating $Z$ through the angle $2\pi r$). Then,

$$(7.1) \qquad (Z + r_1) \cap (Z + r_2) = \varnothing \quad \text{for } r_1 \neq r_2 \text{ modulo } 1$$
$$\text{(i.e., } r_1 - r_2 \neq \text{integer), and}$$

$$(7.2) \qquad m_i(Z + r) = m_i(Z), \quad m_e(Z + r) = m_e(Z).$$

Take all the rational numbers $r_\nu$, $\nu = 1, 2, \ldots$, in the unit interval $I$, and form the set $\bigcup_\nu (Z + r_\nu)$. One has

$$1 = m(I) = m_i\left(\bigcup_\nu (Z + r_\nu)\right) \geq \sum_\nu m_i(Z + r_\nu) = \sum_\nu m_i(Z),$$

$$\sum_\nu m_e(Z) = \sum_\nu m_e(Z + r_\nu) \geq m_e\left(\bigcup_\nu (Z + r_\nu)\right) = m_e(I) = 1.$$

These show that

$$m_i(Z) = 0, \quad m_e(Z) > 0, \tag{7.3}$$

and indeed $Z$ is a non-measurable set.

Now, select a set $Z$ more carefully. Consider the totality of all closed sets of positive measure in the unit interval $I$. The number of these has the cardinal number of the continuum (since their complements in $I$ are all open sets of measure $< 1$, and all these have the cardinal number of the continuum). Let $C_\alpha$ designate a closed set of positive measure in $I$, where $\alpha$ is an ordinal number; and for two different ordinal numbers $\alpha_1, \alpha_2$ the closed sets $C_{\alpha_1}, C_{\alpha_2}$ are different closed sets; and $\alpha$ ranges over all ordinal numbers $< \omega$ where $\omega$ is the smallest ordinal number with the cardinal number of the continuum; and the collection of $C_\alpha$ for all $\alpha < \omega$ consists of all closed sets of positive measure contained in the unit interval $I$. (One could also use perfect sets.) Let $K_\beta \subset I$ designate an equivalence class of real numbers modulo 1, where $\beta$ is an ordinal number $< \omega$, and $K_{\beta_1} \cap K_{\beta_2} = \varnothing$ for two different ordinal numbers $\beta_1, \beta_2$; and $K_\beta$ for all $\beta < \omega$ covers all equivalence classes, so that $\bigcup_{\beta<\omega} K_\beta = I$.

Select $x_1 \in C_1$, and the ordinal number $\beta_1 < \omega$ such that $x_1 \in K_{\beta_1}$. The closed set $C_2$ of positive measure has a continuum number of points, and $K_{\beta_1}$ is countable, so that $C_2 - (C_2 \cap K_{\beta_1}) \neq \varnothing$, and select $x_2 \in (C_2 - (C_2 \cap K_{\beta_1}))$ and $\beta_2$ so that $x_2 \in K_{\beta_2}$. Note that $x_2 \notin K_{\beta_1}$ so that $\beta_2 \neq \beta_1$. Proceed by transfinite induction. Suppose, for an ordinal number $\gamma < \omega$, that real numbers $x_\alpha$, $0 \leq x_\alpha < 1$, and ordinal numbers $\beta_\alpha < \omega$, have been selected for all ordinal numbers $\alpha < \gamma$, with the properties

$$\begin{cases} x_{\alpha_1} \neq x_{\alpha_2} \quad \text{and} \quad \beta_{\alpha_1} \neq \beta_{\alpha_2} \text{ for all } \alpha_1 < \gamma, \\ \qquad\qquad\qquad\qquad \alpha_2 < \gamma \text{ with } \alpha_1 \neq \alpha_2, \quad \text{and} \\ x_\alpha \in (C_\alpha \cap K_{\beta_\alpha}) \quad \text{for all } \alpha < \gamma. \end{cases} \tag{7.4}$$

The set $\bigcup_{\alpha<\gamma} K_{\beta_\alpha}$ has a cardinal number $< \aleph$, where $\aleph$ is the cardinal number of the continuum, since $K_{\beta_\alpha}$ is countable and $\gamma < \omega$. The closed set $C_\gamma$ is of positive measure and so has the cardinal number $\aleph$, so that $C_\gamma - (C_\gamma \cap (\bigcup_{\alpha<\gamma} K_{\beta_\alpha})) \neq \emptyset$, and select $x_\gamma \in (C_\gamma - (C_\gamma \cap (\bigcup_{\alpha<\gamma} K_{\beta_\alpha})))$, and $\beta_\gamma$ such that $x_\gamma \in K_{\beta_\gamma}$. Now, $x_\gamma \notin K_{\beta_\alpha}$ for all $\alpha < \gamma$, so that $x_\gamma \neq x_\alpha$ for all $\alpha < \gamma$ since $x_\alpha \in K_{\beta_\alpha}$. And $\beta_\gamma \neq \beta_\alpha$ for all $\alpha < \gamma$ since $x_\gamma \in K_{\beta_\gamma}$ and $x_\gamma \notin K_{\beta_\alpha}$ for all $\alpha < \gamma$. And $x_\gamma \in C_\gamma$, so that $x_\gamma \in (C_\gamma \cap K_{\beta_\gamma})$. Thus, the real numbers $x_\alpha$, $0 \leq x_\alpha < 1$, and ordinal numbers $\beta_\alpha < \omega$, have been obtained for all $\alpha \leq \gamma$, having the properties (7.4) for all $\alpha \leq \gamma$ and all $\alpha_1 \leq \gamma$, $\alpha_2 \leq \gamma$ with $\alpha_1 \neq \alpha_2$. By the principle of transfinite induction, therefore, real numbers $x_\alpha$, $0 \leq x_\alpha < 1$, and ordinal numbers $\beta_\alpha < \omega$, can be selected for all ordinal numbers $\alpha < \omega$, having the properties stated in (7.4) with $\gamma$ in (7.4) replaced by $\omega$.

Consider the set $\tilde{Z} = \{x_\alpha$, for all $\alpha < \omega\}$. Enlarge the set $\tilde{Z}$ by selecting for every ordinal $\beta \notin \{\beta_\alpha$, for all $\alpha < \omega\}$ and $\beta < \omega$, a real number $x_\beta \in K_\beta$, and uniting the set of all such $x_\beta$ together with $\tilde{Z}$, forming the set $Z$:

$$(7.5) \qquad Z = \{x_\alpha; \alpha < \omega\} \cup \{x_\beta; \beta \notin \{\beta_\alpha; \alpha < \omega\} \text{ and } \beta < \omega\}.$$

The set $Z$ has exactly one point in common with $K_\gamma$ for every $\gamma < \omega$, so that the set $Z$ (and also $\tilde{Z}$) has a continuum number of points.

The set $Z$ (and also $\tilde{Z}$) has interior measure 0, as shown in (7.1), (7.2) and (7.3). And $m_e(Z) = 1$ (also, $m_e(\tilde{Z}) = 1$). For, let $B$ be an open set in the unit interval $I$ for which $B \supset Z$. Then the complement $B' = I - B$ of $B$ is a closed set in $I$. Now, for each ordinal number $\alpha < \omega$, $x_\alpha \in Z$ so that $x_\alpha \in B$ and $x_\alpha \notin B'$. Also, $x_\alpha \in C_\alpha$, so that the closed set $B'$ is *not* $C_\alpha$. But the totality of $C_\alpha$, for all $\alpha < \omega$, is *all* closed sets of positive measure, so that the closed set $B'$ has measure 0. Therefore $m(B) = 1$, and so

$$(7.6) \qquad m_e(Z) = 1, \quad \text{as well as } m_i(Z) = 0.$$

(Likewise $m_i(\tilde{Z}) = 0$ since $\tilde{Z} \subset Z$, and $m_e(\tilde{Z}) = 1$ since the above proof holds for $\tilde{Z}$ as well as $Z$.) Thus, equations (7.1), (7.2), (7.3) hold for $Z$, with (7.3) stating $m_e(Z) = 1$. (And (7.1), (7.2), (7.3) hold for $\tilde{Z}$, with $m_e(\tilde{Z}) = 1$.)

Now, enumerate all the rational numbers $r$, modulo 1, as $r_\nu$, $\nu = 1, 2, \ldots$, and designate $Z + r_\nu$ by $Z_\nu$. The following interesting result has been obtained.

THEOREM 6. *The half open unit interval $I = \{0 \leq x < 1\}$, of measure 1, can be written as the union of a countably infinite number of mutually* *disjoint sets* $Z_\nu$, $\nu = 1, 2, \ldots$,

$$I = \bigcup_{\nu=1}^{\infty} Z_\nu, \quad Z_{\nu_1} \cap Z_{\nu_2} = 0 \quad \textit{for } \nu_1 \neq \nu_2, \textit{ for which}$$

$$m_i(Z_\nu) = 0, \quad m_e(Z_\nu) = 1 \textit{ for all } \nu = 1, 2, \ldots.$$

*Note that* $m_i(I - Z_\nu) = 0$ *(and* $m_e(I - Z_\nu) = 1$*) by* (2.3), $\nu = 1, 2, \ldots$.

The above is on the real number line. For $n$-dimensional Euclidean space, $n \geq 2$, take the Cartesian product of $I$ and of each $Z_\nu$, $\nu = 1, 2, \ldots$, by a half open unit cube in $(n-1)$-dimensional Euclidean space. Specifically, using coordinates $(x_1, x_2, \ldots, x_n)$, take the set of all points $(x_1, x_2, \ldots, x_n)$ with $0 \leq x_2 < 1, \ldots, 0 \leq x_n < 1$ and $x_1 \in I$ or $x_1 \in Z_\nu$ for each $\nu = 1, 2, \ldots$. Calling the resulting sets again $I$ and $Z_\nu$, $\nu = 1, 2, \ldots$, their interior and exterior measures are multiplied by 1, and Theorem 6 holds for the half open unit interval (or cube)

$$I = \{(x_1, x_2, \ldots, x_n), \text{ where } 0 \leq x_j < 1 \text{ for } j = 1, 2, \ldots, n\}$$

in $n$-dimensional Euclidean space.

By translating $I$ and $Z_\nu \subset I$, $\nu = 1, 2, \ldots$, to the half open unit interval

$$I^{(k_1, \ldots, k_n)} = \{k_1 \leq x_1 < k_1 + 1, \ldots, k_n \leq x_n < k_n + 1\}$$

where $k_1, \ldots, k_n$ are integers, one obtains

$$Z_\nu(k_1, \ldots, k_n) \subset I^{(k_1, \ldots, k_n)}, \quad \nu = 1, 2, \ldots,$$

and Theorem 6 holds in $I^{(k_1, \ldots, k_n)}$. Enumerate all the $n$-tuplets $(k_1, \ldots, k_n)$, for all integer values from $-\infty$ to $+\infty$ of $k_1, \ldots, k_n$, and designate the $n$-tuplets $(k_1, \ldots, k_n)$ as $\kappa_\mu$ for $\mu = 1, 2, \ldots$ to $\infty$. Place $\hat{Z}_\nu = \bigcup_{\mu=1}^{\infty} Z_\nu^{\kappa_\mu}$, which is an abbreviated form for $\bigcup_{k_1=-\infty}^{\infty} \cdots \bigcup_{k_n=-\infty}^{\infty} Z_\nu^{(k_1, \ldots, k_n)}$. Then $\bigcup_{\mu=1}^{\infty} I^{\kappa_\mu} = E$ where $E$ is the entire Euclidean $n$-dimensional space, and by Lemma 1,

$$m_i(\hat{Z}_\nu) = \sum_{\mu=1}^{\infty} m_i(Z_\nu^{\kappa_\mu}) = \sum_{\mu=1}^{\infty} 0 = 0, \quad \text{and} \quad E - \hat{Z}_\nu = \bigcup_{\mu=1}^{\infty} (I^{\kappa_\mu} - Z_\nu^{\kappa_\mu})$$

so that

$$m_i(E - \hat{Z}_\nu) = \sum_{\mu=1}^{\infty} m_i(I^{\kappa_\mu} - Z_\nu^{\kappa_\mu}) = \sum_{\mu=1}^{\infty} 0 = 0.$$

The first sentence of the following theorem is established.

THEOREM 7. *The entire n-dimensional Euclidean space E can be written as* $E = \bigcup_{\nu=1}^{\infty} \hat{Z}_\nu$, *where* $\hat{Z}_{\nu_1} \cap \hat{Z}_{\nu_2} = \emptyset$ *for all pairs* $\nu_1, \nu_2$ *with* $\nu_1 \neq \nu_2$, *and* $m_i(\hat{Z}_\nu) = 0$, $m_i(E - \hat{Z}_\nu) = 0$ *for every* $\nu = 1, 2, \ldots$ *to* $\infty$. *For any measurable set* $X \subset E$ *of positive measure,*

$$X = \bigcup_{\nu=1}^{\infty} (X \cap \hat{Z}_\nu), \quad \textit{where}$$

$$(7.6) \qquad \begin{aligned} &m_i(X \cap \hat{Z}_\nu) = 0, m_e(X \cap \hat{Z}_\nu) = m(X) \quad \textit{and} \\ &m_i(X - (X \cap \hat{Z}_\nu)) = 0 \end{aligned}$$

*for every* $\nu = 1, 2, \ldots$.

*Proof.* The first sentence of Theorem 7 has been established above. For the second sentence, $X \cap \hat{Z}_\nu$ is a subset of $\hat{Z}_\nu$ and so $m_i(X \cap \hat{Z}_\nu) = 0$, and the second and third equalities of (7.6) follow from Lemma 3, for $M = E$, $S = \hat{Z}_\nu$, and $L = X$.

Important consequences of Theorem 7 are:

THEOREM 8. . *For any measurable set X of positive measure there is a countably infinite number of mutually disjoint sets*

$$(X \cap \hat{Z}_\nu) \subset X, \qquad \nu = 2, 3, \ldots,$$

*dropping* $\nu = 1$ (*or any particular* $\nu$), *with*

$$\begin{aligned} &m_i(X \cap \hat{Z}_\nu) = 0, \quad m_e(X \cap \hat{Z}_\nu) = m(X) \quad \textit{and} \\ &m_i(X - (X \cap \hat{Z}_\nu)) = 0 \quad \textit{for all } \nu = 2, 3, \ldots, \textit{ and} \end{aligned}$$

$$(7.7) \qquad \begin{cases} m_i\left(\bigcup_{\nu=2}^{\infty} (X \cap \hat{Z}_\nu)\right) = 0; \\ m_e\left(\bigcup_{\nu=2}^{\infty} (X \cap \hat{Z}_\nu)\right) = m(X) \textit{ and} m_i\left(X - \bigcup_{\nu=2}^{\infty} (X \cap \hat{Z}_\nu)\right) = 0. \end{cases}$$

*Proof.* $\bigcup_{\nu=2}^{\infty} (X \cap \hat{Z}_\nu) = X - (X \cap \hat{Z}_1)$, and the first and third equalities in (7.7) follow from (7.6), and the second equality in (7.7) from Lemma 2.

THEOREM 9. *Let X be a measurable set of positive measure. For any positive integer N there are N mutually disjoint sets* $Z_j$, $j = 1, 2, \ldots, N$, *contained in X such that* $m_i(Z_j) = 0$, $m_e(Z_j) = m(X)$ *and*

*$m_i(X - Z_j) = 0$, for $j = 1, 2, \ldots, N$, $m_i(\bigcup_{j=1}^{N} Z_j) = 0$, $m_e(\bigcup_{j=1}^{N} Z_j) = m(X)$ and $m_i(X - \bigcup_{j=1}^{N} Z_j) = 0$.*

*Proof.* Using Theorem 8, pick $Z_j = X \cap Z_{\nu_j}$, $j = 1, 2, \ldots, N$, where $\nu_{j_1} \neq \nu_{j_2}$ for all $j_1 \neq j_2$, $1 \leq j_1 \leq N$, $1 \leq j_2 \leq N$. The first line of the equations in Theorem 9 are stated in Theorem 8. For the second line of the equations in Theorem 9, which involve $\bigcup_{j=1}^{N} Z_j$, one has

$$(X \cap \hat{Z}_{\nu_1}) \subset \bigcup_{j=1}^{N}(X \cap \hat{Z}_{\nu_j}) \subset \bigcup_{\nu=1}^{\infty}(X \cap \hat{Z}_{\nu}).$$

By Theorem 8, both sides of these inclusions have the same interior measure and the same exterior measure, namely 0 and $m(X)$ respectively, and likewise for $X-$ the sets, so the second line of the equation in Theorem 9 is established. Theorem 9 is proved. Note that when $X$ is the entire space $E$, the sets $\hat{Z}_{\nu_j}$ are sets $Z_j$, $j = 1, 2, \ldots, N$; and for any measurable set $X$ of positive measure, the sets $X \cap \hat{Z}_{\nu_j}$ are sets $Z_j$, $j = 1, 2, \ldots, N$.

In the remainder of this article, Theorem 9 will be used for $N = 2$, so that there are two disjoint sets $Z_1, Z_2$ contained in $X$ with the properties stated in Theorem 9 for $N = 2$. It can be stated also that, in obtaining Theorem 9 for $N = 2$, the sets $Z_1$ and $Z_2$ were chosen as $X \cap \hat{Z}_{\nu_1}$ and $X \cap \hat{Z}_{\nu_2}$ for $\nu_1 \neq \nu_2$. Picking another pair $X \cap \hat{Z}_{\nu_3}$ and $X \cap \hat{Z}_{\nu_4}$, with $\nu_3 \neq \nu_4$ and $\nu_3 \neq \nu_1, \nu_2$ and $\nu_4 \neq \nu_1, \nu_2$, gives another pair $Z_1$ and $Z_2$ satisfying Theorem 9 for $N = 2$, which are both disjoint from the first pair. Continuing, there are a countably infinite number of pairs $Z_1, Z_2 \subset X$ satisfying Theorem 9 for $N = 2$ (also for any $N$), and the various sets $Z_1 \cup Z_2$ are mutually disjoint.

Note that the above proofs do not make use of the continuum hypothesis of set theory.

Incidentally, the sets $Z_1$ and $Z_2$ of Theorem 9 for $N = 2$ were obtained by first selecting a set $Z$ in the unit interval $I$ as in (7.5) and (7.6). Sets $Z_1$ and $Z_2$ satisfying Theorem 9 for $N = 2$ can also be obtained from any particular set $Z$ such as described in the paragraph containing formulas (7.1), (7.2), (7.3), by a different kind of construction.

**8. A complete collection of inequalities.** The following main theorem will now be proved.

THEOREM 10. *Let $X$ be a measurable set of positive measure. Given any six non-negative real numbers or $\infty$, namely $a_1$, $a_2$, $d_i$, $h$, $g_1$, $g_2$,* *satisfying*

$$a_1 + a_2 + d_1 + h + g_1 + g_2 \leq m(X), \tag{8.1}$$

*there are two disjoint sets* $S_1, S_2$ *obtained in* $X$,

$$S_1 \subset X, \quad S_2 \subset X, \quad S_1 \cap S_2 = \varnothing,$$

*such that*

$$\begin{cases} m_i(S_1) = a_1, & m_e(S_1) = a_1 + d_i + h + g_1 = b_1, \\ m_i(S_2) = a_2, & m_e(S_2) = a_2 + d_i + h + g_2 = b_2, \\ m_i(S_1 \cup S_2) = a_1 + a_2 + d_i = a, \\ m_e(S_1 \cup S_2) = a_1 + a_2 + d_i + h + g_1 + g_2 = b. \end{cases} \tag{8.2}$$

*Proof.* Given the six non-negative real numbers or $\infty$, namely $a_1$, $a_2$, $d_i$, $h$, $g_1$, $g_2$, satisfying (8.1), six *mutually disjoint measurable* sets $A_1$, $A_2$, $D_i$, $H$, $G_1$, $G_2$ will first be constructed for which

(8.3) $\qquad A_1, A_2, D_i, H, G_1, G_2$ are all $\subset X$, such that

$$\begin{cases} m(A_1) = a_1, & m(A_2) = a_2, & m(D_i) = d_i, \\ m(H) = h, & m(G_1) = g_1, & m(G_2) = g_2. \end{cases} \tag{8.4}$$

This is a consequence of the following Lemmas 7 and 8.

LEMMA 7. *Let* $L$ *be a measurable set and* $c$ *any non-negative real number or* $\infty$ *which is* $\leq m(L)$. *There is a measurable set* $K \subset L$ *for which* $m(K) = c$. *If* $m(L) = \infty$ *and* $c = \infty$, *there is a measurable set* $K \subset L$ *with* $m(K) = \infty$ *and* $m(L - K) = \infty$.

*Proof.* Consider first the case that $c < m(L)$. For the real number line, or Euclidean $n$-dimensional space $(x_1, \ldots, x_n)$, form the sets $K_r$: $K_r = L \cap \{x_1^2 + \cdots + x^2 \leq r^2\}$, where $r \geq 0$. One has, for $r_1 < r_2$, that $K_{r_1} \subset K_{r_2}$ and $K_{r_2} - K_{r_1} = L \cap \{r_1^2 < x_1^2 + \cdots + x_n^2 \leq r_2^2\}$, so that

$$m(K_{r_2}) = m(K_{r_1}) + m(L \cap \{r_1^2 < x_1^2 + \cdots + x_n^2 \leq r_2^2\}) \quad \text{and}$$

$$0 \leq m(K_{r_2}) - m(K_{r_1}) \leq m(\{r_1^2 < x_1^2 + \cdots + x_n^2 \leq r_2^2\}).$$

The right-hand side is a fixed multiple of $(r_2^n - r_1^n)$, which $\to 0$ as $r_2 \to r_1$ or $r_1 \to r_2$. Thus, $m(K_r)$ is a continuous function of $r$, and monotone increasing, and $m(K_0) = 0$, and $m(K_r) \to m(L)$ as $r \to \infty$. The last is true if $L$ is an unbounded set, as is well known, and if $L$ is a bounded set $m(K_r) = m(L)$ for all sufficiently large $r$. Therefore, for any value $c < m(L)$ there is at least one value of $r$ for which

$m(K_r) = c$, and $K$ is such a $K_r$. (If $c = 0$, take $K = \varnothing$ or a single point or several points in $L$; and if $c = m(L)$, take $K = L$.)

If $c = \infty$, and so $m(L) = \infty$, write the entire space as $\bigcup_\nu X^\nu$ as in (3.1). Then $\infty = m(L) = m(\bigcup_\nu (L \cap X^\nu)) = \sum_\nu m(L \cap X^\nu)$, and there is a measurable set $K^\nu \subset (L \cap X^\nu)$ with $m(K^\nu) = \frac{1}{2} m(L \cap X^\nu)$ and $m((L \cap X^\nu) - K^\nu) = \frac{1}{2} m(L \cap X^\nu)$. Then $K = \bigcup_\nu K^\nu$ has

$$m(K) = \sum_\nu m(K^\nu) = \tfrac{1}{2} \sum_\nu m(L \cap X^\nu) = \infty,$$

and

$$L - K = \bigcup_\nu (L \cap X^\nu) - \bigcup_\nu K^\nu = \bigcup_\nu ((L \cap X^\nu) - K^\nu), \quad \text{and}$$
$$m(L - K) = \sum_\nu m((L \cap X^\nu) - K^\nu) = \tfrac{1}{2} \sum_\nu m(L \cap X^\nu) = \infty.$$

Lemma 7 is proved.

LEMMA 8. *Let $L$ be a measurable set, and $c_1, c_2, \ldots, c_k$ be $k$ nonnegative real numbers or $\infty$ with $\sum_{\nu=1}^k c_\nu \leq m(L)$. Then there are $k$ mutually disjoint measurable sets $K_1, K_2, \ldots, K_k$, all $\subset L$, with $m(K_\nu) = c_\nu$ for $\nu = 1, 2, \ldots, k$.*

*Proof.* Since $c_1 \leq m(L)$, from $\sum_{\nu=1}^k c_\nu \leq m(L)$, there is by Lemma 7 a measurable set $K_1 \subset L$ with $m(K_1) = c_1$, and if $c_1 = \infty = m(L)$, with $m(K_1) = \infty = c_1$ and $m(L - K_1) = \infty$. Since $c_2 \leq \sum_{\nu=2}^k c_\nu \leq m(L) - c_1 = m(L - K_1)$, and if $c_1 = \infty$ it is still true that $c_2 \leq \sum_{\nu=1}^k c_\nu \leq m(L - K_1)$ since $m(L - K_1) = \infty$, there is by Lemma 7 a measurable set $K_2 \subset (L - K_1)$ with $m(K_2) = c_2$, and if $c_2 = \infty$, with $m(K_2) = \infty = c_2$ and $m(L - K_1 - K_2) = \infty$. The set $K_2$ is disjoint from the set $K_1$. Continuing in this fashion, successively, one obtains $k$ (true for $k = \infty$ also) mutually disjoint measurable sets $K_\nu$, $\nu = 1, 2, \ldots, k$, all $\subset L$, with $m(K_\nu) = c_\nu$ for $\nu = 1, 2, \ldots, k$. The lemma is proved.

Returning to the proof of Theorem 10, an application of this lemma gives (8.3) and (8.4), by (8.1). Now, consider the case of Theorem 9 for $N = 2$, so that $Z_1$ and $Z_2$ are sets such as in Theorem 9 for $N = 2$. Define the sets $S_1$ and $S_2$ contained in $X$ by

(8.5) $\qquad S_1 = A_1 \cup (Z_1 \cap D_i) \cup (Z_1 \cap H) \cup (Z_1 \cap G_1),$

(8.6) $\qquad S_2 = A_2 \cup (D_i - (Z_1 \cap D_i)) \cup (Z_2 \cap H) \cup (Z_2 \cap G_2).$

Then

$$(8.7)\quad S_1 \cup S_2 = A_1 \cup A_2 \cup D_i \cup ((Z_1 \cup Z_2) \cap H) \cup (Z_1 \cap G_1) \cup (Z_2 \cap G_2),$$

and $S_1 \cap S_2 = \varnothing$ since $Z_1 \cap Z_2 \neq \varnothing$ and $A_1$, $A_2$, $D_i$, $H$, $G_1$, $G_2$ are mutually disjoint. Concerning the set $D_i - (Z_1 \cap D_i)$ in (8.6), one has that $m_i(X - Z_1) = 0$ from Theorem 9, and by Lemma 3, with $M = X$ and $S = Z_1$ and $L = D_i$, that $m_e(Z_1 \cap D_i) = m(D_i)$ and $m_i(D_i - (Z_1 \cap D_i)) = 0$. By Lemma 2, $m_e(D_i - (Z_1 \cap D_i)) + m_i(Z_1 \cap D_i) = m(D_i)$, so that $m_e(D_i - (Z_1 \cap D_i)) = m(D_i)$, since $m_i(Z_1) = 0$. Thus,

$$(8.8)\qquad m_i(D_i - (Z_1 \cap D_i)) = 0, \quad m_e(D_i - (Z_1 \cap D_i)) = m(D_i).$$

If $K$ is any measurable set $\subset X$, then

$$(8.9)\quad m_e(Z_1 \cap K) = m_e(Z_2 \cap K) = m_e((Z_1 \cup Z_2) \cap K) = m(K), \qquad K \subset X,$$

by Theorem 9 for $N = 2$ and Lemma 3.

The four sets on the right-hand side of (8.5) are contained in mutually disjoint measurable sets; likewise for the four sets on the right-hand side of (8.6), and for the six sets on the right-hand side of (8.7). In forming their respective unions, as in (8.5), (8.6) and (8.7), their interior and exterior measures are additive, by Lemma 1. Therefore, from (8.5), since $m_i(Z_1 \cap K) = m_i(Z_2 \cap K) = m_i((Z_1 \cup Z_2) \cap K) = 0$ by Theorem 9 for $N = 2$,

$$(8.10)\quad m_i(S_1) = a_1 + 0 + 0 + 0 = a_1, \quad m_e(S_1) = a_1 + d_i + h + g_1 = b_1$$

by (8.9), and these are the first line of (8.2). From (8.6), using (8.8) and (8.9),

$$(8.11)\quad m_i(S_2) = a_2 + 0 + 0 + 0 = a_2, \quad m_e(S_2) = a_2 + d_i + h + g_2 = b_2,$$

$$(8.12)\quad m_i(S_1 \cup S_2) = a_1 + a_2 + d_i + 0 + 0 + 0 = a_1 + a_2 + d_i = a,$$

$$(8.13)\qquad m_e(S_1 \cup S_2) = a_1 + a_2 + d_i + h + g_1 + g_2 = b,$$

which give the second to fourth lines of (8.2). Theorem 10 is proved.

Theorems 5 and 10 are the main theorems concerning the six quantities $m_i(S)$ and $m_e(S)$ for $S = S_1, S_2$, and $S_1 \cup S_2$, where $S_1$ and $S_2$ are disjoint sets contained in a measurable set $X$. They state that the quantities $a_1$, $b_1$, $a_2$, $b_2$, $a$, $b$, defined in (6.4) and (6.5), are subject to six independent inequalities, that $a_1$, $a_2$, $d_i$, $h$, $g_1$, $g_2$ are each $\geq 0$ and the inequality $a_1 + a_2 + d_i + h + g_1 + g_2 \leq m(X)$. These are valid for every pair of disjoint sets $\subset X$, and any other numerical relation involving $a_1$, $b_1$, $a_2$, $b_2$, $a$, $b$, which is valid for every pair of disjoint sets $\subset X$, is a consequence of these.

Without the use of the quantities $d_i$, $h$, $g_1$, $g_2$, the six inequalities are, besides $b \le m(X)$,

$$\begin{cases} a_1 \ge 0, \quad a_2 \ge 0, \quad a \ge a_1 + a_2 \\ a + b \le a_1 + b_1 + a_2 + b_2 \\ b \ge a_1 + b_2, \quad b \ge a_2 + b_1. \end{cases} \tag{8.14}$$

Theorem 7 states further that the six non-negative quantities $a_1$, $a_2$, $d_i$, $h$, $g_1$,$g_2$, which are the transposed forms of the six inequalities in the finite case $b < \infty$, can have any values independently subject merely to their sum being $\le m(X)$. In the infinite case when $b = \infty$, which is $a_1 + a_2 + d_i + h + g_1 + g_2 = \infty = m(X)$, the inequalities can be written as

$$\begin{cases} a_1, a_2, d_i, h, g_1, g_2 \text{ are each } \ge 0, \quad \text{and} \\ a = a_1 + a_2 + d_i, \\ a_1 + b_1 + a_2 + b_2 = a + b + h, \\ b = a_1 + b_2 + g_1, \quad b = a_2 + b_1 + g_2. \end{cases} \tag{8.15}$$

These are also the inequalities (8.14) in the finite case $b < \infty$.

In words, the non-negativeness of interior and exterior measures, and the relation $m_i(S) \le m_e(S)$, and the monotone increasing property (1.2), and the superadditivity of interior measure and subadditivity of exterior measure (1.3), and Theorems 1 and 4, form a *complete* set of conditions on the quantities $m_i(S)$ and $m_e(S)$ for $S = S_1, S_2$, and $S_1 \cup S_2$, valid for every pair of disjoint sets $S_1$ and $S_2$.

**9. Linear combinations of $m_i(S)$ and $m_e(S)$.** As an immediate application of this article, consider set functions $f(S)$ which are homogeneous linear combinations of $m_i(S)$ and $m_e(S)$, i.e.

$$f(S) = c_1 m_i(S) + c_2 m_e(S), \tag{9.1}$$

where $c_1$ and $c_2$ are constants. Various properties will be considered, such as subadditivity, superadditivity, etc. The quantity $f(S)$ is *subadditive* if

$$f(S_1 \cup S_2) \le f(S_1) + f(S_2) \tag{9.2}$$

for any two disjoint sets $S_1, S_2$. There is the following:

THEOREM 11. *The quantity $f(S) = c_1 m_i(S) + c_2 m_e(S)$ is subadditive for all pairs of disjoint sets, $S_1 \cap S_2 = \varnothing$, if and only if $c_2 \geq 0$ and $c_2 \geq c_1$.*

*Proof.* Suppose $f(X)$ to be subadditive, place (9.1) into (9.2), transpose when $m_e(S_1 \cup S_2)$ is finite, and use (1.4) and (1.5). There results $c_1 d_i(S_1, S_2) \leq c_2 d_e(S_1, S_2)$, or using (6.2),

$$(c_1 - c_2) d_i(S_1, S_2) \leq c_2 h(S_1, S_2). \tag{9.3}$$

The quantities $d_i$ and $h$ can be assigned non-negative values independently, by Theorem 10. Selecting $S_1, S_2$ so that $d_i = 0$, $h > 0$ gives $c_2 \geq 0$; selecting $S_1, S_2$ so that $h = 0$, $d_i > 0$ gives $c_1 - c_2 \leq 0$. Conversely, if $c_1 - c_2 \leq 0$ and $c_2 \geq 0$, then (9.3) holds since 0 is between the two sides of (9.3); and (9.3) gives $c_1 d_i(S_1, S_2) \leq c_2 d_e(S_1, S_2)$, which on transposing is (9.2). The theorem is proved.

Note that if $f(S)$ is subadditive for two particular pairs of sets $S_1, S_2$, such as selected in the proof above, then it is subadditive for all pairs of sets $S_1, S_2$.

For other properties of set functions, $f(S)$ is *monotone increasing* if

$$f(S) \leq f(T) \quad \text{whenever } S \subset T. \tag{9.4}$$

Placing (9.1) for $S$ and $T$ into (9.4), and picking $S = \varnothing$ and $T$ such that $m_i(T) = 0$, $m_e(T) > 0$, (9.4) gives $c_2 \geq 0$; picking $S$ such that $m_i(S) = 0$, $m_e(S) > 0$, and $T \supset S$ to be a measurable set with $m(T) = m_e(S)$, (9.4) gives $c_1 \geq 0$. Conversely, $c_1 \geq 0$ and $c_2 \geq 0$ gives (9.4), since $m_i(S)$ and $m_e(S)$ are monotone increasing. Thus, $f(S)$ is monotone increasing if and only if $c_1 \geq 0$, $c_2 \geq 0$.

A combination of this last result and Theorem 11 yields:

THEOREM 12. *The set function $f(S) = c_1 m_i(S) + c_2 m_e(S)$ is subadditive for disjoint sets, and monotone increasing, if and only if $c_2 \geq c_1 \geq 0$. The set function $f(S)$ may also be written in the form $f(S) = \tilde{c}_1 m_a(S) + \tilde{c}_2 m_e(S)$, where $\tilde{c}_1$ and $\tilde{c}_2$ are constants, and is subadditive for disjoint sets and monotone incrasing if and only if $\tilde{c}_1 \geq 0$, $\tilde{c}_2 \geq 0$.*

*Proof.* The first sentence of Theorem 12 has already been obtained. The quantity $f(S)$ can be written as

$$\begin{aligned} f(S) &= c_1 m_i(S) + c_2 m_e(S) \\ &= 2c_1 \cdot \left( \frac{m_i(S) + m_e(S)}{2} \right) + (c_2 - c_1) m_e(S) \\ &= 2c_1 m_a(S) + (c_2 - c_1) m_e(S), \end{aligned}$$

so that $\tilde{c}_1 = 2c_1, \tilde{c}_2 = c_2 - c_1$. Then $c_2 \geq c_1 \geq 0$ is equivalent to $\tilde{c}_1 \geq 0$, $\tilde{c}_2 \geq 0$. Theorem 12 is proved.

The set function $f(S)$ is *superadditive* if

$$f(S_1 \cup S_2) \geq f(S_1) + f(S_2), \quad S_1 \cap S_2 = \varnothing. \tag{9.5}$$

Then $-f(S)$ is subadditive, and Theorem 11 shows that $f(S)$ is superadditive if and only if $c_2 \leq 0$ and $c_2 \leq c_1$. (Then $f(S)$ can be put in the form $\hat{c}_1 m_i(S) + \hat{c}_2 m_a(S)$ with $\hat{c}_1 \geq 0 \geq \hat{c}_2$.)

THEOREM 13. *The set function $f(S) = c_1 m_i(S) + c_2 m_e(S)$ is superadditive for disjoint sets, and non-negative, if and only if $c_2 = 0$ and $c_1 \geq 0$.*

*Proof.* The *non-negativity* of $f(S)$ is

$$f(S) \geq 0 \quad \text{for all } S.$$

Picking a set $S$ with $m_i(S) = 0$ and $m_e(S) > 0$ gives $c_2 \geq 0$, and picking $S$ to be a measurable set with $m(S) > 0$ gives $c_1 + c_2 \geq 0$. Conversely, $c_2 \geq 0$ and $c_1 + c_2 \geq 0$ gives

$$f(S) = (c_1 + c_2)m_i(S) + c_2(m_e(S) - m_i(S)) \geq 0,$$

so that $c_2 \geq 0$ and $c_1+c_2 \geq 0$ is the condition for nonnegativity of $f(S)$. Combined with the condition $c_2 \leq 0$ and $c_2 \leq c_1$ for superadditivity gives $c_2 = 0$ and $c_1 \geq 0$. The theorem is proved.

Note that a monotone increasing property of $f(S)$ implies its nonnegativity, since $f(S) \geq f(\varnothing) = 0$.

The Theorems 12 and 13 show a difference between interior and exterior measure in an interesting form. Considering set functions $f(S)$ as in (9.1) which are monotone increasing, then only a nonnegative multiple of $m_i(S)$ is superadditive, while $c_1 m_i(S) + c_2 m_e(S)$ is subadditive for $c_2 \geq c_1 \geq 0$. The latter can be put in the form $\tilde{c}_1 m_a(S) + \tilde{c}_2 m_e(S)$ with $\tilde{c}_1 \geq 0$, $\tilde{c}_2 \geq 0$.

Concerning a complementation property, $f(S)$ is *complementary* if

$$f(S) + f(L - S) = f(L) \tag{9.6}$$

where $S \subset L$ and $L$ is a mesurable set of finite positive mesure. Inserting (9.1) into (9.6) gives $(c_2 - c_1)(m_e(S) - m_i(S)) = 0$, using (2.3). For any single non-measurable set $S \subset L$, so $c_2 - c_1 = 0$. Reversing this, $c_2 - c_1 = 0$ implies (9.6). Thus,

THEOREM 14. *The quantity* $f(S) = c_1 m_i(S) + c_2 m_e(S)$ *(or* $\tilde{c}_1 m_a(S) + \tilde{c}_2 m_e(S)$*) is complementary if and only if* $c_1 = c_2$ *(or* $\tilde{c}_2 = 0$*).*

Also, if one desires that $f(S) = m(S)$ when $S$ is measurable, the condition is that $c_1 + c_2 = 1$, or $\tilde{c}_1 + \tilde{c}_2 = 1$. This condition can be added to the theorems above.

## REFERENCES

[1] First shown by G. Vitali, *Sul probleme della misura dei gruppi di punti di una retta*, Bologna (1905).

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