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CONFORMALLY FLAT IMMERSIONS AND FLATNESS OF THE NORMAL CONNECTION

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B. Y. Chen and T. Teng affirmed that almost umbilic isometric immersions of an n-dimensional manifold in $\mathbf{R}^{n+2}$ have zero normal curvature ($R^{\perp} = 0$). In this paper we exhibit a counterexample to this statement and we prove that either $R^{\perp} = 0$ or there exists (locally) an isometric immersion of this manifold in $\mathbf{R}^{n+2}$ with $R^{\perp} = 0$. These immersions are conformally flat and we study their local geometry.

1. Introduction. An n-dimensional Riemannian manifold M^n is *conformally flat* if, for each $x \in M$, there exists a conformal diffeomorphism of a neighborhood of x onto an open set of the Euclidean space $\mathbf{R}^n$. We will call *conformally flat immersion* (CFI for short) an immersion $f: M^n \to \mathbf{R}^{n+p}$ with M conformally flat in the metric induced by f.

A nice characterization of CFI in terms of the second fundamental form was given by Moore in [**4**] and Moore-Morvan in [**5**].

In this paper we want to consider CFI with codimension two. From [**5**], we can conclude that if $n \geq 5$, such an isometric immersion f is *almost umbilic,* i.e., for every $x \in M$ there exists an orthonormal frame of the normal space $T_xM^{\perp}$ such that for each ξ of this frame, the Weingarten operator A_{ξ} has an eigenvalue of multiplicity at least $n-1$.

In [**2**] there exists a false statement which says that "almost umbilic isometric immersions in Euclidean space with codimension two have zero normal curvature". This and [**5**] together would imply that CFI in codimension two has zero normal curvature ($R^{\perp} = 0$). In §2 we will discuss this false result and its counterexample. The aim of this paper is to show that if a conformally flat manifold can be isometrically immersed in $\mathbf{R}^{n+2}$ then either $R^{\perp} = 0$ or there is a local isometric immersion in $\mathbf{R}^{n+2}$ with $R^{\perp} = 0$.

Before stating our results, we will recall some definitions below:

DEFINITION. A submanifold $\Sigma \subset M$ is a *geometric sphere of type ε,* $\varepsilon = 0, 1$, if Σ is an umbilic submanifold with parallel mean curvature vector and such that the sectional curvature of M along planes tangent to Σ are constant k, $k = 0$ if $\varepsilon = 0$, $k > 0$ if $\varepsilon = 1$. If a conformally flat manifold M^n is locally foliated by geometric spheres of type ε and codimension s, we will say that M^n is of type (s, ε).

In the case of CFI with codimension two, it follows from **[4]** and [7] that the connected components of an open dense $V \subset M$ are either umbilic, of type $(1, \varepsilon)$ or of type $(2, \varepsilon)$. It will be easy to prove that a CFI of manifold of type $(1, \varepsilon)$ with $\varepsilon = 1$ has $R^\perp = 0$. Thus, our main result is:

THEOREM. *Let $f: M^n \to \mathbf{R}^{n+2}$ be a CFI where M is of type $(2, \varepsilon)$ and $n \geq 5$. Then M is (locally) foliated by two orthogonal codimension one foliations whose leaves are, in the induced metric, conformally flat manifolds of type $(1, \varepsilon)$ and the intersection of the two foliations gives the codimension two foliation by geometric spheres. Conversely, if M admits such foliations, M can be (locally) isometrically immersed in $\mathbf{R}^{n+2}$ with $R^\perp = 0$.*

The author wants to thank Professor F. Mercuri for bringing our attention to the example (2.1) below.

2. Flat n-dimensional manifold in $\mathbf{R}^{n+2}$ with $R^\perp \neq 0$.

(2.1) EXAMPLE. Let $f: M^2 \to \mathbf{R}^4$ be an isometric immersion with $R^\perp \neq 0$ and M^2 flat (this immersion can be the composition of a cylindrical immersion of $\mathbf{R}^2$ into $\mathbf{R}^3$ with another cylindrical immersion of $\mathbf{R}^3$ into $\mathbf{R}^4$ along a curve whose tangent direction is not perpendicular or equal to the principal direction of the first immersion). Consider the product immersion $\overline{f}: M^2 \times \mathbf{R}^{n-2} \to \mathbf{R}^{n+2}$. We see that $\overline{f}$ satisfies $R^\perp \neq 0$ and that $M^2 \times \mathbf{R}^{n-2}$ is flat, in particular, is conformally flat.

(2.2) ASSERTION. If $n \geq 5$, the example (2.1) is an almost umbilic immersion.

In fact, let (ξ_1, ξ_2) be an orthonormal frame of $T_xM^\perp$ and let $(X_1, \dots, X_n)$ be an orthonormal frame in T_xM which diagonalizes the Weingarten operator A_{ξ_1}. Setting $\xi_\theta = \cos\theta\xi_1 + \sin\theta\xi_2$, we claim that there exists θ such that ξ_θ is almost umbilic. It is enough to take θ as a solution of the following equation:

$$\lambda_1\lambda_2 \cos^2\theta + (a\lambda_2 + c\lambda_1)\cos\theta\sin\theta + (ab - c^2)\sin^2\theta$$

where

$$A_{\xi_1|TM^2} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad A_{\xi_2|TM^2} = \begin{bmatrix} a & c \\ c & b \end{bmatrix}.$$

We observe that the above equation has a solution because $\lambda_1\lambda_2 + (ab - c^2) = 0$. The lemma below allows us to find the other almost umbilic direction.

(2.3) LEMMA. *Let $f: M^n \to \mathbf{R}^{n+2}$ be a CFI, $n \geq 5$ and ξ an almost umbilic direction. Then $\xi^{\perp}$ is almost umbilic.*

Proof. Let $(X_1, \dots, X_n)$ be an orthonormal basis as in [4] and X_1, X_2 such that this basis diagonalizes A_ξ. Then we have

$$A_\xi = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda & & \\ & & \lambda & \\ 0 & & & \lambda \end{bmatrix}, \qquad A_\xi = \begin{bmatrix} a & c & & 0 \\ c & b & & \\ & & \mu & \\ 0 & & & \mu \end{bmatrix}.$$

Now we can use the characterizaiton of conformally flat manifolds in terms of the sectional curvatures given by Kulkarni [3], namely,

$$K(X_1, X_2) + K(X_3, X_4) = K(X_1, X_3) + K(X_2, X_4) \tag{2.4}$$

for any orthonormal vectors $X_1, X_2, X_3, X_4 \in T_xM$ (where $K(X_i, X_j)$ is the sectional curvature of the plane $\{X_i, X_j\}$), together with the Gauss equation to get

$$\mu^2 - (a+b)\mu + (ab - c^2) = 0$$

which implies that μ is an eigenvalue of $A_{\xi^\perp}$.

In [2] the authors affirmed that the example (2.1) contradicts [5] (which, at that time, was a conjecture). But, we have shown that (2.1) contradicts their result.

3. Proof of the theorem. We will start with the necessary part of the theorem. We will denote by ∇ and $\langle\ ,\ \rangle$ the Riemannian connection and metric respectively, α will be the second fundamental form and $\nabla^\perp$ the normal connection.

Let ξ_1 and ξ_2 be the differentiable almost umbilic directions of f. Let $(X_1, \dots, X_n)$ be an orthonormal frame which diagonalizes A_{ξ_1}. Then

$$A_{\xi_1} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda & & \\ & & \lambda & \\ 0 & & & \lambda \end{bmatrix}, \qquad A_{\xi_2} = \begin{bmatrix} a & c & & 0 \\ c & b & & \\ & & \mu & \\ 0 & & & \mu \end{bmatrix}.$$

We have:

$$\alpha(X_i, X_j) = \alpha(X_1, X_i) = \alpha(X_2, X_i) = 0, \qquad i, j \geq 3,\ i \neq j,$$
$$\alpha(X_1, X_2) = c\xi_2.$$

If $\lambda_1 = \lambda$, ξ_1 is umbilic and then $R^\perp = 0$. Thus, we may assume $\lambda_1 \neq \lambda$. Writing the Codazzi equation for X_i, X_j and ξ_1 $(i, j \geq 3, i \neq j)$, we get

$$X_i(\lambda) + \mu\langle\nabla^\perp_{X_i}\xi_2, \xi_1\rangle = (\lambda - \lambda)\langle X_i, \nabla_{X_i}X_j\rangle.$$

Hence, if $\mu \neq 0$ we have

$$\langle\nabla^\perp_{X_i}\xi_2, \xi_1\rangle = -X_i(\lambda)/\mu, \qquad i \geq 3.$$

Similarly, we obtain

$$\langle\nabla^\perp_{X_i}\xi_2, \xi_1\rangle = -X_i(\lambda)/b, \qquad i \geq 3.$$

If $b = \mu$, Gauss and Kulkarni ((2.4)) equations imply $k_1 = K(X_1, X_i) = K(X_1, X_2)$, $i \geq 3$, contradicting that M is of type $(2, \varepsilon)$. Then $b \neq \mu$, and this implies

$$X_i(\lambda) = 0 = \langle\nabla^\perp_{X_i}\xi_2, \xi_1\rangle, \qquad i \geq 3.$$

Now, the Codazzi equation for X_1, X_2, X_i $(i \geq 3)$ and ξ_1 gives

$$\begin{aligned}\langle\nabla_{X_1}X_2, X_i\rangle\lambda + \langle\nabla_{X_1}X_i, X_2\rangle\lambda &= \langle\nabla_{X_2}X_1, X_i\rangle\lambda + \langle\nabla_{X_2}X_i, X_1\rangle\lambda_1 \\ &= \langle\nabla_{X_i}X_1, X_2\rangle\lambda + \langle\nabla_{X_i}X_2, X_1\rangle\lambda_1\end{aligned}$$

and since $\lambda \neq \lambda_1$ we have

$$\langle\nabla_{X_2}X_i, X_1\rangle = \langle\nabla_{X_i}X_2, X_1\rangle = 0. \tag{3.1}$$

Consider the following differentiable distributions:

$$\begin{aligned}D_1 &= \operatorname{span}\{X_1, X_3, \dots, X_n\}, & D_2 &= \operatorname{span}\{X_2, X_3, \dots, X_n\}, \\ D &= \operatorname{span}\{X_3, \dots, X_n\}, & D^\perp &= \operatorname{span}\{X_1, X_2\}.\end{aligned}$$

D is obviously integrable because $X_3, \dots, X_n$ are tangent to the geometric sphere. From (3.1), we conclude that D_2 is integrable and, if we prove that $\langle\nabla_{X_1}X_i, X_2\rangle = 0$, we will have D_1 and $D^\perp$ integrable. For this we apply the Codazzi equation to X_2, X_1, X_i, $(i \geq 3)$, and ξ_2 to get

$$\langle\nabla_{X_1}X_2, X_i\rangle(\mu - b) = c(\langle\nabla_{X_1}X_1, X_i\rangle - \langle\nabla_{X_2}X_2, X_i\rangle).$$

Since $\mu \neq b$, D_1 and $D^\perp$ will be integrable if and only if $c = 0$ $(R^\perp = 0)$ or $\langle\nabla_{X_1}X_1, X_i\rangle = \langle\nabla_{X_2}X_2, X_i\rangle$. Let us suppose $c \neq 0$ in an open set U. We are going to prove that $\langle\nabla_{X_1}X_1, X_i\rangle = \langle\nabla_{X_2}X_2, X_i\rangle$.

First, we will consider the case $\varepsilon = 0$. Then U is a flat manifold. The Codazzi equation for X_1, X_2, X_i and ξ_2 implies

$$\langle\nabla_{X_1}X_1, X_i\rangle c - \langle\nabla_{X_1}X_i, X_2\rangle b = X_i(c) = \langle\nabla_{X_2}X_2, X_i\rangle c.$$

We claim that $X_i(c) = 0$, which implies $\langle \nabla_{X_2} X_2, X_i \rangle = 0$, for $i \geq 3$. In fact, we can take in Σ_2, the maximal leaf of the integrable distribution D_2, local coordinates $(x_2, \cdots, x_n)$ such that $\partial/\partial x_i = \beta_i X_i$, since D is integrable. Because Σ_2 is flat we can choose these coordinates such that $\beta_i = \beta$ and $\partial\beta/\partial x_i = 0$ for each $i = 2, \cdots, n$. Σ_2 is immersed in $\mathbf{R}^{n+2}$ and X_1 is normal vector for Σ_2. Now we compute $X_i(c)$.

$$c = \langle A_{\xi_2} X_2, X_1 \rangle = -1/\beta \langle \tilde{\nabla}_{\partial/\partial x_2} \xi_2, X_1 \rangle$$

where $\tilde{\nabla}$ denotes the connection in $\mathbf{R}^{n+2}$. Then

$$\partial c/\partial x_i = -1/\beta \{ \langle \tilde{\nabla}_{\partial/\partial X_i} \nabla_{\partial/\partial x_2} \xi_2, X_1 \rangle + \langle \tilde{\nabla}_{\partial/\partial x_2} \xi_2, \tilde{\nabla}_{\partial/\partial X_i} X_1 \rangle \}. \tag{3.2}$$

But the last term is zero, because $\varepsilon = 0$. Since

$$\begin{aligned} 0 &= \langle \tilde{R}(\partial/\partial x_2, \partial/\partial x_i)\xi_2, X_1 \rangle \\ &= \langle \tilde{\nabla}_{\partial/\partial x_2} \tilde{\nabla}_{\partial/\partial x_i} \xi_2, X_1 \rangle - \langle \tilde{\nabla}_{\partial/\partial x_i} \tilde{\nabla}_{\partial/\partial x_2} \xi_2, X_1 \rangle \end{aligned}$$

and again because $\varepsilon = 0$, $\tilde{\nabla}_{\partial/\partial X_i} \xi_2 = 0$, implying in (3.2)

$$\partial c/\partial x_i = \beta X_i(c) = 0. \tag{3.3}$$

Now, considering the orthonormal frame $(V_1, V_2, X_3, \ldots, X_n)$ diagonalizing A_{ξ_2} we have

$$A_{\xi_1} = \begin{bmatrix} x & z & & 0 \\ z & y & & \\ & & 0 & \\ 0 & & & 0 \end{bmatrix}, \qquad A_{\xi_2} = \begin{bmatrix} 0 & & & 0 \\ & \mu_1 & & \\ & & 0 & \\ 0 & & & 0 \end{bmatrix}.$$

Similarly, we conclude for $i \geq 3$

$$\langle \nabla_{X_i} V_1, V_2 \rangle = 0, \quad \langle \nabla_{V_1} X_i, V_2 \rangle = 0 \quad \text{and} \tag{3.4}$$
$$X_i(z) = z \langle \nabla_{V_1} V_1, X_i \rangle = 0.$$

Setting $V_1 = \cos\theta X_1 + \sin\theta X_2$ and $V_2 = \sin\theta X_1 - \cos\theta X_2$, because $\langle \nabla_{V_1} X_i, V_2 \rangle = 0$, a straightforward computation shows

$$\langle \nabla_{V_1} V_1, X_i \rangle = \langle \nabla_{X_1} X_1, X_i \rangle \quad \text{and} \quad \langle \nabla_{V_2} V_2, X_i \rangle = \langle \nabla_{X_2} X_2, X_i \rangle. \tag{3.5}$$

Thus, (3.3), (3.4) and (3.5) imply

$$\langle \nabla_{X_2} X_2, X_i \rangle = \langle \nabla_{X_1} X_1, X_i \rangle = 0, \qquad i \geq 3. \tag{3.6}$$

Now, if $\varepsilon \neq 0$, we can take a conformal diffeomorphism Φ from U to an open set O of the Euclidean space. We will denote again by $(V_1, V_2, X_3, \ldots, X_n)$ the orthonormal basis which diagonalizes A_{ξ_2}. Let γ_1 and γ_2 be the integral curves of $d\Phi(X_1)$ and $d\Phi(V_2)$ respectively.

We can immerse O isometrically in $\mathbf{R}^{n+2}$ by composing a cylindrical immersion of $\mathbf{R}^n$ to $\mathbf{R}^{n+2}$ whose principal direction is $\gamma_1'(t)$, with another cylindrical immersion of $\mathbf{R}^{n+1}$ to $\mathbf{R}^{n+2}$ with $\gamma_2'(t)$ as principal direction. Then by (3.6), we have

$$\langle \nabla_{d\Phi(X_1)} d\Phi(X_1), d\Phi(X_i)\rangle = \langle \nabla_{d\Phi(X_2)} d\Phi(X_2), d\Phi(X_i)\rangle.$$

Since Φ is conformal, writing the Riemannian connection in terms of the Riemannian metric, we get

$$\langle \nabla_{X_1} X_1, X_i\rangle = \langle \nabla_{X_2} X_2, X_i\rangle.$$

This proves that the distributions D_1 and $D^\perp$ are integrable. Now Lemma (3.3) of **[6]** finishes the first part of the proof.

The second part of the proof follows from Theorem (1.12) of **[6]**.

REMARK. If M is of type $(1, \varepsilon)$ with $\varepsilon = 1$, taking $\xi_1, \xi_2, X_1, \dots, X_n$ as in the above theorem, we have

$$A_{\xi_1} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda & & \\ & & \lambda & \\ 0 & & & \lambda \end{bmatrix}, \qquad A_{\xi_2} = \begin{bmatrix} a & c & & 0 \\ c & b & & \\ & & \mu & \\ 0 & & & \mu \end{bmatrix}$$

with $b = \mu$, because M is of type $(1, \varepsilon)$. Now, the Gauss and Kulkarni equations imply $R^\perp = 0$.

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