PACIFIC JOURNAL OF MATHEMATICS
Vol. 128, No. 1, 1987

JÓNSSON ω_0-GENERATED ALGEBRAIC FIELD EXTENSIONS

ROBERT GILMER AND WILLIAM HEINZER

A field K algebraic over its subfield F is said to be a J-extension (for Jónsson ω_0-generated extension) of F if K/F is not finitely generated, but E/F is finitely generated for each proper intermediate field E. We seek to determine the structure of a given J-extension and to determine the class of fields that admit a J-extension. Consideration of Galois J-extensions plays a special role in each of these problems. In §2, we show that a Galois extension K/F is a J-extension if and only if $\mathrm{Gal}(K/F) \simeq \varprojlim Z/p^nZ$ for some prime p. In §3, we show that F admits a J-extension if the algebraic closure of F is infinite over F—that is, F is neither algebraically closed nor real closed.

1. Introduction. Assume that α is an infinite cardinal. In universal algebra, an algebra A is said to be a *Jónsson α-algebra* if $|A| = \alpha$ while $|B| < \alpha$ for each proper subalgebra B of A [**CK**, p. 469]. This terminology has been extended in [**GH**] to generating sets, as follows. The algebra A is said to be a *Jónsson α-generated algebra* if A has a generating set of cardinality α, no generating set of smaller cardinality, and each proper subalgebra B of A has a generating set of cardinality less than α. In considering these concepts, special attention has been given to the cases where $\alpha = \omega_0$, the first infinite cardinal, and where $\alpha = \omega_1$, the first uncountable cardinal. For $\alpha = \omega_0$, we consider here a problem of this type for field extensions. Specifically, suppose that F is a subfield of the field K. Shortening the phrase "K is a Jónsson ω_0-generated F-algebra", we say that K is a *J-extension of F* if $[K:F] = \infty$ while $[E:F] < \infty$ for each proper intermediate field E. The condition that $[E:F] < \infty$ for each proper intermediate field implies that K/F is countably generated, so in fact, $[K:F] = \omega_0$ if K is a J-extension of F. We begin by noting three examples of J-extension; these are labelled below as (E1), (E2), and (E3).

(E1) An absolutely algebraic field F of characteristic $p \neq 0$ is uniquely determined, up to isomorphism, by specifying its Steinitz number, a formal product $N = \prod p_i^{k_i}$ over all primes p_i, where $k_i \in Z_0 \cup \{\infty\}$ for each i [**J**, p. 147]. If F is not algebraically closed, then $k_t \neq \infty$ for some t. If K is the absolutely algebraic extension field of F with Steinitz number $\prod p_i^{h_i}$, where $h_t = \infty$ and $h_i = k_i$ for $i \neq t$, then K is a J-extension of F;

the proper intermediate fields, in this case, are the fields corresponding to the Steinitz numbers $\prod p_i^{v_i}$, where $v_t \in Z_0$, $v_t \geq k_t$, and $v_i = k_i$ for $i \neq t$.

(E2) If F is an imperfect field of characteristic $p \neq 0$ and if $s \in F \setminus F^p$, then $K = F(\{s^{1/p^n}\}_{n=1}^{\infty})$ is a J-extension of F. The proper intermediate fields are the fields $F(s^{1/p^n})$, where $n \in Z_0$ (see Lemma 2.8).

(E3) Assume that the field E is not algebraically closed and let K be an algebraic closure of E. Choose $s \in K \setminus E$ and let F be an intermediate field maximal with respect to failure to contain s. If $[K:F] = \infty$, then K is a J-extension of F; the set of proper intermediate fields forms a chain $F = F_0 < F_1 < F_2 < \cdots$, where $[F_n:F] = q^n$, with q prime, for each $n \in Z^+$ [Q].

In each of the examples above, the set of fields between F and K is linearly ordered, and given adjacent intermediate fields $E_1 < E_2$, $[E_2:E_1]$ is a prime integer independent of E_1 and E_2. While examples of this type are the easiest to produce, we subsequently show (Examples 2.11, 2.18) that there are other examples of J-extensions with neither of the properties cited. On the other hand, Theorem 2.5 shows that if K/F is Galois, then the two conditions are satisfied. There are two primary emphases of this paper, as follows: (1) to determine possible structures of a J-extension (§2), and (2) to determine the class of fields that admit a J-extension or a Galois J-extension (§3). The structure of a Galois J-extension is completely determined in Theorem 2.5, but subsequent examples in §2 show that a variety of structures are possible in the non-Galois case.

Suppose F is a field that is neither algebraically closed nor real closed. In §3, we initially investigate the question of whether F admits a Galois J-extension if F admits a nontrivial cyclic extension. We observe that the answer is negative in general for a cyclic extension of degree two, but in many other cases we obtain an affirmative answer. We subsequently show (Theorem 3.9) that F admits a J-extension if F is neither algebraically closed nor real closed. If F is the quotient field of a non-trivial valuation domain with principal maximal ideal, we present in Theorem 3.11 a concrete construction of a J-extension of F.

In §4, we show that each subfield of A, the abelian closure of Q, admits a Galois J-extension. The final section of the paper contains comments on two open questions that merit, in our opinion, further consideration.

2. Structure of J-extensions. Suppose K is a J-extension of F. In this section we focus on the lattice $\mathcal{S}$ of intermediate fields. In particular, we consider two questions concerning $\mathcal{S}$ that have been mentioned in the introduction: (1) Is $\mathcal{S}$ linearly ordered under inclusion? (2) What possibil-

ities exist for the set $\{[E_2:E_1]\}$ as $E_1 < E_2$ ranges over all adjacent elements of $\mathscr{S}$? Theorem 2.5 shows that if K/F is Galois, then (1) has an affirmative answer, and in (2), $[E_2:E_1]$ is a prime integer independent of E_1 and E_2. We begin the section with a general result concerning J-extensions; the proof of Proposition 2.1 is routine, and hence is omitted.

PROPOSITION 2.1. *Assume that F is a subfield of the field K.*

(1) *If K is a J-extension of F, then K is the union of a strictly ascending sequence of intermediate fields. In fact, if $\{K_i\}_{i=1}^{\infty}$ is any strictly ascending sequence of intermediate fields, then $K = \bigcup_{i=1}^{\infty} K_i$.*

(2) *Suppose K is expressed as the union of a strictly ascending sequence $\{K_i\}_{i=1}^{\infty}$ of intermediate fields, where $[K_i:F] < \infty$ for each i. The following conditions are equivalent.*

(a) *K is a J-extension of F.*

(b) *Each proper intermediate field is contained in some K_i.*

(c) *If $x_i \in K \setminus K_i$ for each i, then $K = F(\{x_i\}_{i=1}^{\infty})$.*

COROLLARY 2.2. *Assume that K is an extension field of the field F and that K is expressed as the union of a strictly ascending sequence $\{K_i\}_{i=1}^{\infty}$ of intermediate fields, where $[K_i:F] < \infty$ for each i. If for each i the set of subfields of K_i containing F is linearly ordered under inclusion, then K is a J-extension of F.*

COROLLARY 2.3. *Assume that F is a field of characteristic $p \neq 0$. Let M be an extension field of F and let K, L be intermediate fields with K/F purely inseparable and L/F a separable J-extension. Then KL is a J-extension of K.*

Proof. According to Proposition 2.1, there exists an ascending sequence $F = L_0 < L_1 < \cdots L = \bigcup_{i=0}^{\infty} L_i$ of finite extensions of F with the property that if $x_i \in L \setminus L_i$ for each i, then $L = F(\{x_i\}_0^{\infty})$. We show that the sequence $\{KL\}_{i=0}^{\infty}$ of fields between K and KL has the same properties; this suffices to show that KL is a J-extension of K. Since K and L_i are linearly disjoint over F, we have $[KL_i:K] = [L_i:F] < \infty$. Hence $K = KL_0 < KL_1 < \cdots KL = \bigcup_0^{\infty} KL_i$. Take $y_i \in KL \setminus KL_i$ for each i. There exists a finite extension E_i of F in K so that $y_i \in E_iL$, and if E_i has exponent e_i over F, then $y_i^{p^{e_i}} \in FL = L$. Since KL is separable over $K \subseteq KL_i$, it is also true that $y_i^{p^{e_i}} \notin KL_i$, and hence $y_i^{p^{e_i}} \notin L_i$. By hypothesis we have $F(\{y_i^{p^{e_i}}\}) = L$, so $K(\{y_i\}_i^{\infty}) \supseteq K(\{y_i^{p^{e_i}}\}) \supseteq L$, and hence $K(\{y_i\}) = KL$.

Can the roles of "separable" and "purely inseparable" be interchanged in Corollary 2.3? We have been unable to answer this question. If K/F is separable and L/F is a *standard* purely inseparable J-extension, as defined in the paragraph preceding the statement of Proposition 2.19, then it is easily seen that KL/K is a standard purely inseparable J-extension.

Suppose K is a Galois extension of the field F. Our determination of equivalent conditions for K to be a J-extension of F uses properties of the group $\varprojlim Z/p^nZ$, where $p \in Z^+$ is prime. We denote this group by W_p. Much is known about the structure of W_p; we list below some of the properties of W_p that we use.[1]

(W1) W_p is isomorphic to the additive group of the ring Q_p of p-adic integers [**F**, p. 62]; also $W_p \simeq \operatorname{End} Z(p^\infty)$, the endomorphism group of the p-quasicyclic group $Z(p^\infty)$ [**F**, p. 181].

(W2) W_p is q-divisible for each prime $q \neq p$ [**Wn**, p. 174]; hence $\{nW_p \mid n \in Z^+\} = \{p^kW_p\}_{k=0}^{\infty}$.

(W3) $W_p/p^kW_p \simeq Z/p^kZ$ for each k [**F**, p. 19]. Hence $\{p^tW_p\}_{t=0}^{k}$ is the set of subgroups of W_p containing p^kW_p.

(W4) W_p is torsion-free [**Wn**, p. 174], so $W_p \simeq p^kW_p$ for each $k \in Z^+$.

(W5) If H is a subgroup of W_p of finite index, then $H \simeq W_p$; this follows from (W2), (W3), and (W4).

(W6) Considering W_p as the additive group of Q_p, each endomorphism of W_p is multiplication by a fixed element of Q_p [**F**, p. 181]. Each automorphism of W_p is multiplication by a unit of Q_p.

The proof of Theorem 2.5 uses one new result concerning the group W_p; this result is established in Theorem 2.4. We are indebted to Murad Özaydın for the proof of Theorem 2.4.

THEOREM 2.4. *Assume that G is a torsion-free group containing W_p as a subgroup of finite index. Then $G \simeq W_p$.*

Proof. Because G is not assumed to be abelian, we write the group operation in G as multiplication. However, we consider W_p as the additive group of Q_p, and hence we write the group operation within W_p as addition. It turns out that no confusion results from this dichotomy of notation.

---

[1] Fuchs [**F**] and Weinstein [**Wn**] denote this group by J_p. In class field theory, the notation $\mathbf{Z}_p$ is frequently used, and a Galois extension of K with Galois group $\mathbf{Z}_p$ is called a $\mathbf{Z}_p$-extension of K [**I**], [**Wa**].

Since $[G: W_p] = s$ is finite, W_p contains a normal subgroup H of G with $[G:H]$ finite. (W5) shows that $H \simeq W_p$, so we assume without loss of generality that W_p is normal in G. Thus, if $g \in G$, then conjugation by g induces an automorphism φ_g of W_p; (W6) shows that φ_g is multiplication by a unit u of the ring Q_p—that is, $\varphi_g(x) = ux$ for each $x \in W_p$. Since $[G: W_p] = s < \infty$, then $g^s \in W_p$, so $ug^s = \varphi_g(g^s) = gg^s g^{-1} = g^s$. Since G is torsion-free, $g^s \neq 0$ and we conclude that $u = 1$. Therefore $\varphi_g = 1$ and g commutes with each element of W_p. We conclude that W_p is contained in the center of G; this implies that the transfer τ: $G \to W_p$ is a group homomorphism that is the sth power map: $\tau(g) = g^s$ for each $g \in G$ [**Ro**, p. 155]. Because G is torsion-free, τ is injective and $W_p \supseteq \tau(G) \supseteq sW_p$. From (W2), (W3) and (W4), it then follows that $G \simeq \tau(G) \simeq W_p$. This completes the proof.

THEOREM 2.5. *Assume that K is an algebraic extension of the field F and that K/F is Galois.*

(1) *If K is a J-extension of F, then* $\mathrm{Gal}(K/F) = W_p$ *for some prime p.*

(2) *Conversely, assume that* $\mathrm{Gal}(K/F) \simeq W_p$. *Then for each $n \in Z_0$ there exist a unique intermediate field K_n with $[K_n: F] = p^n$; $\{K_n\}_{n=0}^{\infty}$ is the set of proper intermediate fields. K_n/F is cyclic and $K_n \subseteq K_{n+1}$ for each n. In particular, K is a J-extension of F.*

Proof. (1): Set $G = \mathrm{Gal}(K/F)$. Choose $x \in K \setminus F$ and let E be an intermediate field maximal with respect to failure to contain x. Let L be any proper finite extension of E in K and let M be the normal closure in K of L/E. Because $E(x)$ is the unique minimal proper extension of E in M, the Galois group H of M/E has a unique maximal subgroup. Consequently, H is cyclic of prime-power order p^n for some prime p. Because each subgroup of G is normal in G, each field between E and M is normal over E. In particular, L is normal over E and $L = M$. If L_1 and L_2 are distinct finite proper extensions of E in K of degrees p_1^a and p_2^b, respectively, over E, then $p_1 = p_2$ since $[L_1L_2: E]$ is also a prime power. We denote by p this prime integer associated with E. Since K is the union of a chain of extension fields E_n of E such that E_n is cyclic over E of degree p^n for each $n \in Z^+$, it follows that the Galois group of K over E is isomorphic to $\varprojlim Z/p^nZ = W_p$ [**Bol**, Prop. 8, p. V. 61], [**L**, p. 351]. Because $[E:F] < \infty$, the index of W_p in G is at most $[E:F]$ [**H**, Lemma 2.8, p. 247]. Moreover, G is torsion-free since no proper intermediate field has finite codimension in K [**H**, Lemma 2.10, p. 249]. Therefore Theorem 2.4 shows that $G \simeq W_p$.

(2): We assume that $\mathrm{Gal}(K/F) \simeq W_p$. Then K is not a finite extension of F. Let P be any finite extension of F in K and let $H = \mathrm{Gal}(K/P)$. Since G is Abelian, H is normal in G, so P is normal over F. The group $G/H \simeq \mathrm{Gal}(P/F)$ is finite so $H \supseteq p^kG$ for some k and G/H is a homomorphic image of $G/p^kG \simeq Z/p^kZ$. Therefore G/H is cyclic, and we have proved that P/F is cyclic of degree p^m for some $m \in Z_0$. Because K/F is not finite-dimensional, the set of degrees of finite extensions of F in K is unbounded, and hence for arbitrarily large integers $n \in Z^+$, there exists an intermediate field K_n of degree p^n over F. Moreover, since K_n/F is cyclic, there exists an intermediate field K_i of degree p^i over F for $0 \leq i \leq n$. Thus, there exists an intermediate field K_n of degree p^n over F for each $n \in Z^+$. If E is an intermediate field of degree p^n over F, then EK_n is cyclic over F of degree p^r for some $r \geq n$; since $\mathrm{Gal}(EK_n/F)$ admits a unique subgroup of index p^n, it follows that $E = K_n$. To complete the proof, we show that $K = \bigcup_{n=0}^{\infty} K_n$ and that K is the only intermediate field distinct from each K_n. Thus, if $y \in K$, then $[F(y):F] = p^r$ for some r, so $F(y) = K_r$ and $y \in \bigcup_0^{\infty} K_n$. This proves that $K = \bigcup_0^{\infty} K_n$. If $E \neq K_n$ for $n \in Z^+$, then choose $y_n \in E \setminus K_n$. We have $E \supseteq F(y_n) \supseteq K_n$. Therefore if $E \notin \{K_n\}_0^{\infty}$, then $E \supseteq \bigcup_0^{\infty} K_n = K$ so $E = K$ and this completes the proof of (2).

COROLLARY 2.6. *Assume that K and L are subfields of a field M, that F is a subfield of $K \cap L$, and that K is a Galois J-extension of F. If $L \not\supseteq K$, then LK is a Galois J-extension of L.*

Proof. The proof of Theorem 2.5 shows that K is expressible as the union of a strictly ascending sequence $\{K_i\}_{i=1}^{\infty}$ of intermediate fields such that K_i/F is cyclic of degree p^i for each i, where p is a fixed prime depending only upon K and F. We have $L \cap K = L \cap (\bigcup K_i) = \bigcup_{i=1}^{\infty}(L \cap K_i) < K$. Hence $L \cap K = K_j$ for some j. Consider the ascending sequence $L \subseteq LK_{j+1} \subseteq LK_{j+2} \subseteq \cdots$ of subfields of LK. We have $LK = \bigcup_{i=1}^{\infty} LK_{j+1}$. Moreover, LK_{j+i}/L is Galois since K_{j+i}/F is Galois and

$$\mathrm{Gal}(LK_{j+i}/L) \simeq \mathrm{Gal}\big(K_{j+i}/(K_{j+i} \cap L)\big) = \mathrm{Gal}(K_{j+i}/K_j)$$

is cyclic of order p^i for each i. Therefore, the subfields of LK_{j+i} containing L are linearly ordered, and LK, L and $\{LK_{j+i}\}_{i=1}^{\infty}$ satisfy the hypothesis of Corollary 2.2. Consequently, LK is a Galois J-extension of L.

We remark that Corollary 2.6 does not extend to the case where K/P is a separable J-extension; an example showing this is noted after the presentation of Example 2.15.

Suppose the field F admits a Galois J-extension. Corollary 2.6 implies that if L is an extension field of F such that the algebraic closure E of F in L is finite-dimensional over F, then L also admits a Galois J-extension. We investigate more thoroughly in §3 the problem of determining the class of fields that admit Galois J-extensions, but for the present we merely review a concrete construction of J-extensions of the field Q and certain related fields (see [**I**, §2] or [**Wa**, §§7.2, 7.3]).

EXAMPLE 2.7. Let $p \in Z^+$ be prime and for each $r \in Z^+$, let ζ_r be a primitive complex p^rth root of unity. If p is odd, the field $Q(\zeta_r)$ is cyclic over Q of degree $p^{r-1}(p-1)$ [**J**, p. 113]; hence $Q(\zeta_r)$ is cyclic over $Q(\zeta_1)$ of degree p^{r-1}, so Corollary 2.2 implies that $K = \bigcup_{r=1}^{\infty} Q(\zeta_r)$ is a J-extension of $Q(\zeta_1)$. However, since $[Q(\zeta_1)::E]$ is not divisible by p for each proper subfield E of $Q(\zeta_1)$, it follows that K is not a J-extension of E for any such E. In particular, K is not a J-extension of Q. On the other hand, the field $Q(\zeta_r)$ contains a unique subfield L_r of degree p^{r-1} over Q, L_r/Q is cyclic, and $L_r \supseteq L_{r-1}$ for $r \geq 2$. Therefore $L = \bigcup_{r=1}^{\infty} L_r$ is a J-extension of Q contained in K.

If $p = 2$ and $r \geq 3$, then $\mathrm{Gal}(Q(\zeta_r/Q)$ is the direct product of a cyclic group of order 2^{r-2} and a group of order 2. The real subfield of $Q(\zeta_r)$ is the field $F_r = Q(\zeta_r + \zeta_r^{-1}) = Q(\cos(2\pi/2^r))$; it is cyclic over Q of degree 2^{r-2}, and $F_r \supseteq F_{r-1}$ for $r \geq 4$. Consequently, $F = \bigcup_{r=3}^{\infty} F_r$ is a J-extension of Q. As in the case where p is odd, the field $\bigcup_{r=3}^{\infty} Q(\zeta_r)$ is a Galois J-extension of $Q(\zeta_2) = Q(i)$.

Some aspects of Example 2.7 seem worthy of comment. First, Theorem 2.5 shows that any Galois J-extension of Q is abelian, and hence is contained in the abelian closure of Q. We observe that if F is a subfield of the real field $\mathbf{R}$ and if K is a Galois J-extension of F, then K is also contained in $\mathbf{R}$, for if not, then complex conjugation induces an automorphism μ of K with fixed subfield E of codimension 2 in K, a contradiction to the assumption that K/F is a J-extension. In particular, any Galois J-extension of Q is real. This means, for example, that no nonreal cyclic extension of Q of prime degree (for example, $Q(\sqrt{-2})$) can be extended to a *Galois* J-extension of Q. It follows from the proof of Theorem 3.11, however, that certain fields of this type (for example, $Q(\sqrt{-2})$) can be extended to a J-extension of Q.

What can be said about the structure of a J-extension K/F in the case where K/F is not Galois? This question can be partially reduced to the case where K/F is either separable or purely inseparable as follows. If K/F is not separable, then the separable part K_s of K/F is a finite extension of F and K/K_s is a purely inseparable J-extension. The reduction is partial because for fields $F \subseteq E \subseteq K$ with E/F finite separable and K/E a purely inseparable J-extension, it need not follow that K/F is a J-extension. For example, let $F = F_0(X^3)$, where $\operatorname{char} F_0 = 2$, let $E = F_0(X)$, and let $K = F_0(\{X^{1/2^n}\}_{n=1}^{\infty})$; $F_0(\{X^{3/2^n}\}_{n=1}^{\infty})$ is a proper intermediate field that is not finite-dimensional over F. One question that naturally arises is whether a J-extension must, in fact, be either separable or purely inseparable. If K/F is a normal J-extension, this question has an affirmative answer, for in that case K is the composite of K_s and K_i, the separable and purely inseparable parts of K/F, respectively [**Ba**, p. 88]. Hence K_s/F or K_i/F is not finite-dimensional, which implies that $K = K_s$ or $K = K_i$. In general, however, Example 2.11 shows that this question has a negative answer. The presentation of Example 2.11 uses two auxiliary results. The first of these, Lemma 2.8, must be known, but we haven't located an appropriate reference. The statement of Lemma 2.10 uses the following terminology from [**Re**]. An extension field K of F is an *exceptional extension* of F if K/F is algebraic, not separable, and F is the purely inseparable part of K/F.

LEMMA 2.8. *Assume that* $\operatorname{char} F = p \neq 0$ *and consider a simple purely inseparable extension* $F(\theta)$ *of* F, *where* θ *has exponent* $e > 1$ *over* F.

(1) *For* $1 \leq i \leq e$, $F(\theta^{p^{e-i}}) = \{x \in F(\theta) \mid x^{p^i} \in F\}$.

(2) *If* $\alpha \in F(\theta^{p^{e-i}}) \setminus F(\theta^{p^{e-i+1}})$, *then* $F(\alpha) = F(\theta^{p^{e-i}})$.

(3) $\{F(\theta^{p^j})\}_{j=0}^{e}$ *is the set of intermediate fields.*

Proof. In (1), we use induction on e. The case where $e = 1$ is obvious. At the inductive step, suppose θ has exponent $e + 1$ over F and the result is known for exponent e. Let $E = \{x \in F(\theta) \mid x^{p^e} \in F\}$. The inclusions $F(\theta^p) \subseteq E$ and $E < F(\theta)$ are clear. Since $[F(\theta) : F(\theta^p)] = p$, we conclude that $E = F(\theta^p)$, where θ^p has exponent e over F. The equality $F(\theta^{p^{e+1-i}}) = \{x \in F(\theta) \mid x^{p^i} \in F\}$ is obvious for $i = e + 1$. If $1 \leq i \leq e$ and if $x^{p^i} \in F$, then $x^{p^e} \in F$ and $x \in F(\theta^p)$. Thus, the induction hypothesis implies that $x \in F((\theta^p)^{p^{e-i}}) = F(\theta^{p^{e+1-i}})$.

(2) If $\alpha \in F(\theta^{p^{e-i}}) \setminus F(\theta^{p^{e-i+1}})$, then (1) shows that α has exponent i over F. Therefore $[F(\alpha) : F] = p^i = [F(\theta^{p^{e-i}}) : F]$, and consequently, $F(\alpha) = F(\theta^{p^{e-i}})$.

(3) follows immediately from (2).

COROLLARY 2.9. *Assume that* $\operatorname{char} F = p \neq 0$ *and* $F(t)/F$ *is a simple inseparable extension of exponent* $e > 1$. *Then* $F(t)/F$ *is exceptional if and only if* $F(t^{p^{e-1}})/F$ *is exceptional.*

Proof. Clearly $F(t^{p^{e-1}})/F$ is exceptional if $F(t)/F$ is exceptional. For the converse, we show that if $x \in F(t)$ is such that $x^p \in F$, then $x \in F$. This statement follows from Lemma 2.8: $F(t)/F(t^{p^e})$ is of exponent e, so Lemma 2.8 implies that $x \in F(t^{p^{e-1}})$, and hence $x \in F$ since $F(t^{p^{e-1}})/F$ is exceptional.

LEMMA 2.10. *Assume that* F *is a field of characteristic* $p \neq 0$ *and let* K *be an algebraic extension of* F *containing elements* a, b, s *with the following properties:* a^p, $b^p \in F$, $[F(a,b):F] = p^2$, *and* $F(s)/F$ *is a nontrivial Galois extension. Let* $t = a + bs$ *and let* $L = F(t)$. *Then* $F(t)/F$ *is an exceptional extension,* $F(s)$ *is the separable part of* $F(t)/F$, *and if* $\alpha \in F(t) \setminus F(s)$, *then* $F(s) = F(\alpha^p)$.

Proof. We have $t^p = a^p + b^p s^p$ so $F(t^p) = F(s^p) = F(s)$. The inclusion $F(s) \subseteq F(t)$ is proper, for if not, then $F(s,b) \supseteq F(s,b,a+bs) = F(s,b,a)$, so that $F(s,b) = F(s,b,a)$. This is a contradiction since $[F(s,b):F]_i = p$ while $[F(s,b,a):F]_i = p^2$. Therefore $F(s) < F(t)$, $F(s)$ is the separable part of $F(t)/F$, and $[F(t):F]_i = p$. We next show that t is the unique conjugate in $F(t)$ of t over F. To do so, let K be a normal closure of $F(t)/F$. If t is not the only conjugate of t in $F(t)$, then there exists an F-automorphism σ of K such that $\sigma(t) \in F(t)$, $\sigma(t) \neq t$. Since $t = a + bs$ with a, b purely inseparable over F, then $\sigma(t) = a + bu$, where $u = \sigma(s)$. We have $F(t) = F(\sigma(t))$, and by the argument given above, $F(s) = F(u)$ is the separable part of $F(t)/F$. Moreover, $t \neq \sigma(t)$ implies that $s \neq u$. Thus, $b(u - s) = a + bu - t \in F(t)$ so $b \in F(t)$ and $a = t - bs \in F(t)$, contrary to the fact that $[F(t):F]_i < [F(s,a,b):F]_i$. We conclude that $F(s)$ is the unique proper subfield of $F(t)$ containing F over which $F(t)$ is normal. In particular, $F(t)/F$ is exceptional, for if not, then $F(t)$ is the composite of $F(s)$ and the purely inseparable part of $F(t)$ over F, and hence is normal over F, a contradiction. Finally, we show that $F(s) = F(\alpha^p)$ for each $\alpha \in F(t) \setminus F(s)$. The inclusion $F(\alpha^p) \subseteq F(s)$ holds since $F(t)/F(s)$ is purely inseparable of exponent 1. Note, however, that $F(t)$ is the composite of $F(s)$ and the purely inseparable part of $F(t)/F(\alpha^p)$, implying that $F(t)/F(\alpha^p)$ is normal. We conclude that $F(\alpha^p) = F(s)$, and this completes the proof of Lemma 2.10.

EXAMPLE 2.11. Let the notation and hypothesis be as in the statement of Lemma 2.10. If $E = \bigcup_{n=1}^{\infty} F(t^{1/p^n})$, then E is an exceptional J-extension of F. Moreover, if $\alpha \in E \setminus F(s)$, then $F(s) \subseteq F(\alpha)$, so each subfield of E containing F compares with $F(s)$ under inclusion.

Proof. To prove that E is a J-extension of F, we choose $x_n \in E \setminus F(t^{1/p^n})$ for each n and we show that $E = F(\{x_i\}_{i=1}^{\infty})$. Thus, for a fixed n, choose m so that $x_n \in F(t^{1/p^m}) \setminus F(t^{1/p^{m-1}})$; note that $m > n$. Considering $F(t^{1/p^m})$ as an extension of $F(t^p) = F(s)$, Lemma 2.8 shows that $x_n^{p^m} \in F(t) \setminus F(s)$. Therefore $F(x_n^{p^m}) = F(t)$ by Lemma 2.10 so that $F(x_n) = F(t)(x_n) = F(t^{1/p^m}) \supseteq F(t^{1/p^n})$. We conclude that E is a J-extension of F. Moreover, the preceding proof, together with Lemma 2.10, shows that $F(\alpha) \supseteq F(s)$ for each $\alpha \in E \setminus F(s)$. Clearly E/F is not separable; to show that E/F is exceptional, it suffices to show that $F(t^{1/p^n})/F$ is exceptional for each n. This last assertion follows at once from Lemma 2.10 and Corollary 2.9.

One example of fields F and K as in the statement of Lemma 2.10 can be obtained by taking $F = L(a^p, b^p)$ and $K = L(s, a, b)$, where L is a field of characteristic $p \neq 0$, a and b are indeterminates over L, and $L(s)/L$ is Galois. The presentation of Example 2.11 shows that the lattice of fields between F and $E = \bigcup_{n=1}^{\infty} F(t^{1/p^n})$ consists of an isomorphic copy of the lattice of fields between L and $L(s)$, topped by a well-ordered countably infinite chain. Since the lattice of fields between L and $L(s)$ is as arbitrary as the lattice of subgroups of a finite group under $\supseteq$, it follows that at least at the "bottom", little can be said about the lattice of subfields of an inseparable J-extension.

We turn to the problem of examining the structure of a separable J-extension K/F. While it seems reasonable to think that K/F might share properties in common with Galois J-extensions, as recorded in Theorem 2.5, this turns out to not be the case. One reason that the usual procedure for passing from Galois to separable extensions fails in this case is that no proper extension L of K is a J-extension of F; in particular, the normal closure L of K/F is not a J-extension of F if $L \neq K$. We present two constructions of separable J-extensions K/F. In the first (Example 2.15), the set $\{K_i\}_0^{\infty}$ of intermediate fields forms a chain $F = K_0 < K_1 < \cdots$, but the degrees $[K_{i+1}:K_i]$ need not be prime and the set $\{[K_{i+1}:K_i]\}_{i=0}^{\infty}$ may be infinite. Each example of a J-extension K/F considered thus far in the paper has been such that there exists a proper intermediate field E such that the fields between E and K form a

chain. In Example 2.18, we give an example of a separable J-extension that fails to satisfy this condition. In both Example 2.15 and 2.18, we use results of Fried and MacRae from [**FM**]. For sake of reference, we record the results used from [**FM**] as Theorem 2.12; it is a composite of Proposition 3.4 and Theorem 3.6 of [**FM**].

THEOREM 2.12 (*Fried-MacRae*). *Let K be a field and let $f(X) \in K[X] \setminus K$ be such that* $\operatorname{char} K$ *does not divide the degree of $f(X)$. Each field between $K(X)$ and $K(f(X))$ is of the form $K(g(X))$, where $g(X) \in K[X]$ is such that $f(X) \in K[g(X)]$. If M_1 and M_2 are intermediate fields of degrees d_1 and d_2, respectively, over $K(f(X))$, then $[M_1 \cap M_2 : K(f(X))] = \gcd\{d_1, d_2\}$ and $[M_1 M_2 : K(f(X))] = \operatorname{lcm}\{d_1, d_2\}$. In particular, $M_1 \subseteq M_2$ if and only if $d_1 \mid d_2$.*

We use two lemmas in presenting Example 2.15.

LEMMA 2.13. *Assume that $r > 1$ is an integer not divisible by the characteristic of the field F. If $g = X^r + X^{r-1}$, then there exsits no intermediate field properly between $F(g)$ and $F(X)$.*

Proof. By Theorem 2.12, it suffices to show that if $f \in F[X]$ is such that $g = h(f)$ for some $h \in F[X]$, then either $\deg f = 1$ or $\deg f = r$. Without loss of generality we assume that $f(0) = 0$. Then considering f, g, and h as elements of $F[[X]]$, we have $\operatorname{ord} g = r - 1 = \operatorname{ord} h \cdot \operatorname{ord} f$, where ord denotes the order function. Moreover, $\deg g = r = \deg h \cdot \deg f$. These equations are impossible for $\operatorname{ord} h < \deg h$ and $\operatorname{ord} f < \deg f$. Hence either $\operatorname{ord} h = \deg h$ or $\operatorname{ord} f = \deg f$; in the first case, $\deg h = 1$ and $\deg f = r$, while the second implies $\deg f = 1$. This completes the proof.

LEMMA 2.14. *Assume that F is a field and $f(X) \in F[X]$ is such that* $\operatorname{char} F$ *does not divide* $\deg f$. *Assume that there exists a sequence $\{F_i\}_{i=0}^n$ of subfields of $F(X)$ such that $F(f) = F_0 < F_1 < \cdots < F_n = F(X)$ and the following conditions are satisfied for $0 \le i \le n - 1$.*

(1) *There are no fields properly between F_i and F_{i+1},*

(2) *Each prime divisor of $[F_i : F_0]$ divides $[F_{i+1} : F_i]$.*

Then $\{F_i\}_{i=0}^n$ is the set of subfields of F_n containing F_0.

Proof. We use induction on n, the case $n = 1$ being obvious. Assume the result for $n - 1$. Let the prime factorization of $[F_{n-1} : F_0]$ be $p_1^{e_1} \cdots p_k^{e_k}$ and let $[F_n : F_0] = (p_1^{h_1} \cdots p_k^{h_k})b$, where no p_i divides b and $e_j < h_j$ for

each j. Let K be a subfield of F_n containing F_0. If $K \subseteq F_{n-1}$, then $K \in \{F_i\}_{i=0}^{n-1}$ by the induction hypothesis. Otherwise, let $[K:F_0] = p^{u_1} \cdots p_k^{u_k} c$, where c is divisible by no p_i. Since $F_{n-1} < KF_{n-1}$, (1) implies that $KF_{n-1} = F_n$, and Theorem 2.12 shows that $[F_n:F_0] = (p_1^{h_1} \cdots p_k^{h_k})b = \text{lcm}\{p_1^{u_1} \cdots p_k^{u_k} c, p_1^{e_1} \cdots p_k^{e_k}\}$. Since $e_i < h_i$ for each i, this implies that $c = b$ and $u_i = h_i$ for each i, and hence $F_n = K$.

EXAMPLE 2.15. Let F be a field and let $\{k_i\}_{i=1}$ be a sequence of integers, $k_i > 1$, such that (i) $\text{char}\, F$ does not divide k_i for each i, and (ii) each prime divisor of $k_1 k_2 \cdots k_n$ divides k_{n+1} for each n. We prove existence of a separable J-extension K of $F(X)$ with the following properties: (a) the set of proper intermediate fields forms a chain $F(X) = F_0 < F_1 < F_2 < \cdots$, and (b) $[F_i:F_{i-1}] = k_i$ for each i. Thus. let y_1 be a root of $Y^{k_1} + Y^{k_1-1} - X$ in an extension field of $F(X)$ and let $F_1 = F(y_1)$. The element y_1 is transcendental over F and $X = y_1^{k_1} + y_1^{k_1-1}$. Therefore Lemma 2.13 implies that there exist no proper intermediate field between F_0 and F_1 and $[F_1:F_0] = k_1$. Let y_2 be a root of $Y^{k_2} + Y^{k_2-1} - y_1$ in an extension field of F_1. As above, if $F_2 = F(y_2)$, then $[F_2:F_1] = k_2$ and there exists no proper intermediate field between F_1 and F_2. Now X is a polynomial in y_2 of degree $k_1 k_2$, so Lemma 2.14 shows that $\{F_0, F_1, F_2\}$ is the set of subfields of F_2 containing F_0. We inductively continue this process, obtaining a strictly ascending sequence $\{F_i\}_{i=0}^{\infty}$ such that $\{F_i\}_{i=0}^{n}$ is the set of subfields of F_n containing F_0 for each n and $[F_n:F_{n-1}] = k_n$ for each n. Corollary 2.2 shows that K is a J-extension of $F(X)$, and $K/F(X)$ is separable since $\text{char}\, F$ does not divide $[F_n:F(X)] = k_1 k_2 \cdots k_n$ for each n.

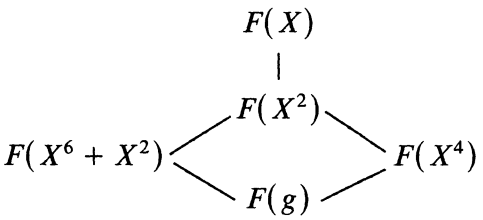
Corollary 2.6 states that if K/F is a Galois J-extension, if $F \subseteq L$, and if $K \not\subseteq L$, then LK/L is also a J-extension. Lemma 2.14 can be used, as in Example 2.15, to show that this result does not extend to the case where K/F is a separable J-extension. To see this, let C be the field of complex numbers, let $F = C(X)$, and let $K = F(y_1, y_2 \cdots)$, where $y_1^7 + y_1^6 - X = 0$, $y_2^7 + y_2^6 - y_1 = 0, \cdots$. Lemma 2.14 implies that K/F is a separable J-extension. Let $L = C((X))$, the quotient field of the formal power series ring $C[[X]]$. For $n \in Z^+$, it is known that $C((X^{1/n})) = L(X^{1/n})$ is the unique algebraic extension of L of degree n [**Co**, Cor. 5.339, p. 418], and $\bigcup_{n=1}^{\infty} L(X^{1/n})$ is the algebraic closure of L. We note that $X = y_1^7 + y_1^6 = y_1^6(1 + y_1)$ and $1 + y_1$ has a sixth root b in $C[[y_1]] = C[[y_1^6 + y_1^7]][y_1] = C[[X]][y_1] \subseteq L(y_1)$; here the equality $C[[y_1^6 + y_1^7]][y_1] = C[[y_1]]$ follows, for example, from [**Bo2**, p. 309]. It follows that $X = (by_1)^6$, and hence $L(y_1) = L(X^{1/6})$. Similarly, $L(y_1, \ldots, y_n)$ has degree

6^n over L, so $L(y_1, \ldots, y_n) = L(X^{1/6^n})$ for each $n \in Z^+$. Because L contains the 6^nth roots of unity, it follows that $LK = L(y_1, y_2, \cdots) = \bigcup_{n=1}^{\infty} L(X^{1/6^n})$ is a Galois extension of L. Since $L(X^{1/6})$ is an intermediate field of degree 6 over L, Theorem 2.5 implies that LK/L is not a J-extension; more directly, $\bigcup_{n=1}^{\infty} L(X^{1/2^n})$ is a proper intermediate field that is not finitely generated over L.

By passing from K/F to $K(T)/F(T)$, where T is a transcendence basis for $L = C((X))$ over $F = C(X)$, one sees in the above example that a J-extension need not lift under a separable algebraic field extension, even in the case where the intermediate fields of the J-extension are linearly ordered and the degree of any adjacent pair of intermediate fields is a fixed prime (in this case 7). Here we are making use of the fact that if K/F is a separable algebraic field extension and T is a family of indeterminates over K, then the lattice of fields between F and K is isomorphic to the lattice of fields between $F(T)$ and $K(T)$. To see this fact, one may reduce to the case where E/F is a finite separable algebraic field extension and use Galois theory. If M/F is the normal closures of E/F, then $M(T)/F(T)$ is the normal closure of $E(T)/F(T)$, the Galois groups are isomorphic, and any field between $F(T)$ and $E(T)$ is generated over $F(T)$ by an element of E.

In comparison to Example 2.15, Example 2.18 is more specific in nature. Also, while Example 2.15 is designed to produce a linearly ordered set of intermediate extensions in a separable J-extension, the main point of Example 2.18 is to show that branching may occur infinitely often in such an extension. Again we require two lemmas in Example 2.18.

LEMMA 2.16. *Let F be a field of characteristic distinct from* 2 *and* 3 *and let $g = (X^4 + 1)^2 X^4 \in F[X]$. The lattice of fields between $F(X)$ and $F(g)$ is as follows.*



Proof. We have $g = (X^6 + X^2)^2$, and clearly g is a polynomial in both X^2 and X^4, so there exist distinct intermediate fields as indicated in the diagram. Theorem 2.12 shows that for each divisor d of 12, there

exists at most one intermediate field of degree d over $F(g)$. Hence, to complete the proof, it suffices to show that there exists no intermediate field of degree 4 over $F(g)$. Equivalently, we need to show that $g \notin F[f(X)]$ for each cubic polynomial $f(X) \in F[X]$. Assume, to the contrary, that $g = f^4 + af^3 + bf^2 + cf + d$ where, without loss of generality, we assume that f is monic with constant term 0. As a power series in X, g has order 4, so $c = d = 0$ and $g = (X^4 + 1)^2X^4 = f^2(f^2 + af + b)$. The polynomials $(X^4 + 1)^2$ and X^4 are relatively prime; if $b \neq 0$, then f^2 and $f^2 + af + b$ are relatively prime, and in this case $(X^4 + 1)^2$ divides either f^2 or $f^2 + af + b$, an impossibility by a degree argument. Hence $b = 0$ and $g = f^3(f + a)$. Now $a = 0$ is impossible because g is not a fourth power in $F[X]$. But if $a \neq 0$, then f^3 and $f + a$ are relatively prime and we conclude that f^3 is an associate of either $(X^4 + 1)^2$ or X^4, neither of which is possible by a degree argument. We conclude that there exists no intermediate field of degree 4 over $F(g)$.

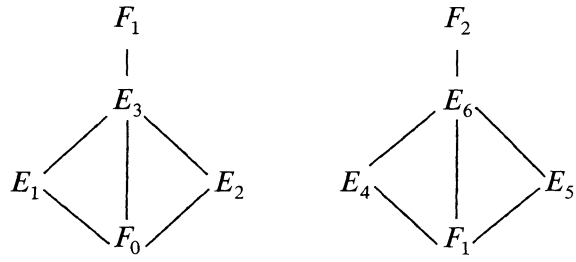
LEMMA 2.17. *Assume that F is a field of characteristic distinct from* 2 *and* 3. *The lattice of fields between $F(X)$ and $F((X + 1)^4X^2)$ is as follows.*

$$
\begin{array}{c}
F(X) \\
| \\
F\big((X + 1)^2X\big) \\
| \\
F\big((X + 1)^4X^2\big)
\end{array}
$$

Proof. As in Lemma 2.16, we need only show that $(X + 1)^4X^2 \notin F[f(X)]$ for each quadratic $f \in F[X]$. Because of its similarity to the proof of Lemma 2.16, we omit the proof of this statement.

EXAMPLE 2.18. Let F be a field of characteristic distinct from 2 and 3. To obtain a separable J-extension with infinite branching, we iterate infinitely many times the extension suggested by Lemma 2.16. To wit, let $F_0 = F(X)$ and let $F_1 = F_0(y_1)$, where y_1 is a root of $(Y^4 + 1)^2Y^4 - X$ in an extension field of F_0, let $F_2 = F_1(y_2)$, where $(y_2^4 + 1)^2y_2^4 - y_1 = 0$, and so forth. We take $K = \bigcup_{i=1}^{\infty} F_i$ and intend to show that K/F_0 is a separable J-extension. That K/F_0 is separable is clear. The key result in showing that KF_0 is a J-extension is a proof that each subfield of F_n containing F_0 lies between F_i and F_{i-1} for some $i \in \{1, 2, \ldots, n\}$. The proof is by induction on n, and the case $n = 2$ is crucial in the process.

Thus, label the lattice of fields between F_0 and F_1 and between F_1 and F_2 as follows.



Letting $t = y_2^4$, we examine the following piece of the joined lattice.

$$\begin{array}{lcl} & & E_5 = F\left(y_2^4\right) = F(t) \\ F_1 & = & F\left((t+1)^2 t\right) \\ | & & \\ E_3 & = & F\left((t+1)^4 t^2\right) \end{array}$$

According to Lemma 2.17, no subfield of E_5 containing E_3 has degree 3 over E_3. Since X is a polynomial in y_2 of degree 144, Theorem 2.12 implies that there exists no intermediate field E, $F_0 \subseteq E \subseteq F_2$, with $[E:F_0] = 18$, for since $6 \mid 18$ and $18 \mid 36$, such a field E would lie between E_3, of degree 6 over F_0, and E_5, of degree 36 over F_0, and would have degree 3 over E_3, an impossibility. Now let $S = \{d \mid d = [E:F_0]$ for some field E with $F_0 \subseteq E \subseteq F_2\}$. The set S contains $T = \{1, 2, 3, 6, 12, 24, 36, 72, 144\}$, it is closed under taking greatest common divisors and least common multiples, and we know that $4, 18, 48 \notin S$. It is then straightforward to show that $S = T$, and hence the assertion holds for $n = 2$. Assume by induction that each subfield of F_n containing F_0 lies between F_{i-1} and F_i for some i between 1 and n. Since the map $y_n \to y_{n+1}$ determines an F-isomorphism of F_n onto F_{n+1} that maps F_0 to F_1, we know also that each subfield of F_{n+1} containing F_1 lies between F_{i-1} and F_i for some i between 2 and $n+1$. Let E be a subfield of F_{n+1} containing F_0 and let $h = [E:F_0]$. If $12 \mid h$ or if $[F_n:F] = 2^{2n}3^n \mid h$, then E lies between some F_{i-1} and F_i. Otherwise, $EF_n > F_n$ and $[EF_n:F_0] = \operatorname{lcm}\{h, 2^{2n}3^n\}$ is a proper multiple of $2^{2n}3^n$. Therefore either (i) $2^{2n+1} \mid h$ and $3 \nmid h$ or (ii) $3^{n+1} \mid h$. If (i) is true, we obtain the contradiction that $E \cap F_1$ has degree 4 over F_0. Suppose (ii) holds. If $2 \nmid h$, we obtain the contradiction that $E \cap E_5$ has degree 9 over F_0, and if $2 \mid h$, then $[E \cap E_5 : F_0] = 18$ yields a contradiction. Hence E lies between some F_{i-1} and F_i.

Finally we conclude that K/F_0 is a J-extension, for if $x_n \in K \setminus F_n$ for each n, then the above result shows that $F_0(x_n) \supseteq F_n$, so $F_0(\{x_n\}_{n=1}^{\infty}) = K$. Also, the fields between E and K are not linearly ordered for any proper intermediate field E, for any such E is contained in some F_n.

We remark that Example 2.18 has the following group-theoretic implication. If E is an intermediate field of K/F_0 such that E is maximal with respect to failure to contain $a \in K \setminus F_0$, the $E(a)$ is the unique minimal proper extension of E in K. If K_s is the separable closure of K/E, $G = \mathrm{Gal}(K_s/E)$ and $H = \mathrm{Gal}(K_s/K)$, then H is contained in a unique maximal subgroup of G. Since the fields between E and K are not linearly ordered, neither are the subgroups of G containing H linearly ordered. By taking finite extensions of E in K, examples similar to the above can be obtained, where G is a finite group. On the other hand, it is easy to see that if H is a normal subgroup of G contained in a unique maximal subgroup of G, then the subgroups of G containing H are linearly ordered.

We conclude this section with some remarks concerning the structure of purely inseparable J-extensions. For sake of reference, we call a purely inseparable J-extension constructed as in example (E2) of the introduction *standard*. Nonstandard examples are not easy to construct, but they do exist. Deveney in [**De**] constructs, in fact, a purely inseparable J-extension U/V, where $\mathrm{char}\, V = p$, so that if $U_i = \{x \in U \mid x^{p^i} \in V\}$, then $[U_i : V] = p^{2i}$ for each i; the set of intermediate fields is not linearly ordered, nor is there a proper intermediate field W such that the set of subfields of U containing W is linearly ordered. If $x \in U \setminus V$ and if W is an intermediate field maximal with respect to failure to contain x, then U/W provides an example of a nonstandard J-extension such that $\{u \in U \mid u^p \in W\}$ has degree p over W. Proposition 2.19 contains some positive information concerning purely inseparable J-extensions.

PROPOSITION 2.19. *Assume F is a field of characteristic $p \neq 0$ and let K be a purely inseparable J-extension of F. For $n \in Z_0$, let $K_n = \{x \in K \mid x^{p^n} \in F\} = F^{1/p^n} \cap K$.*

(1) *The sequence $\{K_n\}_{n=1}^{\infty}$ is strictly ascending—that is, K does not have finite exponent over F.*

(2) *K/F is standard if and only if there exists $s \in F \setminus F^p$ such that $s^{1/p^n} \in K$ for each $n \in Z^+$.*

Proof. (1): Suppose $K = K_n$ for some n. Clearly $n > 0$. We assume that n is minimal so that $K = K_n$. Then K/K_{n-1} is a J-extension of exponent 1. This is impossible, for if B is a p-basis for K/K_{n-1} and if

$b \in B$, then $K_{n-1}(B \setminus \{b\})$ is an infinite-dimensional extension of K_{n-1} properly contained in K. Therefore $K_n < K$ for each n.

(2) If K/F is standard, then the stated condition is satisfied by definition. Conversely, if K contains $\{s^{1/p^n}\}_{n=1}^{\infty}$ for some $s \in F \setminus F^p$, then $F(\{s^{1/p^n}\}_1^{\infty})$ is an infinite-dimensional extension of F in K, and hence is equal to K.

While a standard purely inseparable J-extension is such that the set of intermediate fields is linearly ordered, a purely inseparable J-extension L/E with linearly ordered intermediate fields need not be standard. For example, we can take

$$E = M\left(\{X_i\}_{i=1}^{\infty}\right), \quad \text{where char } M = p \neq 0, \quad \text{and}$$

$$L = E\left(\left\{X_1^{1/p^n} + X_2^{1/p^{n-1}} + \cdots + X_n^{1/p}\right\}_{n=1}^{\infty}\right).$$

3. Fields that admit a J-extension. We consider in this section the problem of determining those fields that admit a J-extension or a Galois J-extension. We begin with a consideration of the Galois case. Theorem 2.5 answers the question in one sense: F admits a Galois J-extension if and only if W_p can be realized as a Galois group over F for some prime p. This criterion, however, is elusive, and we seek to give it more substance. If W_p can be realized as a Galois group over F, then F admits a cyclic extension of degree p. The converse fails (for example, let F be the real field and let $p = 2$), but we are able to establish the converse in enough cases to provide a large class of fields that admit Galois J-extensions. In fact, there are three related questions here for a field F. We list these as Q(1), Q(2), and Q(3); their answers depend upon the fields F and K. Q(1) is most clearly related to the problem at hand.

Q(1) If F admits a cyclic extension of prime degree, does F admit a Galois J-extension?

Q(2) If K/F is cyclic of prime degree p, can W_p can be realized as a Galois group over F?[2]

Q(3) If K/F is cyclic of prime degree, can K be extended to a Galois J-extension of F?

We note that an affirmative answer to question Q(n) implies an affirmative answer to Q($n-1$); a more general version of Q(3) asks whether K can be extended to a Galois J-extension of F if K/F is cyclic of prime-power degree.

---

[2]For an odd prime p, Q(2) has an affirmative answer; see the appendix to this paper.

Cyclic extension fields play a classical role in Galois theory, and papers of Artin and Schreier [**AS**] and Albert [**A1**], [**A2**] are landmarks in the area. In fact, part (1) of Theorem 3.1 listed below comes from [**AS**], and part (2) comes from [**A1**]. Theorem 3.1 provides a case in which Q(3) has an affirmative answer.

THEOREM 3.1 (*Artin-Schreier*; *Albert*). *Assume that F is a field of characteristic $p \neq 0$.*

(1) *F admits a cyclic extension of degree p if and only if $F \neq \{a^p - a \mid a \in F\}$.*

(2) *If K is a cyclic extension of F of degree p^e, than K can be extended to a cyclic extension of degree p^{e+1}. Consequently, K can be extended to a Galois J-extension of F.*

A second case in which Q(3) has an affirmative answer is provided by Proposition 3.2.

PROPOSITION 3.2. *If K/F is cyclic of prime degree p and if F contains the p^nth roots of unity for all $n \in Z^+$, then K is contained in a Galois J-extension of F.*

Proof. If $\operatorname{char} F = p$, then the result follows from Theorem 3.1. If $\operatorname{char} F \neq p$, then $K = F(\sqrt[p]{a})$, where $a \in F$ [**Ba**, p. 174]. The Vahlen-Capelli Theorem [**K**, Theorem 51] then implies that $X^{p^n} - a$ is irreducible over F for each $n \in Z^+$, except in the case where $p = 2$, $\operatorname{char} F \neq 2$, and $-4a$ is a fourth power in F. Since F contains a primitive fourth root of unity i, this exceptional case does not occur, for $-4a = b^4 = -i^2b^4$ implies $a = (ib^2/2)^2$, so a has a square root in F, a contradiction. Therefore $X^{p^n} - a$ is irreducible over F for each n, and since F contains the p^nth roots of unity, $F(\sqrt[p^n]{a}) = K_n$ is cyclic over F of degree p^n for each n [**Ba**, p. 175]. Therefore $E = \bigcup_{n=1}^{\infty} K_n$ is a Galois J-extension of F containing K.

In passing to the case where F does not contain the p^nth roots of unity for all n, some distinction must be made between the cases where $p = 2$ and where p is odd. The following lemma, which fails for $p = 2$ and $k = 1$, is at the root of the distinction. The routine proof of Lemma 3.3 is omitted.

LEMMA 3.3. *If k, $r \in Z^+$, $b \in Z$, and p is prime, then $(1 + p^k b)^{p^r} \equiv 1 + p^{k+r}b \pmod{p^{k+r+1}}$ if p is odd. If $p = 2$, this congruence is valid for $k \geq 2$.*

PROPOSITION 3.4. *Assume that p is prime and that F is a field that contains the p^kth roots of unity but not the p^{k+1}st roots of unity, where $k \geq 1$ and where $k \geq 2$ if $p = 2$. Let ζ be a primitive p^kth root of unity in F and for $r \in Z^+$, let ζ_r be a root of $X^{p^r} - \zeta$ in an extension field of F. Then ζ_r is a primitive $p^{k+r} - th$ root of unity and $F(\zeta_r)/F$ is cyclic of order p^r. Therefore $F(\{\zeta_r\}_{r=1}^{\infty})$ is a Galois J-extension of F with Galois group W_p.*

Proof. We have $(\zeta_r)^{p^{k+r}} = (\zeta_r^{p^r})^{p^k} = \zeta^{p^k} = 1$ and $(\zeta_r)^{p^{k+r-1}} = \zeta^{p^{k-1}} \neq 1$, so ζ_r has order p^{k+r}. Since F contains the pth roots of unity, the equation $X^p - \zeta$ is either irreducible over F or it has a root in F [**vdW**, p. 180], but the latter condition fails by choice of k. Hence $X^p - \zeta$ is irreducible over F, and as in the proof of Proposition 3.2, the Vahlen-Capelli Theorem implies that $[F(\zeta_r):F] = p^r$ for each r. To show that $F(\zeta_r)/F$ is cyclic, observe that $(\zeta_r^{1+p^k})^{p^r} = \zeta_r^{p^r}\zeta_r^{p^{k+r}} = \zeta$. Hence, there exists $\sigma \in \mathrm{Gal}(F(\zeta_r)/F)$ such that $\sigma(\zeta_r) = \zeta_r^{1+p^k}$. The order of σ is the order of $1 + p^k$ modulo p^{k+r}, which, according to Lemma 3.3, is p^r. Thus, σ generates $\mathrm{Gal}(F(\zeta_r)/F)$.

Proposition 3.4 may fail for $p = 2$ and $k = 1$; for example, it fails if $F = Q$. The distinction between the cases $p = 2$ and $p > 2$ carries over to the statement of the next result.

PROPOSITION 3.5. *Let F be a field.*

(a) *Assume that p is an odd prime and that F does not contain the pth roots of unity. Let ζ be a primitive pth root of unity in an extension field of F, and assume that $F(\zeta)$ does not contain the p^nth roots of unity for some n. Then F admits a Galois J-extension with Galois group W_p.*

(b) *If $p = 2$, the statement of* (a) *remains valid if F does not contain the fourth roots of unity and if $F(i)$, where i is a primitive fourth root of unity in an extension field of F, does not contain a 2^nth root of unity for some n.*

Proof. The proof is similar to the one used in Example 2.7 to show that Q admits Galois J-extensions, and indeed, Example 2.7 is a special case of Proposition 3.5. To prove (a), assume that $k \in Z^+$ is such that $F(\zeta)$ contains the p^kth, but not the p^{k+1}st roots of unity. Fix an algebraic closure L of $F(\zeta)$ and for each $r \in Z^+$, let ζ_r be a primitive p^{k+r}th root of unity in L. Proposition 3.4 shows that $F(\zeta_r)$ is cyclic over $F(\zeta)$ of degree p^r. Hence $F(\zeta_r)$ is cyclic over F of degree divisible by p^r. It follows that $F(\zeta_r)$ admits a unique subfield K_r of degree p^r over F. Uniqueness also implies that $K_{r-1} \subseteq K_r$ for $r \geq 2$. hence $K = \bigcup_{r=1}^{\infty} K_r$ is a Galois J-extension of F with Galois group W_p.

The proof of (b) differs slightly from that of (a). Thus, assume that $F(i)$ contains the 2^kth but not the 2^{k+1}st roots of unity. If ζ_r is a primitive 2^{k+r}th root of unity in a fixed algebraic closure of $F(i)$, then $F(\zeta_r)/F(i)$ is cyclic of degree 2^r. Therefore $F(\zeta_r)/F$ is a Galois extension of degree 2^{r+1}. The automorphism σ_r of $F(\zeta_r)/F$ determined by $\zeta_r \to \zeta_r^{-1}$ has fixed subfield $K_r = F(\zeta_r + \zeta_r^{-1})$ of codimension 2 in $F(\zeta_r)$. Moreover, $\sigma_r(i) \neq i$, so $F(i) \not\subseteq K_r$. It follows that $F(i)$ and K_r are linearly disjoint subfields of $F(\zeta_r)$ with composite field $F(\zeta_r)$. Therefore $\mathrm{Gal}(K_r/F) \simeq \mathrm{Gal}(F(\zeta)/F(i))$ is cyclic of order 2^r. The definition of K_r implies that $K_r \subseteq K_{r+1}$ for each r; hence $\bigcup_{r=1}^{\infty} K_r$ is a Galois J-extension of F and $\mathrm{Gal}(K/F) \simeq W_2$.

We note that in the notation of part (a) of the proof of Proposition 3.5, the field K_r does not contain the pth roots of unity, but $K_r(\zeta)$ contains the p^{r+k}th roots of unity; hence $\zeta \notin K$, but $K(\zeta)$ contains the p^rth roots of unity for all r. A similar example for the fourth and 2^rth roots of unity can be obtained from the proof of part (b) of Proposition 3.5.

PROPOSITION 3.6. *If F is a field of positive characteristic and if F admits a cyclic extension of prime degree p, then F admits a Galois J-extension.*

Proof. Let E be the algebraic closure in F of the prime subfield of F. If F contains the p^nth roots of unity for all n, then the result follows from Proposition 3.2. Otherwise, E is a field as described in Example (E1) of the introduction, so E admits a Galois J-extension L. By Corollary 2.6, LF is a Galois J-extension of F.

Suppose the field F admits a cyclic extension of prime degree p. The results of this section leave one case in which we have not given an affirmative answer to Q(1); this is the case where $\mathrm{char} F = 0$, F does not contain a primitive pth root of unity ζ, but $F(\zeta)$ contains the p^nth roots of unity for all n. In this case it may happen that F is not real closed and yet does not admit a Galois J-extension. If S is the solvable closure of Q, then S admits no Galois J-extension since solvability of field extensions is a transitive property [**G1**, Cor. 3.9]. Since S/Q is normal and S is not contained in the field $\mathbf{R}$ of real numbers, $F = S \cap \mathbf{R}$ is a subfield of S such that S/F is cyclic of degree two. But Corollary 2.6 implies that F does not admit a Galois J-extension. It would be interesting to know

whether a field F that admits a cyclic extension of degree $p > 2$ necessarily admits a Galois J-extension (See the appendix.)

Our next result shows that there exists a finite extension of S that does admit a Galois J-extension. Proposition 3.7 will also be useful in proving that a field that is neither algebraically closed nor real closed admits a J-extension.

PROPOSITION 3.7. *If the field F is imperfect, then F admits a standard purely inseparable J-extension. If the separable algebraic closure F_s of F is such that F_s/F is infinite, then there exists a finite separable extension E of F such that E admits a Galois J-extension.*

Proof. If F is imperfect, then as in example (E2) of the introduction. F admits a standard purely inseparable J-extension. If there exists a finite separable algebraic extension of F of degree over F divisible by an odd prime p, then the normal closure of this extension is a finite Galois extension K/F such that p divides $|G|$, where $G = \mathrm{Gal}(K/F)$. Let H be a subgroup of G of order p, and let E be the fixed field of H acting on K. If E contains a primitive pth root of unity, then E admits a Galois J-extension by Propositions 3.2 and 3.4. Otherwise, let $E_1 = E(\zeta)$, where ζ is a primitive pth root of unity. We note that KE_1/E_1 is cyclic of order p since $[K:E]$ and $[E_1:E]$ are relatively prime, and hence the previous case implies that E_1 admits a Galois J-extension.

The remaining case is where each finite separable algebraic extension of F is of degree over F a power of 2. The field $F(i)$, where i is a primitive fourth root of unity, has this same property. Since any group of order 2^k contains a subgroup of index 2, it follows that $F(i)$ admits a cyclic extension of degree 2, and Propositions 3.2 and 3.4 again show that $F(i)$ admits a Galois J-extension. This completes the proof of Proposition 3.7.

We move to a consideration of fields that admit a J-extension. In this connection, it is useful to note that if L/K is a J-extension, then the fields between K and L satisfy the descending chain condition (d.c.c.). Moreover, if L/F is a field extension that is not finitely generated but is such that the fields between F and L satisfy d.c.c., then there exists a J-extension of F contained in L. For if E is an intermediate field such that E is minimal with respect to the property that E/F is not finitely generated, then E/F is a J-extension.

Concerning d.c.c. on intermediate fields, we note the following easy fact.

LEMMA 3.8. *If K/F is a finite algebraic field extension and L/K is a field extension such that the intermediate fields satisfy* d.c.c., *then the fields between F and L also satisfy* d.c.c.

Proof. Suppose there exists a strictly descending chain of fields $E_1 > E_2 > \cdots$ between F and L. Since the fields between K and L satisfy d.c.c., the chain $E_1K \supseteq E_2K \supseteq \cdots$ stabilizes, so there exists a positive integer n such that $E_nK = E_{n+i}K$ for all $i \geq 0$. If $[K:F] = s$, then $[E_iK:E_i] \leq s$ for each i. But $[E_n:E_{n+s}] > s$ since the chain $\{E_i\}$ strictly descends. This gives $E_{n+s} \subseteq E_n \subseteq E_nK = E_{n+s}K$ with $[E_{n+s}K:E_{n+s}] \leq s$ and $[E_n:E_{n+s}] > s$, a contradiction. We conclude that the fields between F and L satisfy d.c.c..

Proposition 3.7 and Lemma 3.8 yield the following general result.

THEOREM 3.9. *If the field F is neither algebraically closed nor real closed, then F admits a J-extension. More precisely, if the separable algebraic closure F_s of F is such that F_s/F is infinite, then F admits a separable J-extension. If F is not perfect, then F admits a standard purely inseparable J-extension.*

Theorem 3.9 shows that the class of fields that admit a J-extension properly contains the class of fields that admit a Galois J-extension. We record this formally in the following corollary.

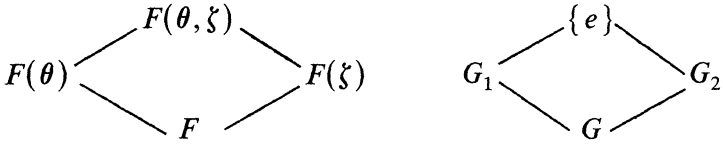
COROLLARY 3.10. *There exist fields F, such as the solvable closure S of Q, that admit a J-extension but do not admit a Galois J-extension.*

Theorem 3.9 is genuinely an existence theorem. For a field F that is the quotient field of a nontrivial valuation domain with principal maximal ideal, we present in Theorem 3.11 a more explicit result and construction.

THEOREM 3.11. *Assume that F is the quotient field of a nontrivial valuation domain V with principal maximal ideal $M = mV$. Then F admits a separable J-extension for which the intermediate fields are linearly ordered with respect to inclusion.*

Proof. Choose a prime $p \neq \text{char}(V/M)$ and a sequence $\theta_1, \theta_2, \cdots$ of elements in a fixed algebraic closure of F such that $\theta_1^p = m$, $\theta_2^p = \theta_1$, $\theta_3^p = \theta_2, \cdots$. We show that $K = F(\theta_1, \theta_2, \cdots) = \bigcup_{i=1}^{\infty} F(\theta_i)$ has the desired properties. Separability is no problem; to prove the theorem, it will

suffice to show that the set of subfields of $F(\theta_n)$ containing F is linearly ordered for each $n \in Z^+$. Let ζ_n be a primitive p^nth root of unity in an extension field of K. We show below that (i) $[F(\theta_n):F] = p^n$, and (ii) $F(\theta_n) \cap F(\zeta_n) = F$. Assuming (i) and (ii) for the moment, we complete the proof of Theorem 3.11. Thus, since $F(\zeta_n)/F$ is Galois, it follows that $F(\theta_n)$ and $F(\zeta_n)$ are linearly disjoint over F. Write θ and ζ for θ_n and ζ_n, and consider the following diagram, where G denotes the Galois group of $F(\theta, \zeta)/F$ and G_1 and G_2 are subgroups of G corresponding to the fields $F(\theta)$ and $F(\zeta)$, respectively.



We have $|G_2| = p^n = [G:G_1]$, and $G = G_1G_2$, where G_2 is normal in G and $G_1 \cap G_2 = \{e\}$. The subfields of $F(\theta)$ are in 1-1 order-reversing correspondence with the subgroups of G containing G_1, and each such subgroup H is of the form $G_1(G_2 \cap H)$. Since G_2 is cyclic of order p^n, its subgroups form a chain; hence the subgroups of G containing G_1 form a chain, and consequently, the subfields of $F(\theta)$ containing F are linearly ordered. Since $[F(\theta_n):F] = p^n$, where $\theta_n^{p^n} \in F$, it follows easily that $[F(\theta_n^{p^i}):F] = p^{n-i}$ for $0 \le i \le n$, and hence $\{F(\theta_n^{p^i})\}_{i=0}^n$ is the set of intermediate fields of $F(\theta_n)/F$.

We proceed to establish the equality $F(\theta) \cap F(\zeta) = F$ in (ii) by showing that V is totally ramified with respect to E if $F \subseteq E \subseteq F(\theta)$; in the process, we also prove (i). Let v be a valuation associated with V and let $w_1, \ldots, w_g$ be the extensions of v to $F(\theta)$. Let e_i and f_i, respectively, denote the reduced ramification index and the relative degree of w_i with respect to v. We have $p^n w_i(\theta) = v(m) > 0$ for each i, and since $v(m)$ is the smallest positive element of the value group of v, it follows that $\{jw_i(\theta)\}_{j=1}^{p^n}$ is a set of representatives of distinct cosets of the value group of v in the value group of w_i. Therefore $e_i \ge p^n$ for each i. Since $\sum_{i=1}^g e_i f_i \le [F(\theta):F] \le p^n$ [**Ri**, p. 228], we conclude that $g = 1$ and $e_1 = p^n$, $f_1 = 1$— that is, V is totally ramified with respect to $F(\theta)$ and $[F(\theta):F] = p^n$. If E is a subfield of $F(\theta)/F$, then standard techniques and two applications of the inequality cited in the preceding sentences show that V is totally ramified with respect to E—that is, v admits a unique extension u to E and the reduced ramification index of u with respect to v is $[E:F]$. Next we consider the behavior of V with respect to $F(\zeta)$. Since p is a unit of V/M, Theorem 10.18 of [**N**] shows that $V[\zeta]$ is the integral closure of V in $F(\zeta)$; thus, each extension of V to $F(\zeta)$ is a

quotient ring of the Prüfer domain $V[\zeta]$. Then [**N**, (38.6)] implies that V is unramified with respect to $F(\zeta)$; this means that for each extension (V^*, M^*) of V to $F(\zeta)$, V^*/M^* is separable over V/M, and V^* and V have the same value group. Since these two properties are obviously inherited by extensions of V to an intermediate field E, then V is unramified with respect to each such field E. We conclude that V is both unramified and totally ramified with respect to $F(\zeta) \cap F(\theta)$, so $F(\zeta) \cap F(\theta) = F$ as we wished to prove.

What fields F are *not* the quotient field of a nontrivial valuation domain V with principal maximal ideal? A sufficient condition for this to occur is that the equation $X^n - t = 0$ has a solution in F for each $t \in F$ and for infinitely many integers $n \in Z^+$. Thus, if $\operatorname{char} F = p \neq 0$, then the hypothesis of Theorem 3.11 implies that F is imperfect, and hence F also admits a standard purely inseparable J-extension. It seems reasonable to ask whether F is the quotient field of a rank-one discrete valuation domain if F satisfies the hypothesis of Theorem 3.11. We proceed to show (Example 3.13) that this question has a negative answer. The presentation of Example 3.13 involves the notion of a Henselian domain, defined as follows (see [**N**, p. 103]). Let (D, M) be a quasi-local domain with quotient field K. We say that (D, M) is *Henselian* and that K is *Henselian with respect to* D if the following condition is satisfied. If f, g_0, h_0 are monic polynomials in $D[X]$ such that $f \equiv g_0 h_0 \pmod{M[X]}$, where g_0 and h_0 are relatively prime modulo $M[X]$, then there exist monic polynomials g, $h \in D[X]$ such that $f = gh$, $g \equiv g_0 \pmod{M[X]}$, and $h \equiv h_0 \pmod{M[X]}$. We use a result (Satz 2.3.11, p. 60) from [**BKKN**] in Example 3.13. This result, labelled below as Theorem 3.12, is attributed to F. K. Schmidt [**S**] in [**BKKN**].

THEOREM 3.12 (*Schmidt*). *If (D, M) is a Henselian integral domain with quotient field K and if V is a rank-one discrete valuation domain on K, then $D \subseteq V$.*

EXAMPLE 3.13. We exhibit a field L that is the quotient field of a valuation domain with principal maximal ideal, but not the quotient field of a rank-one discrete valuation domain. Existence follows from Theorem 3.12 once we give an example of a Henselian valuation domain with principal maximal ideal that is not contained in a rank-one discrete valuation domain of its quotient field. It is easy to construct a valuation domain V such that V has principal maximal ideal and V is not contained

in a rank-one discrete valuation domain of its quotient field. For example, if X, Y, Z are indeterminates over a field k, we define a rank-one nondiscrete valuation domain W on the field $k(X, Y, Z)$ over $k(X)$ by defining $W(Y) = 1$ and $W(Z) = \sqrt{2}$. Then $W = k(X) + N$, where N is the maximal ideal of W [**G2**, Exer. 12, p. 271]. Therefore $V = k[X]_{(X)} + N$ is a rank-two valuation domain with the required properties. The Henselization V^* of V is a localization of the integral closure D of V in a appropriate algebraic extension field L [**N**, p. 180]; hence D is a Prüfer domain and V^* is a Henselian valuation domain with principal maximal ideal $M^* = MV^*$ [*ibid*]. The rank-one valuation overring of V^* is D_P, where P is the rank-one prime of D contained in $M^* \cap D$. Hence D_P is an extension of V to L, and since V is nondiscrete, so is D_P.

We remark that the sufficient condition given in Theorem 3.11 for F to admit a separable J-extension with linearly ordered intermediate fields is not necessary. This is shown, for example, by the fields of example (E1) of the introduction; other examples can be obtained from Proposition 3.7 or by taking $F = K(\{X^{1/p^n}\}_{n=1}^{\infty})$, where K has characteristic $p \neq 0$ and K admits a Galois J-extension.

REMARK 3.14. A field F that is the quotient field of a rank-one discrete valuation domain need not admit a Galois J-extension. To see this, let $\mathbf{R}$ denote the field of real numbers, and let $F = \mathbf{R}((X))$ be the quotient field of the formal power series ring $\mathbf{R}[[X]]$. Then f does not admit an abelian field extension of degree $n > 4$, and hence does not admit a Galois J-extension. For suppose F_n/F is an abelian field extension of degree $n > 4$, and consider the compositum $K_n = F_n\mathbf{C}$, where $\mathbf{C}$ is the field of complex numbers. Since $\mathbf{C}((X))/F$ is cyclic of degree 2, K_n/F is an abelian extension and $[K_n : \mathbf{C}((X))] = m$, where $m = n$ or $m = n/2$. By Puiseaux's Theorem [**Co**, p. 418], $K_n = C((Y))$, where $Y^m = X$. Therefore K_n contains the subfield $\mathbf{R}((Y))$. But $\mathbf{R}((Y))/F$ is not Galois for $m > 2$, which means that K_n/F cannot be abelian. We conclude that F does not admit an abelian extension of degree $n > 4$, so by Theorem 2.5, F does not admit a Galois J-extension.

4. Abelian extensions of Q. Denote by A the abelian closure of Q. This section deals with the same general theme as §3, but is more concrete in nature; to wit, the main result of the section is Corollary 4.11, which states that each subfield F of A admits a Galois J-extension.

To begin, we introduce some notation that will be used throughout this section. For $n \in Z^+$, let ζ_n be a primitive nth root of unity over Q. Let $\mathbf{P} = \{p_j\}_{j=0}^{\infty}$ be the sequence of primes, where $p_0 = 2$, and for $p \in \mathbf{P}$,

denote by Δ_p the Galois J-extension of Q with Galois group W_p exhibited in Example 2.7. Recall that for p odd, $\Delta_p = \bigcup_{j=1}^{\infty}\Delta_{p,j}$, where $\Delta_{p,j}$ is the unique subfield of $Q(\zeta_{p^{j+1}})$ of degree p^j over Q, while $\Delta_2 = \bigcup_{j=1}^{\infty}\Delta_{2,j}$, where $\Delta_{2,j}$, the real subfield of $Q(\zeta_{2^{j+2}})$, is cyclic over Q of degree 2^j. We have $Q(\zeta_{2^{j+2}}) = \Delta_{2,j}(i)$, while $Q(\zeta_{p^{j+1}}) = \Delta_{p,j}(\zeta_p)$. Hence $A = \Delta(\zeta_4, \zeta_3, \zeta_5, \cdots)$, where Δ is the compositum of the family $\{\Delta_p \mid p \in \mathbf{P}\}$.

The first three results of the section are used in the proof of Theorem 4.4.

LEMMA 4.1. *Assume that $F(\alpha, \beta)$ is an abelian extension of the field F and that $F(\alpha) \cap F(\beta) = F$. If K is a subfield of $F(\alpha)$ containing F, then $[F(\alpha, \beta) : K(\beta)] = [F(\alpha) : K]$.*

Proof. Since $F(\alpha) \cap F(\beta) = F$, then $[F(\alpha, \beta) : F(\beta)] = [F(\alpha) : F]$. Also, $K \cap F(\beta) = F$ implies $[K(\beta) : F(\beta)] = [K : F]$. Therefore

$$\begin{aligned}[F(\alpha, \beta) : K(\beta)] &= [F(\alpha, \beta) : F(\beta)]/[K(\beta) : F(\beta)] \\ &= [F(\alpha) : F]/[K : F] = [F(\alpha) : K].\end{aligned}$$

If $r, s \in Z^+$ have greatest common divisor d and least common multiple m, then $Q(\zeta_r) \cap Q(\zeta_s) = Q(\zeta_d)$ and $Q(\zeta_r, \zeta_s) = Q(\zeta_m)$ [**C**, Th. 1], [**L**, p. 314]. We use this result in the proof of Lemma 4.2.

LEMMA 4.2. *Assume that $k, n \in Z^+$ with $k < n$, and that $\{e_j\}_{j=0}^n$ is a subset of Z^+. Let*

$$r = 2^{e_0+2}\prod_{j=1}^{k} p_j^{e_j+1}, \qquad s = \prod_{k+1}^{n} p_j^{e_j+1},$$

$$t = 4p_1 \cdots p_k, \qquad u = p_{k+1} \cdots p_n.$$

Let $K_1 = \prod_{j=0}^{k}\Delta_{p_j,e_j}$, $K_2 = \prod_{j=k+1}^{n}\Delta_{p_j,e_j}$, where $\prod$ denotes compositum, and let $K = K_1K_2$ be the compositum of K_1 and K_2. Then $[K(\zeta_u, \zeta_t)K(\zeta_u)] = [K_1(\zeta_t) : K] = \phi(t) = [Q(\zeta_t)Q]$.

Proof. We note that $K_1(\zeta_t) = Q(\zeta_r)$ and $K_2(\zeta_u) = Q(\zeta_s)$, where $Q(\zeta_r) \cap Q(\zeta_s) = Q$. Since $K(\zeta_u, \zeta_t)$ is the compositum of $K_1(\zeta_t)$ and $K_2(\zeta_u)$ and since $K(\zeta_u)$ is the compositum of K_1 and $K_2(\zeta_u)$, it follows that

$$\begin{aligned}[K(\zeta_u, \zeta_t) : K(\zeta_u)] &= [K(\zeta_u, \zeta_t) : Q]/[K(\zeta_u) : Q] \\ &= [K_1(\zeta_t) : Q][K_2(\zeta_u) : Q]/[K_1 : Q][K_2(\zeta_2) : Q] \\ &= [K_1(\zeta_t) : Q]/[K_1 : Q] = [K_1(\zeta_t) : K_1].\end{aligned}$$

To see that $K_1(\Delta_t)$ has degree $\phi(t)$ over K_1, note that the equality $\Delta_{p_j,e_j} \cap (\prod\{\Delta_{p_v,e_v} | 0 \le v \le k, v \ne j\}) = Q$ (which follows since $\Delta_{p_j,e_j} \subseteq Q(\zeta_{p_j e_j} + 2)$ for each j) implies that $[K_1 : Q] = \prod_{j=0}^{k} [\Delta_{p_j,e_j} : Q] = \prod_{j=0}^{k} p_j^{e_j}$. Therefore

$$[K_1(\zeta_t) : K_1] = [Q(\zeta_r) : K_1] = [Q(\zeta_r) : Q]/[K_1 : Q]$$
$$= \phi(r)/[K_1 : Q] = \phi(t) = [Q(\zeta_t) : Q].$$

COROLLARY 4.3. *Let* $U = \{4\} \cup \{p_j\}_{j=1}^{\infty}$.

(1) *If* $\{u_j\}_{j=1}^{n}$ *is a finite nonempty subset of* U, *if* $u = u_1 u_2 \cdots u_n$, *and if* $V = U \setminus \{u_j\}_{j=1}^{n}$, *then* ζ_u *has degree* $\phi(u)$ *over* $\Delta(\{\zeta_v | v \in V\})$; *hence the minimal polynomial for* ζ_u *over* $\Delta(\{\zeta_v | v \in V\})$ *is the* u*th cyclotomic polynomial* $\Phi_u(X)$.

(2) *If* $U = V \cup W$ *is a partition of* U *into nonempty subsets* V *and* W, *and if* $V^* = \{\zeta_v | v \in V\}$, $W^* = \{\zeta_w | w \in W\}$, *then* $\Delta(V^*)$ *and* $\Delta(W^*)$ *are linearly disjoint over* Δ.

Proof. In (1), assume first that $4 \in U$. The minimal polynomial $f(X)$ for ζ_u over $\Delta(\{\zeta_v | v \in V\})$ has coefficients in a finite extension E of Δ in $\Delta(\zeta_v\})$. There exists a finite subset $\{v_j\}_{j=1}^{m}$ of V and a set $\{e_j\}_{j=1}^{h}$ of positive integers such that $E \subseteq K(\zeta_w)$, where K is the compositum of $\{\Delta_{p_j,e_j}\}_{j=1}^{h}$ and $w = v_1 v_2 \cdots v_m$. Since the degree of ζ_u over $\Delta(\{\zeta_v\})$ is the same as its degree over any subfield of $\Delta(\{\zeta_v\})$ containing E, we assume without loss of generality that $\{p_j\}_{j=1}^{h} = \{u_j\}_{j=1}^{n} \cup \{v_j\}_{j=1}^{m}$. Then Lemma 4.2 implies that the degree of $f(X)$, which is the degree of ζ_u over $K(\zeta_w)$, is $\phi(u)$, and hence $f(X) = \Phi(X)$. If $4 \notin U$, let $F = \Delta(\{\zeta_v | v \in V, v \ne 4\})$. By the case just considered, $F(\zeta_{4u}) = F(\zeta_4, \zeta_u)$ has degree $\phi(4u) = 2\phi(u)$ over F and ζ_4 has degree 2 over $F(\zeta_u)$. Therefore ζ_u has degree $\phi(u)$ over $F(\zeta_4) = \Delta(\{\zeta_v | v \in V\})$, and again this implies that $\Phi_u(X)$ is the minimal polynomial for ζ_u over $\Delta(\{\zeta_v\})$.

(2): If V is finite, the assertion in (2) follows from (1). Otherwise, let $V = \{v_j\}_{j=1}^{\infty}$. Since $\Delta(V^*) = \bigcup_{j=1}^{\infty} \Delta(\zeta_{v_1 \cdots v_j})$ and since $\Delta(\zeta_{v_1 \cdots v_j}) \cap \zeta(W^*) = \Delta$ for each j, it follows that $\Delta(V^*) \cap \Delta(W^*) = \Delta$—that is, $\Delta(V^*)$ and $\Delta(W^*)$ are linearly disjoint over Δ.

THEOREM 4.4. *Let* F *be a proper subfield of* Δ *and let* L *be the compositum of the set of Galois* J*-extensions of* F *in* A. *Then* $L = \Delta$.

Proof. Since $F < \Delta$, there exists p such that $\Delta_p \not\subseteq F$. Then $F\Delta_p \subseteq L$, and hence $\Delta_p \subseteq L$ for each p with $\Delta_p \not\subseteq F$. Since the inclusion $\Delta_p \subseteq L$ is clear if $\Delta_p \subseteq F$, we conclude that $\Delta \subseteq L$. Suppose that the inclusion is

proper, and let $E \subseteq A$ be a Galois J-extension of F that is not contained in Δ. Choose $\theta \in E \setminus \Delta$. There exists $k \in Z^+$ such that $\theta \in \Delta(\zeta_u)$, where $u = 4p_1p_2 \cdots p_k$. Let $M = \Delta(\{\zeta_{p_j}\}_{j=k+1}^{\infty})$; part (1) of Corollary 4.3 shows that $B = M(\zeta_u)$ is Galois over F of degree $\phi(u)$. Since $\theta \notin M$ by part (2) of Corollary 4.3, it follows that there exists an M-automorphism σ of B such that $\sigma(\theta) \neq \theta$. Since $F \subseteq M$ and E/F is Galois, σ induces an F-automorphism μ of E such that μ has finite order. Since E/F is a J-extension, it follows that $\mu = 1$, contrary to the fact that $\mu(\theta) \neq \theta$. Consequently, $L \subseteq \Delta$, equality holds, and this completes the proof of Theorem 4.4.

COROLLARY 4.5 (*Iwasawa* [**I**]). *The field Δ_p is the unique Galois J-extension of Q with Galois group W_p.*

Proof. If E is a Galois J-extension of Q with Galois group W_p, then by Theorem 4.4, $E \subseteq \{\theta \in \Delta \mid [Q(\theta):Q]$ is a power of $p\} = \Delta_p$. Hence $\Delta_p = E$ since Δ_p/Q is a J-extension.

We have observed in more than one place that Q(3) fails in general, but Corollary 4.5 shows that this failure occurs on a wide scale. To wit, (4.5) implies that for each prime p, there exists a unique extension E_p of Q such that E_p/Q is cyclic of degree p and E_p can be extended to a Galois J-extension of Q.

By a result of Brumer [**Br**], (4.5) is known to extend to a finite totally real[3] abelian extension F of Q—that is, $F\Delta_p$ is the unique Galois J-extension of F with Galois group W_p. Thus, the phrase "in A" in the statement of Theorem 4.4 is redundant if $[F:Q]$ is finite and F is totally real[4]. On the other hand, if $Q \subseteq F < \Delta$ and if $[F:Q]$ is infinite, then F may admit a Galois J-extension that is not contained in Δ; for example, slight modifications of the proofs of (4.8) and (4.9) show that Δ_2 admits a Galois J-extension E with Galois group W_2 such that $E \not\subseteq \Delta$.[5]

---

[3] Recall that a finite algebraic extension L of Q is *totally real* if each of the conjugate fields of L over Q is real; hence if L/Q is Galois, then L is totally real if and only if L is real.

[4] If F is a finite abelian extension of Q that is not totally real, then it is known that F admits infinitely many W_p-extensions (see [**I**, p. 253], [**Bm**, p. 248]).

[5] In fact, using Remark 4.6, it follows that $E \not\subseteq A$ for any such field E.

REMARK 4.6. The proof of Theorem 4.4 shows that each element $\theta \in A \setminus \Delta$ is moved by an automorphism σ_θ of A of finite order; any such σ_θ induces the identity map on Δ. Using this fact and an argument similar to that of (4.4), it can be shown that if $\Delta \subseteq F \subseteq A$, then F admits no Galois J-extension within A. On the other hand, Corollary 4.11 shows that F admits a Galois J-extension.

REMARK 4.7. For each $n \in Z^+$, there exists a unique extension Σ_n of Q in Δ such that $[\Sigma_n : A] = n$. Specifically, if $n > 1$ and $n = \prod_{j=1}^{k} p_j^{e_j}$ is the prime factorization of n, then Σ_n is the composite of the fields Δ_{p_j, e_j} for $1 \leq j \leq k$. For the sake of future reference, we note that $\Sigma_2 = Q(\sqrt{2})$ and $\Sigma_3 = Q(\zeta_9 + \zeta_9^{-1})$; thus, Σ_3 is the splitting field over Q of the polynomial $X^3 - 3X + 1$.

We turn to a proof of the result that each subfield of A admits a Galois J-extension. The proof is obtained by showing (Corollary 4.10) that Δ admits a Galois J-extension with Galois group W_2.

THEOREM 4.8. *If there exist* $a \in \Delta$, $|a| < 1$, *such that* $\sqrt{(1 - a^2)} \in \Delta$ *and the polynomial* $s_n(X) = X^{2^n} - 2aX^{2^{n-1}} + 1$ *is irreducible over* Δ *for each* $n \in Z^+$, *then* Δ *admits a Galois J-extension with Galois group* W_2.

Proof. Choose elements $\beta, \theta_1, \theta_2, \cdots$ so that $\beta = a + i\sqrt{(1 - a^2)}$ is a root of $s_1(X)$, $\theta_1 = \sqrt{\beta}$, and $\theta_{n+1} = \sqrt{\theta_n}$ for each $n \in Z^+$. We show for each n that (1) $\Delta(\theta_n) = \Delta(\theta_n, i)$ is Galois over Δ with Galois group $Z_2 \times Z_2 n$, (2) $\Delta(\theta_n + \theta_n^{-1})$ is a subfield of $\Delta(\theta_n)$ that is cyclic over Δ of degree 2^n, and (3) $\Delta(\theta_n + \theta_n^{-1}) \subseteq \Delta(\theta_{n+1} + \theta_{n+1}^{-1})$. Once these statements have been proved, it then follows that $\bigcup_{n=1}^{\infty} \Delta(\theta_n + \theta_n^{-1})$ is a Galois J-extension of Δ with Galois group W_2. The assertion in (3) follows at once from the equality $\theta_n + \theta_n^{-1} = (\theta_{n+1} + \theta_{n+1}^{-1})^2 - 2$. To prove (1), fix n and write θ in place of θ_n. Since θ is a root of $s_{n+1}(X)$, it has degree 2^{n+1} over Δ. Also, $\theta^{2^n} = \beta$, so $\Delta(\beta) = \Delta(i)$ is a subfield of $\Delta(\beta)$, and $\Delta(i)$ contains the 2^kth roots of unity for all k; in particular, $\Delta(i)$ contains a primitive 2^nth root of unity ζ. Since θ has degree 2^n over $\Delta(i)$, it follows that $\{\zeta^j \theta\}_{j=0}^{2^n - 1}$ is the set of conjugates of θ over $\Delta(i)$; hence $\{\zeta^j \theta\}_{j=0}^{2^n - 1}$ is a set of 2^n conjugates of θ over Δ. What are the other 2^n conjugates of θ over Δ? To answer this question, let G be the galois group of $s_{n+1}(X)$ over Δ. There exists $\sigma \in G$ such that $\sigma(i) = -i$; thus $\sigma(\theta^{2^n}) = \sigma(\beta) = \bar{\beta} = [\sigma(\theta)]^{2^n} = \beta^{-1}$. Since $(\theta^{-1})^{2^n} = \beta^{-1}$, it follows that the other 2^n conjugates of θ over Δ are the elements $\zeta^j \theta^{-1}$ for $0 \leq j \leq 2^n - 1$. Consequently, $\Delta(\theta)/\Delta$ is normal. An element of G is completely determined by its

action on θ; hence $G = \{\sigma_i\}_{j=0}^{2^n-1} \cup \{\tau_j\}_{j=0}^{2^n-1}$, where $\sigma_j(\theta) = \zeta^j\theta$ and $\tau_j(\theta) = \zeta^j\theta^{-1}$. We note $\sigma_j(\theta^{2^n}) = (\zeta^j\theta)^{2^n} = \theta^{2^n}$ and $\tau_j(\theta^{2^n}) = \theta^{-2^n}$, the complex conjugate of θ^{2^n}. It follows that σ_j induces the identity automorphism on $\Delta(i)$ and τ_j restricted to $\Delta(i)$ is complex conjugation on this field. In particular, $\sigma_j(\zeta^j) = \zeta^j$ and $\tau_j(\zeta^j) = \zeta^{-j}$. It is then straightforward to show that $\sigma_j^k(\theta) = \zeta^{jk}\theta$ and $\tau_j^{2^k}(\theta) = \zeta^{-j2^k}\theta$ for each $k \in Z^+$ and that G is abelian. In particular, the order of σ_j and of τ_j is the additive order of j in the group $Z/2^nZ$. Hence G contains elements, such as σ_1, of order 2^n, but no element of order 2^{n+1}. It follows that $G \simeq Z_{2^n} \times Z_2$, and in fact, $G = \langle\sigma_1\rangle \times \langle\tau_0\rangle$, where $\tau_0(\theta) = \theta^{-1}$. This completes the proof of (1).

Continuing the notation of the preceding paragraph, it is clear that $\Delta(\theta + \theta^{-1})$ is contained in the fixed subfield of τ_0. On the other hand, θ satisfies the quadratic polynomial $X^2 - (\theta + \theta^{-1})X + 1$ over $\Delta(\theta + \theta^{-1})$, and consequently, $\Delta(\theta + \theta^{-1})$ is the fixed field of τ_0. Therefore, the Galois group of $\Delta(\theta + \theta^{-1})$ over Δ is isomorphic to $(\langle\sigma_1\rangle \times \langle\tau_0\rangle)/\langle\tau_0\rangle \simeq \langle\sigma_1\rangle$—that is, $\Delta(\theta + \theta^{-1})$ is cyclic over Δ of degree 2^n, as asserted in (2).

PROPOSITION 4.9. *The polynomial* $t_n(X) = X^{2^n} - (8/5)X^{2^{n-1}} + 1$ *is irreducible over the field* Δ *for each* $n \in Z^+$.

Proof. The roots of $t_1(X)$ are $\beta = (4 + 3i)/5$ and $\bar{\beta} = \beta^{-1} = (4 - 3i)/5$. Since Δ is a real field, it follows that t_1 is irreducible over Δ. Let $\theta = \sqrt{\beta}$. The roots of $t_2(X)$ are $\pm\theta$ and $\pm\theta^{-1}$. We show that $\theta \notin \Delta(\beta) = \Delta(i)$; this will imply that $\Delta(\theta) = \Delta(i, \theta)$ has degree 4 over Δ, and hence $t_2(X)$ is irreducible over Δ. Note that $(\theta + \theta^{-1})^2 = \theta^2 + 2 + \theta^{-2} = 18/5$, and hence $\sqrt{10} \in Q(\theta)$. Thus, if $\theta \in \Delta(i)$, then $\sqrt{10} \in \Delta(i)$. On the other hand, Remark 4.7 shows that $Q(\sqrt{2})$ is the unique quadratic extension of Q in Δ, so $\sqrt{10} \notin \Delta$ and $\Delta(i) = \Delta(\sqrt{10})$; this is a contradiction, for $\Delta(\sqrt{10})$ is a real field.

We have shown that $X^2 - \beta$ is irreducible over $\Delta(i)$, and hence the Vahlen-Capelli Theorem implies that $X^{2^n} - \beta$ is irreducible over $\Delta(i)$ for each $n \in Z^+$. If θ_n is a root of $X^{2^n} - \beta$, then θ_n is also a root of $t_{n+1}(X)$ and $\Delta(\theta_n) = \Delta(i, \theta_n)$ has degree 2^{n+1} over Δ, so t_{n+1} is irreducible, as asserted.

COROLLARY 4.10. *The field* Δ *admits a Galois J-extension with Galois group* W_2.

Proof. Apply (4.8) and (4.9).

COROLLARY 4.11. *Each subfield F of the abelian closure A of Q admits a Galois J-extension.*

Proof. If there exists a Galois J-extension K of Q such that $K \not\subseteq F$, then FK is a Galois J-extension of F. On the other hand, if F contains each Galois J-extension of Q, then $\Delta \subseteq F$. Let M be a Galois J-extension of Δ; existence is assured by Corollary 4.10. Remark 4.6 shows that $M \not\subseteq A$, and hence $M \not\subseteq F$ as well. Consequently, FM is a Galois J-extension of F.

Using the description of the unique cubic extension Σ_3 of Q in Δ given in Remark 4.7, we have been able to prove that W_3 can be realized as a Galois group over Δ. In fact, if β is a root of $X^2 - (2/7)X + 1$ and if $\theta_1, \theta_2, \cdots$ are chosen so that $\theta_1 = \beta^{1/3}$, $\theta_2 = \theta_1^{1/3}, \ldots$, then $\bigcup_{j=1}^{\infty} \Delta(\theta_j + \theta_j^{-1})$ is a Galois J-extension of Δ with Galois group W_3. For $p > 3$, we have been unable to determine whether W_p can be realized as a Galois group over Δ. Since Δ does admit a cyclic extension of degree p^k for each $k \in Z^+$, the field $F = \Delta$ represents a test case for Q(2). (See the appendix.)

5. Concluding remarks. We conclude with some remarks concerning two open questions that seem to merit further study. The first question concerns the restriction to the case of algebraic extensions in the definition of a J-extension. To wit, if K is an extension field of F, then consider the following condition (#): K is not finitely generated as an extension field of F, but E/F is finitely generated over F for each proper intermediate field E. If K/F is algebraic, then a (#)-extension is the same as a J-extension, but we have been unable to determine whether a (#)-extension is necessarily algebraic. If not, then it is straightforward to show that there exists a (#)-extension K/F of transcendence degree 1 with F algebraically closed in K. In this case, $K/F(t)$ is a J-extension for each $t \in K \setminus F$; moreover, either (1) $[K:F(t)]_s = \infty$ for each $t \in K \setminus F$, or else (2) $[K:F(t)]_i = \infty$ for each $t \in K \setminus F$. Condition (1) leads to a contradiction in the case where $\operatorname{char} F \neq 0$, and (2) is impossible in the case where $\operatorname{char} F \neq 0$ and F is perfect. Beyond this, we have no additional information.

The second question is the case of Q(2), §3, where p is odd: if F admits a cyclic extension of degree p, can W_p be realized as a Galois group over F? While the corresponding case of Q(3) has a negative answer, the answer to Q(2) has been shown to be "yes" except possibly in the case where $\operatorname{char} F = 0$, $\zeta_p \notin F$, and $F(\zeta_p)$ contains the p^nth roots of

unity for all n. As mentioned in §4, the case where $F = \Delta$ and $p > 3$ is one test case for this second question. (See the appendix.)

References

[A1] A. A. Albert, *Cyclic fields of degree p^n over F of characteristic p*, Bull. Amer. Math. Soc., **40** (1934), 625–631.

[A2] _____, *On cyclic fields*, Trans. Amer. Math. Soc., **37** (1935), 454–462.

[A3] _____, *Modern Higher Algebra*, Univ. Chicago Press, Chicago, 1937.

[AS] E. Artin and O. Schreier, *Eine Kennzeichnung der reell abgeschlossenen Körper*, Abh. Math. Sem. Univ. Hamburg, **5** (1927), 225–231.

[Ba] J. Bastida, *Field Extensions and Galois Theory*, Encyclopedia of Mathematics and Its Applications, Vol. 22, Addison-Wesley, Reading, Mass., 1984.

[BKKN] R. Berger, R. Kiehl, E. Kunz, and J.-J. Nastold, *Differentialrechnung In Der Analytischen Geometrie*, Lecture Notes Math., Vol. 38, Springer-Verlag, New York, 1967.

[Bm] J. Bloom, *On the invariants of some Z_p-extensions*, J. Number Theory, **11** (1979), 239–256.

[Bol] N. Bourbaki, *Algèbre*, Chap. V, Corps Commutative, Masson, Paris, 1981.

[Bo2] _____, *Commutative Algebra*, Addison-Wesley, Reading, Mass., 1972.

[Br] A. Brumer, *On the units of algebraic number fields*, Mathematika, **14** (1967), 121–124.

[C] R. Chalkley, *A lattice of cyclotomic fields*, Math. Mag., **48** (1975), 42–44.

[CK] C. C. Chang and H. J. Keisler, *Model Theory*, North-Holland, Amsterdam, 1973.

[Co] Ian Connell, *Modern Algebra A Constructive Introduction*, North Holland, New York, 1982.

[D] R. A. Dean, *A rational polynomial whose group is the quaternions*, Amer. Math. Monthly, **88** (1981), 42–45.

[De] J. K. Deveney, *ω_0-generated field extensions*, Arch. d. Math., (to appear).

[DD] J. Diller and A. Dress, *Zur Galoistheorie Pythagoriescher Körper*, Arch. Math., **16** (1965), 148–152.

[FM] M. D. Fried and R. E. MacRae, *On the invariance of chains of fields*, Illinois J. Math., **13** (1969), 165–171.

[F] L. Fuchs, *Infinite Abelian Groups*, Vol. I, Academic Press, New York, 1970.

[G1] R. Gilmer, *On solvability by radicals of field extensions*, Math. Ann., **199** (1972), 263–277.

[G2] _____, *Multiplicative Ideal Theory*, Marcel Dekker, New York, 1972.

[GH] R. Gilmer and W. Heinzer, *Cardinality of generating sets for modules over a commutative ring*, Math. Scand., **52** (1983), 41–57.

[H] T. W. Hungerford, *Algebra*, Springer-Verlag, New York, 1980.

[I] K. Iwasawa, *On Z_l-extensions of algebraic number fields*, Ann. Math., **98** (1973), 246–326.

[J] N. Jacobson, *Lectures in Abstract Algebra, Vol. III, Theory of Fields and Galois Theory*, Springer-Verlag, New York, 1981.

[K] I. Kaplansky, *Fields and Rings*, Univ. Chicago Press, Chicago, 1972.

[L] S. Lang, *Algebra*, Addison-Wesley, Reading, Mass., 1984.

[N] M. Nagata, *Local Rings*, Wiley Interscience, New York, 1962.

[Q] F. Quigley, *Maximal subfields of an algebraically closed field not containing a given element*, Proc. Amer. Math. Soc., **13** (1962), 562–566.

[Re] J. D. Reid, *A note on inseparability*, Michigan Math. J., **13** (1966), 219–223.

[Ri] P. Ribenboim, *Theorie des Valuations*, Presses Univ. Montreal, Montreal, 1965.
[Ro] J. J . Rotman, *An Introduction to the Theory of Groups*, Third edition, Allyn and Bacon, Boston, 1984.
[S] F. K. Schmidt, *Mehrfach perfekte Korper*, Math. Ann., **108** (1933), 1–25.
[vdW] B. L. van der Waerden, *Algebra*, Vol. I, Frederick Ungar, New York, 1970.
[Wa] L. C. Washington, *Introduction to Cyclotomic Fields*, Springer-Verlag, New York, 1982.
[Wn] M. Weinstein, *Examples of Groups*, Polygonal Publ. House, Passaic, N. J., 1977.
[ZS] O. Zariski and P. Samuel, *Commutative Algebra*, Vol. I, Springer-Verlag, New York, 1975.

APPENDIX

Professor David Saltman has informed us that Q(2) has an affirmative answer for an odd prime p, and this may have been previously known. The argument given below was supplied by him.

THEOREM 1. *If p is an odd prime and if the field F admits a cyclic extension of degree p, then W_p can be realized as a Galois group over F. If F admits a cyclic extension of degree 4, then W_2 can be realized as a Galois group over F.*

Proof. We begin with some general comments. Let F be any field and $G = \mathrm{Gal}(F_s/F)$ the Galois group of F in its separable closure. The character group $\chi(F)$ is the group of continuous homomorphisms $\mathrm{Hom}_c(G, Q/Z)$, where Q is the additive group of the rational field and Z is the subgroup of integers. Q/Z is supplied with the discrete topology. Let $\chi(F)_p$ be the p-primary component of $\chi(F)$. If $f\colon G \to Q/Z$ is in $\chi(F)$, then $f^{-1}(0)$ is open in G, so $f(G)$ is finite. But all finite subgroups of Q/Z are cyclic, so if N is the kernel of f, then G/N is cyclic of order, say, n and $f^{-1}(1/n + Z) = \sigma N$ is a generator of G/N. Conversely, if L/F is cyclic of degree n and τ generates $\mathrm{Gal}(L/F)$, set $N = \mathrm{Gal}(F_s/L) \subseteq G$. N is normal in G and $G/N = \mathrm{Gal}(L/F)$. Viewed in G/N, let τ be σN. Define $f\colon G \to Q/Z$ by setting $f(\eta) = i/n + Z$ if $n \in \sigma^i N$. It is not hard to see that $f \in \chi(F)$. In this way we have a bijection between $\chi(F)$ and pairs $(L/F, \tau)$, where L/F is cyclic and τ generates the Galois group of L/F. Note that in this correspondence, the order of f is the degree of L/F. The following lemma is the way we find W_p-extensions.

LEMMA. *Suppose $f \in \chi(F)_p$ and that f corresponds to $(L/F, \tau)$. Then L/F embeds in a W_p-extension if and only if f is in a divisible subgroup of $\chi(F)_p$.*

Proof. Let f, $(L/F, \tau)$ be as above, assume that L/F has degree q, and let $L \subseteq L'$, where L'/F is a W_p-extension. Then there are $L \subseteq L_1 \subseteq L_2 \subseteq \cdots \subseteq L'$ such that L_i/F is cyclic of degree qp^i. Furthermore, $\mathrm{Gal}(L'/F)$ has a topological generator σ such that σ restricted to L is τ. Let τ_i be the restriction of σ to L_i. Define $f_i \in \chi(F)_p$ to be the character associated with $(L_i/F, \tau_i)$. Then $p^i f_i = f$ for all i. That is, f is in a divisible subgroup of $\chi(F)_p$.

Conversely, suppose f is in a divisible subgroup of $\chi(F)_p$. Then there are $f_i \in \chi(F)_p$ such that $pf_{i+1} = f_i$ and $pf_1 = f$. Then $L_i \subseteq L_{i+1}$ and the restriction of τ_{i+1} to L_i is τ_i. If $L' = \bigcup L_i$, then L'/F is a W_p-extension. This proves the lemma.

We require one more general observation. Let F, G be as above and let $H \subseteq G$ be a subgroup of finite index m. The restriction map defines a homomorphism res: $\mathrm{Hom}_c(G, Q/Z) \to \mathrm{Hom}_c(H, Q/Z)$. On the other hand, if we supply Q/Z with the trivial G- (and hence H-) action, then $\mathrm{Hom}_c(G, Q/Z) = H^1(G, Q/Z)$ and $\mathrm{Hom}_c(H, Q/Z) = H^1(H, Q/Z)$. By, for example, [**1**, p. 82], there is a corestriction homomorphism cor: $\mathrm{Hom}_c(H, Q/Z) \to \mathrm{Hom}_c(G, Q/Z)$ such that the composition (cor)(res) is just multiplication by m. Using our definition of $\chi(F)_p$, we have that if F'/F is a separable extension of degree m, there are maps res: $\chi(F)_p \to \chi(F')_p$ and cor: $\chi(F')_p \to \chi(F)_p$ such that (cor)(res) is multiplication by m.

We are ready to prove the theorem. Let F be a field with a cyclic extension L/F of degree q, where $q = p$ if p is odd and $q = 4$ if p is 2. Assume that F has no W_p-extension. Let F' be the field gotten by adjoining all p^s roots of one to F. Let the group U be cyclic of order $p - 1$ if p is odd and cyclic of order 2 if p is 2. Then $\mathrm{Gal}(F'/F) \cong U' \oplus B$, where U' is a subgroup of U and B is either (0) or W_p. Since F has no W_p-extensions, $B = (0)$ and F'/F is finite of degree m, where m is prime to p if p is odd and m is 1 or 2 if p is 2. Over F', all cyclic p-extensions can be embedded in a W_p-extension. Thus $\chi(F')_p$ is divisible. If p is odd, the fact that (cor)(res) is multiplication by an integer prime to p implies that cor: $\chi(F')_p \to \chi(F)_p$ is a surjection. Thus $\chi(F)_p$ is nonzero and divisible. This case is done by the lemma. If p is 2, let f be a character associated with L/F, so $f \in \chi(F)_2$ has order 4. $\mathrm{cor}(\mathrm{res}(f)) = f$ or $2f$, which are not 0. Thus $\mathrm{cor}(\chi(F')_p)$ is a nonzero divisible subgroup of $\chi(F)_p$, and we are again done by the lemma. Thus the theorem is proved.

[1] K. Brown, *Cohomology of Groups*, Springer-Verlag, New York and Berlin 1982.

Professor David Leep has pointed out to us some consequences of Theorem 1 that we proceed to record below as Theorems 2–4. The first of these theorems is related to Q(3).

THEOREM 2. *Assume that F is a field and p is prime. Denote by ζ either a primitive pth root of unity, for p odd, or a primitive fourth root of unity if $p = 2$. Assume that $F(\zeta)$ contains the p^nth roots of unity for all n, and that K/F is cyclic of degree p.*

(1) *If p is odd, K can be extended to a Galois J-extension of F.*

(2) *If $p = 2$, then K can be extended to a Galois J-extension of F if and only if K can be extended to a cyclic extension of F of degree* 4.

Proof. The proof of Theorem 2 is essentially contained in the proof of Theorem 1. In fact, in the notation of that proof, the hypothesis on F implies that for p even or odd, $B = (0)$. Thus, (1) follows exactly as in the proof of Theorem 1. In (2), let L be an extension field of K such that L/F is cyclic of degree 4. If f is the character associated with L/F, the proof of Theorem 1 shows that either f or $2f$ belongs to a nonzero divisible subgroup of $\chi(F)_p$, and hence in either case $2f$, the character associated with K/F, belongs to such a subgroup. Hence K/F can be extended to a Galois J-extension of F, as asserted.

Another consequence of Theorem 1 is a result which represents a stronger form of Corollary 4.11.

THEOREM 3. *Each subfield F of A admits a Galois J-extension with Galois group W_p for each prime p.*

Proof. Fix p. If $\Delta_p \not\subseteq F$, then $F\Delta_p/F$ is Galois with Galois group W_p. Suppose $\Delta_p \subseteq F$. We show first that Δ_p admits a Galois J-extension with Galois group W_p. To do so, it suffices, in view of Theorem 1, to show that Δ_p admits a cyclic extension of degree p^2. This is easy to do: take q prime such that $q \equiv 1 \pmod{p^2}$. If ζ is a primitive qth root of unity then $\Delta_p(\zeta)/\Delta_p$ is cyclic of degree $q - 1$, and hence $\Delta_p(\zeta)$ contains a subfield that is cyclic over Δ_p of degree p^2. Let K be a W_p-extension of Δ_p. Since no element of Δ has degree p over Δ_p, it follows that $K \not\subseteq \Delta$, and hence $\Delta K/\Delta$ is also a J-extension. Remark 4.6 then implies that $\Delta K \not\subseteq A$; hence $K \not\subseteq A$, $K \not\subseteq F$, and FK is a Galois J-extension of F with Galois group W_p.

In relation to part (2) of Theorem 2, we remark that even under the hypothesis of that result, it need not be true that a cyclic extension K/F

of degree 4 can be extended to a Galois J-extension of F. (Thus, if $2f$ belongs to a divisible subgroup of $\chi(F)_2$, f itself need not belong to such a subgroup.) For example, let $F = \Delta_2$. If ζ is a primitive fourth root of unity over F, then $F(\zeta)$ contains the 2^nth roots of unity for all n. If μ is a primitive fifth root of unity over F, then $F(\mu)/F$ is cyclic of degree 4, and the unique subfield $E = F(\cos 2\pi/5)$ of $F(\mu)$ of degree 2 over F is a subfield of the reals. We claim that $F(\mu)$ cannot be extended to a field L that is cyclic over F of degree 8. We argue by contradiction. If such a field L exists, then L/E is cyclic of degree 4, and it is known that this implies that the intermediate field $F(\mu)$ is of the form $E(\sqrt{t})$, where t is the sum of two squares in E [**A3**, Exer. 1, p. 208], [**D**, p. 43]. This leads to the contradiction that $F(\mu) = E(\sqrt{t})$ is a subfield of the reals. Therefore K/F cannot be extended to a J-extension of F.

Finally, Leep pointed out to us that the following known result (*) which is included in Satz 1 of [**DD**], yields a more complete answer to Q(2) than that provided by Theorem 1. In the statement of (*), recall that a field F is *Pythagorean* if each sum of squares in F is a square in F.

(*) *If F admits a cyclic extension of degree* 2, *then F fails to admit a cyclic extension of degree* 4 *if and only if* $\operatorname{char} F \neq 2$ *and F is Pythagorean.*

Theorem 4 follows immediately from Theorem 1 and (*).

THEOREM 4. *If F admits a cyclic extension of degree* 2, *then W_2 can be realized as a Galois group over F unless* $\operatorname{char} F \neq 2$ *and F is Pythagorean.*

Added in proof. We have learned that Theorem 2 of the paper *Algebraic extensions of arbitrary fields*, Duke Math. J., **24** (1957), 201–204, by G. Whaples, also answers (Q1) and (Q2). Propositions 3.2, 3.4 and 3.5, as well as Theorems 1, 2 and 4 of the Appendix, can be obtained from Whaples' theorem and its proof.

Received November 6, 1985. The first author's research was supported by NSF Grant DMS-8501003 and the second author by NSF Grant DMS-8320558.

FLORIDA STATE UNIVERSITY
TALLAHASSEE, FL 32306

AND

PURDUE UNIVERSITY
W. LAFAYETTE, IN 47907