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REALIZING CERTAIN POLYNOMIAL ALGEBRAS AS COHOMOLOGY RINGS OF SPACES OF FINITE TYPE FIBERED OVER $\times BU(d)$

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The problem of constructing topological spaces whose cohomology ring with coefficients in the field of p elements is a polynomial algebra has attracted the attention of algebraic topologists for many decades. Apart from the naturally occurring examples, classifying spaces of Lie groups away from their torsion primes, rather little progress was made until the construction of Clark and Ewing of a vast number of new non-modular examples. The completeness of their construction in the non-modular case was shown by Adams and Wilkerson (see Smith and Switzer for a compact-proof). One interest in the construction of spaces with polynomial cohomology is that they are related to the study of finite H-spaces, which should appear as their loop spaces; "should" because the construction of Clark and Ewing does not yield a simply connected CW complex of finite type. On the contrary the construction of Clark and Ewing yields non-simply connected spaces that are p-adically complete. By forming their finite completion they can be made simply connected. But considerably more effort would be required to show that they have the homotopy type of the p-completion of a simply connected CW complex of finite type.

We will avoid these drawbacks by constructing for certain of the examples of Clark and Ewing a simply connected space of finite type with the requisite cohomology.

Recall that the construction of [**6**] depends on a group $G < \mathrm{GL}(V)$, $V = \oplus_n \mathbf{F}_p$, where $\mathbf{F}_p$ is the field of p-elements, and which satisfies: G is generated by pseudo reflections and $|G| \not\equiv O(p)$. A theorem of Chevalley [**2**; V §5 no. 5, 3, Thm. 3] [**24**] shows that:

$$P(V^*)^G \simeq P[\rho_1, \dots, \rho_n]$$

where $P(V^*)$ denotes the ring of polynomials on the dual vector space V^* of V upon which G acts, and $P(V^*)^G \leq P(V^*)$ is the ring of invariant polynomials. We will say that $P(V^*)^G$ satisfies the *weak splitting principle* iff we can find polynomial generators $\rho_1, \dots, \rho_n \in P(V^*)^G$, and polynomials $f_1(X), \dots, f_b(X) \in P(V^*)^G[X]$, where X is an indeterminate of degree 2, such that:

(1) $\rho_1, \dots, \rho_n$ are among the coefficients of $f_1(X), \dots, f_b(X)$, and

(2) $f_i(X)$ splits into a product of linear factors in $P(V^*)[X]$ for $i = 1, \dots, b$. If we can choose $b = 1$ then we say $P(V^*)^G$ satisfies the *splitting principle*. (N.B. The polynomials $f_i(X)$ are homogeneous and their roots lie in V^*.)

THEOREM. *Suppose $G < \mathrm{GL}(V)$ is generated by pseudo reflections. $|G| \not\equiv O(p)$, and $P(V^*)^G$ satisfies the weak splitting principle. Then there exist integers $d_1, \dots, d_a$ depending on $G < \mathrm{GL}(V)$ and a convergent tower of fibrations*

$$\cdots \to X_n \to X_{n-1} \to \cdots \to X_1 \to \mathop{\times}_{i=1}^{a} BU(d_i)$$

such that: if W is the space at the top of the tower then W is simply connected, of the homotopy type of a CW complex of finite type, and further

$$H^*(W, \mathbf{F}_p) \simeq P(V^*)^G$$

and the induced map

$$H^*\left(\mathop{\times}_{i=1}^{a} BU(d_i); \mathbf{F}_p \right) \to P(V^*)^G$$

is surjective.

There are two aspects to the construction of the tower to which we wish to draw attention. The first is the simple idea of "trapping" $P(V^*)^G$, which we wish to realize as a cohomology ring, as the image of an induced map inside $H^*(BV; \mathbf{F}_p)$; where BV is the classifying space of the elementary abelian p group V. (See Lemma 5 and the discussion around it.) This allows us to solve the sort of extension problems that occur in **[8]**, **[9]**, **[14]**, **[15]**, etc.

The second aspect of the construction worthwhile remarking is the use of the Carlsson **[4]**, Miller **[10]** result that $H^*(BV; \mathbf{F}_p)$ is an injective object in the category of unstable modules over the Steenrod algebra (see also **[23]**) to continue trapping $P(V^*)^G < H^*(BV; \mathbf{F}_p)$ as the image of an induced map as we move up the tower.

The method employed should be of use in other situations, and is perhaps more important than the actual results obtained here. A number of further applications will be appearing in a joint paper with Paul Goerss.

We postpone a discussion of examples of groups satisfying the splitting principles to the end of this paper and **[17]**, where, among other things, we show that, when p is large compared to $|G|$, $P(V^*)^G$ satisfies weak splitting.

The remainder of this paper is organized as follows. After some preliminary maneuvers we take up the construction of the tower required for the proof of the main theorem. We then complete the proof of the main theorem, deferring to the end the proofs of the technical results, so as to not interrupt the flow of the argument.

The tower construction. We denote by BV the classifying space of the abelian p-group $V = \oplus_n \mathbf{F}_p$, and identify

$$H^2(BV, \mathbf{Z}) \simeq V^* \simeq \mathrm{Ext}_{\mathbf{Z}}(V, \mathbf{Z}).$$

We may then write

$$H^*(BV; \mathbf{F}_p) \simeq E(\beta^{-1}V^*) \otimes P(V^*)$$

where β is the mod p Bockstein operator and $E(\ )$ the exterior algebra functor. If $G < \mathrm{GL}(V)$ then there is an induced action of G on BV (which we may assume to be free) and hence on $H^*(BV; \mathbf{F}_p)$. In this way we obtain an inclusion

$$P(V^*)^G \hookrightarrow H^*(BV; \mathbf{F}_p)^G$$

and an isomorphism (see **[1]** for more information)

$$P(V^*)^G \simeq \frac{H^*(BV; \mathbf{F}_p)^G}{\sqrt{O}},$$

where $\sqrt{O}$ denotes the ideal of nilpotents.

PROPOSITION 1. *Suppose $G < \mathrm{GL}(V)$, $V = \oplus_n \mathbf{F}_p$, and $P(V^*)^G$ satisfies the weak splitting principle. Then there exist G vector bundles $\xi_i \downarrow BV$, $i = 1, \dots, a$, such that*

$$c_i(\xi_j) \in P(V^*)^G; \qquad i = 1, \dots, \dim \xi_j;\ j = 1, \dots, a$$

and moreover $\{c_i(\xi_j)\}$ generate $P(V^*)^G$ as an algebra.

Proof. By hypothesis there are polynomials

$$f_i(X)S \in P(V^*)^G[X]$$

such that:

$$f_i(X) = \prod_{v \in A_i} (X + v) \in P(V^*)[X]; \qquad A_i \subset V$$

and $P(V^*)^G$ is generated as an algebra by the coefficients of $f_1(X), \dots, f_a(X)$. Since $f_1(X), \dots, f_a(X) \in P(V^*)^G[X]$, it follows that $A_1, \dots, A_a$ are G invariant subsets of $V^* \simeq H^2(BV; \mathbf{Z})$. Corresponding to each element $v \in A_j$ we may find a (unique) line bundle $\lambda(v) \downarrow BV$ with $c_1(\lambda(v)) = v$. The vector bundle $\xi_i = \oplus_{v \in A_i} \lambda(v) \downarrow BV$ is then a G bundle and satisfies

$$c(\xi_i) = f(1) \in P(V^*)^G,$$

where $c(\ )$ denotes the total Chern class as required. □

COROLLARY 2. *Suppose $G < \mathrm{GL}(V)$, $V = \oplus_n \mathbf{F}_p$ and $P(V^*)^G$ satisfies the weak splitting principle. Then there is an epimorphism of algebras over the Steenrod algebra*

$$\bigotimes_{i=1}^{a} H^*(BU(d_i); \mathbf{F}_p) \to P(V^*)^G. \qquad \square$$

The preceding corollary suggests a rather obvious way to attempt to construct a space W with $H^*(W; \mathbf{F}_p) \simeq P(V^*)^G$. Namely, we let

$$f\colon \bigtimes_{i=1}^{a} BU(d_i) \to K = \bigtimes_{i=1}^{c} K(\mathbf{H}Z, m_j)$$

be a map such that the classes

$$f^*(i_{m_j}) \in H^*\left(\bigtimes_{i=1}^{a} BU(d_i); \mathbf{F}_p \right): \qquad j = 1, \ldots, c$$

are a minimal set of ideal generators for the kernel of an epimorphism

$$\bigotimes_{i=1}^{a} H^*(BU(d_i); \mathbf{F}_p) \to P(V^*)^G.$$

Introduce the fibre square

$$(\mathscr{F}) \qquad \begin{array}{ccc} X & \to & L \\ \downarrow \tau & & \downarrow \\ \bigtimes_{i=1}^{a} BU(D_i) & \xrightarrow{f} & K \end{array}$$

where $L \downarrow K$ is the usual pathspace fibration. It is of course too much to hope that

$$H^*(X; \mathbf{F}_p) \simeq P(V^*)^G \simeq H^*\left(\bigtimes_{i=1}^{a} BU(d_i); \mathbf{F}_p \right) // f^*.$$

It is however not too much to hope that $H^*(X; \mathbf{F}_p)$ is a good approximation to $P(V^*)^G$, and in fact this is the case as we now proceed to show.

LEMMA 3. *In the preceding diagram set $S^* = \operatorname{Im} f^*$. Then S^* is a polynomial algebra and $H^*(\times_{i=1}^{s} BU(d_i); \mathbf{F}_p)$ is a free S^*-module.*

LEMMA 4. *$\operatorname{Ker} f^* < H^*(K; \mathbf{F}_p)$ is a Borel ideal.*

We may now apply [**14**; II.2] to compute $H^*(X; \mathbf{F}_p)$. We obtain an isomorphism of algebras over the Steenrod algebra $\mathscr{A}^*$

$$H^*(X; \mathbf{F}_p) \simeq \bigcup_{R^*} (F^{-1}),$$

a coexact sequence of $\mathscr{A}^*$ algebras (for a discussion of coexactness see [**14**, II] or [**15**])

$$1 \to R^* \to H^*(X; \mathbf{F}_p) \to \mathsf{U}(M) \to 1,$$

and an exact sequence of $\mathscr{A}^*$ modules

$$1 \to R^* \to F^{-1} \to E_\infty^{-1,*} \to 0$$

where:

(a) $R^* =: H^*(\times_{i=1}^a BU(d_i); hF_p)//f^* \simeq P(V^*)^G$;

(b) $F^{-1} < H^*(X; hF_p)$ is the submodule of elements of filtration -1 with respect to the Eilenberg-Moore spectral sequence;

(c) $M = \mathbf{F}_p \otimes_{R^*} E_\infty^{-1,*}$;

(d) $\mathsf{U}(M) \simeq \operatorname{Im}\{i^*\colon H^*(X; \mathbf{F}_p) \to H^*(\Omega K; \mathbf{F}_p)\}$.

(For a discussion of the functors $\mathsf{U}(\ )$ and $\mathsf{U}_{R^*}(\ )$ see [**8**] and [**9**].) All this is standard. What we would now like to do is to choose a submodule $N < H^*(X; \mathbf{F}_p)$ mapping isomorphically onto M; represent the elements of N by a map

$$X \xrightarrow{f'} \mathop{\times}_{i=1}^{\infty} K(N^i, i) = K',$$

form the pullback diagram

$$\begin{array}{ccc} X' & \to & L' \\ \downarrow \tau' & & \downarrow \\ X & \xrightarrow{f'} & K' \end{array}$$

etc. The problem is that we do not a priori know that N can be chosen to be an $\mathscr{A}^*$ submodule, and so we cannot assure that τ'^* maps $P(V^*)^G \simeq R^* < H^*(X; \mathbf{F}_p)$ monomorphically. To deal with this problem let us decorate the fibre square $(\mathscr{F})$ to a diagram

$$(\vartheta) \qquad \begin{tikzcd} & X \arrow[r] \arrow[d, "\tau"] & L \arrow[d] \\ BV \arrow[ur, dashed, "\sigma"] \arrow[r, "g"] & \mathop{\times}_{i=1}^{a} BU(d_i) \arrow[r, "f"] & K \end{tikzcd}$$

where

$$g_j\colon BV \xrightarrow{g} \mathop{\times}_{i=1}^{a} BU(d_i) \xrightarrow{\mathrm{pr}_j} BU(d_j)$$

classifies the bundle $\xi_j \downarrow BV$ of Proposition 1. By construction $f \circ g$ is null homotopic, so there is a lift σ as indicated.

LEMMA 5. *There is a choice of σ such that*

$$\operatorname{Im}\sigma^* \simeq P(V^*)^G.$$

We obtain therefore a diagram

$$\begin{tikzcd}
 & H^*(X,\mathbf{F}_p) \arrow[dl, "\sigma^*"'] & \\
 & & R^* \simeq P(V^*)^G \arrow[ul, hook] \\
H^*(BV;\mathbf{F}_p) > P(V^*)^G & \bigotimes_{i=1}^{a} H^*(BU(d_i);\mathbf{F}_p) \arrow[l, "g^*"'] \arrow[uu, "\tau^*"'] \arrow[ur] &
\end{tikzcd}$$

the composite

$$R^* \hookrightarrow H^*(X;\mathbf{F}_p) \xrightarrow{\sigma^*} P(V^*)^G$$

being the identity. The map σ^* therefore provides a splitting over $R \odot \mathscr{A}^*$ of both of the sequences

$$0 \to R^* \to F^{-1}H^*(X;\mathbf{F}_p) \to E_\infty^{-1,*} \to 0,$$

$$1 \to R^* \to H^*(X;\mathbf{F}_p) \to \mathsf{U}(M) \to 0.$$

If we set

$$N = \operatorname{Ker}\left\{\sigma^*\colon F^{-1}H^*(X;\mathbf{F}_p) \to P(V^*)^G\right\},$$

we then obtain:

PROPOSITION 6. *With the notations preceding there is an isomorphism*

$$H^*(X;\mathbf{F}_p) \simeq R^* \otimes \mathsf{U}(N)$$

such that

$$\begin{array}{ccc}
H^*(\Omega K;\mathbf{F}_p) & \hookleftarrow & \mathsf{U}(N) \\
\uparrow i^* & & \uparrow \varepsilon \otimes 1 \\
H^*(X;\mathbf{F}_p) & \simeq & R^* \otimes \mathsf{U}(N) \\
\downarrow \sigma^* & & \downarrow 1 \otimes \varepsilon \\
H^*(BV;\mathbf{F}_p) & \hookleftarrow & P(V^*)^G \simeq R^*
\end{array}$$

where ε is the augmentation. The module N is at least 3 connected. If $x \in N$ is a non-zero element of minimal degree, then some higher order Bockstein applied to x is non-zero.

REMARK. For a discussion of higher order Bockstein operators β_r see [3]. If $u \in H^d(X, \mathbf{F}_p)$ and $\beta_r u = 0$, $r < s$, $\beta_s u \neq 0$, then

$$u \in \operatorname{Im}\{H^*(X; \mathbf{Z}/p^s) \to H^*(X, \mathbf{Z}/p)\},$$

but

$$u \notin \operatorname{Im}\{H^*(X; \mathbf{Z}/p^{s+1}) \to H^*(X; \mathbf{Z}/p)\}.$$

This implies that we can represent u by a map

$$\varphi\colon X \to K(\mathbf{Z}/p^s, d)$$

such that $\varphi^*(i) = u$, and hence $\ker \varphi^* = 0$ in degrees $< d + 2$.

PROPOSITION 7. *With the notations preceding we may choose f, such that*

$$H^*(X; \mathbf{Q}) \simeq H^*\left(\mathop{\times}_{i=1}^{a} BU(d_i), \mathbf{Q}\right) // f^*$$

is a polynomial algebra of the same type as $P(V^*)^G$. In fact there is a subalgebra $R^* < H^*(X; \mathbf{Z})$ such that $R^* \otimes_{\mathbf{Z}} \mathbf{Q} \simeq H^*(X; \mathbf{Q})$ and $R^* \otimes_{\mathbf{Z}} \mathbf{Z}/p \simeq R^* \simeq P(V^*)^G$.

Proof of the main theorem. We are going to construct inductively a convergent tower of fibrations and lifts

$$\begin{tikzcd}
 & & {} \arrow[d] \\
 & & X_n \arrow[d, "\tau_n"] \\
 & & \vdots \arrow[d, "\tau_2"] \\
 & & X_1 \arrow[d, "\tau_1"] \\
g\colon BV \arrow[r, "\pi"] & B \arrow[uuur, "\sigma_n"] \arrow[ur, "\sigma_1"] \arrow[r, "\bar{g}"] & \mathop{\times}_{i=1}^{a} BU(d_i)
\end{tikzcd}$$

where $B = BV/G$ and $\bar{g}$ factors g over the orbit map $\pi\colon BV \downarrow B$, and

(a) $\operatorname{Im} g^* = \operatorname{Im} \pi^*\sigma_1^* = \cdots = \operatorname{Im} \pi^*\sigma_n^* = P(V^*)^G$;

(b) $H^*(X_n; \mathbf{F}_p) \simeq P(V^*)^G \otimes \mathrm{U}(N(n))$ as algebra over the Steenrod algebra; and

(c) the diagram

$$\begin{array}{ccc} H^*(X_n; \mathbf{F}_p) & \simeq & P(V^*)^G \otimes \mathrm{U}(N(n)) \\ \downarrow \pi^*\sigma_n^* & & \downarrow 1 \otimes \varepsilon \\ H^*(BV; \mathbf{F}_p) & \hookleftarrow & P(V^*)^G \end{array}$$

commutes.

For $n = 1$ this is just Proposition 6, so the induction starts and we turn to the inductive step. We will first describe the construction of X_{n+1} from X_n. We then locate inside $H^*(X_{n+1}; \mathbf{F}_p)$ a well-placed copy of $P(V^*)^G$ (condition (a)). Next we show how to split this copy of $P(V^*)^G$ off (condition (b)), and finally we adjust the lift σ_{n+1} (condition (c)) so that we may repeat the splitting argument at the next stage.

Construction of X_{n+1}. Let

$$f_{n+1}\colon X_n \to K_{n+1} := \times K\left(N(n)^j; j\right)$$

be defined by requiring that

$$f_{n+1}^* : H^*\left(K_{n+1}; \mathbf{F}_p\right) \to H^*\left(X_n; \mathbf{F}_p\right)$$

map $N(n)$ isomorphically. There is then the fibre square

$$(\mathscr{F}_n) \qquad \begin{array}{ccc} X_{n+1} & \to & L \\ {\scriptstyle \tau_{n+1}}\downarrow & & \downarrow \\ X_n & \xrightarrow{f_{n+1}} & K_{n+1} \end{array}$$

defining X_{n+1} where $L_{n+1} \downarrow K_{n+1}$ is the path space fibration.

Locating $P(V^*)^G$ *inside* $H^*(X_{n+1}; \mathbf{F}_p)$. An Eilenberg-Moore spectral sequence argument (of the type to be found in [**15**; §4] say) applied to $(\mathscr{F}_n)$ yields an isomorphism of algebras over the Steenrod algebra

$$H^*\left(X_{n+1}; \mathbf{F}_p\right) \simeq \bigcup_{R^*} (F^{-1}),$$

a coexact sequence of $\mathscr{A}^*$ algebras

$$1 \to R^* \to H^*\left(X_{n+1}; \mathbf{F}_p\right) \to \mathrm{U}(M(n+1)) \to 1$$

and an exact sequence of $\mathscr{A}^*$ modules

$$1 \to R^* \to F^{-1} \to E_\infty^{-1,*} \to 0,$$

where

$$R^* = H^*\left(X_n; \mathbf{F}_p\right) \simeq P(V^*)^G,$$

and

$$\mathsf{U}(M(n+1)) \simeq \operatorname{Im}\{H^*(X_n;\mathbf{F}_p) \to H^*(\Omega K_{n+1};\mathbf{F}_p)\}.$$

Splitting $H^*(X_{n+1};\mathbf{F}_p)$. By property (c) of the inductive hypothesis we may decorate $(\mathscr{F}_n)$ to a diagram

$$\begin{tikzcd} & & X_{n+1} \arrow[r] \arrow[d, "\tau_{n+1}"] & L_{n+1} \arrow[d] \\ BV \arrow[r, "\pi"] & B \arrow[r, "\sigma_n"] \arrow[ur, dashed, "\sigma_{n+1}"] & X_n \arrow[r, "f_{n+1}"] & K_{n+1} \end{tikzcd}$$

because the composite $f_{n+1} \cdot \sigma_n$ is null homotopic by construction. Since $|G| \not\equiv O(p)$ a transfer argument shows

$$H^*(B;\mathbf{F}_p) \simeq H^*(BV;\mathbf{F}_p)^G.$$

From [1] it follows

$$H^*(BV;\mathbf{F}_p)^G/\sqrt{O} \simeq P(V^*)^G.$$

The composite

$$H^*(X_{n+1};\mathbf{F}_p) \xrightarrow{\sigma_{n+1^*}} H^*(B;\mathbf{F}_p) \to P(V^*)^G$$

splits the above exact sequences to give an isomorphism of $R^* \odot \mathscr{A}^*$ modules

$$F^{-1} \simeq R^* \oplus E_\infty^{-1,*}.$$

Since the functor $\mathsf{U}_{R^*}(\ )$ sends direct sums into tensor products and $E_\infty^{-1,*}$ is a free R^* module, we obtain

$$H^*(X_{n+1};\mathbf{F}_p) \simeq R^* \otimes \mathsf{U}(M(n+1))$$

where

$$M(n+1) = \mathbf{F}_p \otimes_{R^*} E_\infty^{-1,*}.$$

Adjusting the lift σ_{n+1}. It remains to show that we can choose the lift σ_{n+1} so that $\sigma_{n+1}^*(N(n+1)) = 0$. To this end consider the diagram (P denotes primitives)

$$\begin{tikzcd} & & PH^*(K_{n+1};\mathbf{F}_p) \arrow[dll, dashed, "\alpha"'] \\ H^*(B;\mathbf{F}_p) & H^*(X_{n+1};\mathbf{F}_p) \arrow[l, "\sigma_{n+1}^*"] & N(n) \arrow[l, hook'] \arrow[u, "i^*"'] \end{tikzcd}$$

By the dual of a theorem of Carlsson [4] and Miller [10], [23] $H^*(BV; \mathbf{F}_p)$ is an injective object in the category of unstable modules over the Steenrod algebra. By a transfer argument $H^*(B; \mathbf{F}_p)$ is an $\mathscr{A}^*$ direct summand in $H^*(BV; \mathbf{F}_p)$. Therefore $H^*(B; \mathbf{F}_p)$ is also an injective in the category of unstable modules over the Steenrod algebra, and therefore we obtain a dotted map α. Cutting α down to fundamental classes defines a map φ: $B \to K_{n+1}$ such that

$$\sigma_{n+1*} = \varphi^* i^*: N(n) \to H^*(B; \mathbf{F}_p).$$

Since the fibration

$$\Omega K_{n+1} \to X_{n+1} \to X_n$$

is a principal bundle, we can form the twisted lift

$$\sigma'_{n+1}: B \xrightarrow{(\varphi, \sigma_{n+1})} \Omega K_{n+1} \times X_{n+1} \xrightarrow{\mu} X_{n+1}$$

where μ is the principal action. The adjusted lift σ'_{n+1} satisfies

$$\sigma'^*_{n+1}(N(n+1)) = 0$$

by construction.

This completes the inductive construction of the tower. It remains to verify that the tower converges. To this end, note that as a consequence of Propositions 6 and 7 the space X_{n+1} arises from X_n by using an $\mathbf{F}_p$ vector space to annihilate p-torsion classes in $H_*(X_n; \mathbf{Z})$ of finite order. Thus the connectivity of $N(n)$ goes to infinity with n and the tower converges. □

It remains to prove (3)–(7).

Proof of (3). We first show that $H^*(\times_{i=1}^a BU(d_i); \mathbf{F}_p)$ is a free S^*-module. Note by construction we have a coexact sequence of commutative graded algebras

$$1 \to S^* \to H^* \to R^* \to 1,$$

where we have abbreviated $H^*(\times_{i=1}^a BU(d_i); \mathbf{F}_p)$ to H^*, and $P(V^*)^G$ to R^*. Polynomial algebras being free commutative, this sequence splits to yield an isomorphism of algebras $H^* \simeq S^* \otimes R^*$. Hence H^* is a free S^*-module.

To verify that S^* is a polynomial algebra we employ Serre's converse to Hilbert's syzygy theorem [12] (see also [20; Thm. 8]). According to this result a graded connected algebra over a field is a polynomial algebra iff it has finite projective dimension. So it will suffice to show that the projective dimension of S^* is finite. To this end suppose M is a graded S^*-module. Set $d = d_1 + \cdots + d_a$. Let

$$(\mathscr{P}) \qquad 0 \to K_{d+1} \to P_d \to \cdots \to P_0 \to M \to 0$$

be a partial projective resolution of M, that is, $P_0, \ldots, P_d$ are projective S^*-modules and the sequence is exact. Since H^* is a free S^*-module the functor $\otimes_{S^*} H^*$ is exact and sends projective S^*-modules into projective H^*-modules. Thus

$$0 \to K_{d+1} \otimes_{S^*} H^* \to P_d \otimes_{S^*} H^* \to \cdots \to M \otimes_{S^*} H^* \to 0$$

is an exact sequence of H^*-modules, where all but perhaps the last term are projective. By Hilbert's syzygy theorem H^* has projective dimension d and hence the last term $K_{d+1} \otimes_{S^*} H^*$ is also H^* projective [5; VI.2.1]. Since H^* is free over S^*, projective H^*-modules are also projective S^*-modules. Note

$$S^* \hookrightarrow H^* \simeq S^* \otimes R^* \xrightarrow{1 \otimes \varepsilon} S^*$$

where ε is the augmentation, presents S^* as an S^*-module direct summand in H^*. Thus

$$K_{d+1} \simeq K_{d+1} \otimes_{S^*} S^* \rightleftarrows K_{d+1} \otimes_{S^*} H^*$$

represents K_{d+1} as an S^* direct summand in the projective S^*-module $K_{d+1} \otimes_{S^*} H^*$. Hence K_{d+1} is a projective S^*-module, and thus $(\mathscr{P})$ is a projective resolution of M, so the projective dimension of S^* is at most d. □

Proof of (4). From the epimorphism

$$f^*\colon H^*(K; \mathbf{F}_p) \to S^*$$

we obtain by passing to the indecomposable quotients an epimorphism

$$Qf^*\colon QH^*(K; \mathbf{F}_p) \to QS^*.$$

Since polynomial algebras are free commutative algebras there is an algebra splitting of f^*

$$s\colon S^* \to H^*(K; \mathbf{F}_p)$$

by Lemma 3. Let T be the kernel of Qf^*. By choosing representatives in $H^*(K; \mathbf{F}_p)$ for a basis of T and using the splitting s, we can define

$$\varphi\colon P(T) \otimes S^* \to H^*(K; \mathbf{F}_p),$$

which is an isomorphism on the module of indecomposables. Since the domain and range of φ are polynomial algebras φ is an isomorphism. Thus $\varphi(T)$ generates the kernel of f^*. The iso φ shows that $H^*(K; \mathbf{F}_p)$ is a free $P(\varphi T)$-module, so the result follows from [**14**; II.1.1]. $\square$

Proof of (5) (after Zabrodsky). The action of G on V induces an action on BV, which, without loss of generality we may assume to be free. Let π: $BV \downarrow B$ be the orbit map. By Proposition 1 the bundles $\xi_1, \dots, \xi_a$ descend to B so we may form the diagram

$$\begin{tikzcd} & & X \arrow[r] \arrow[d, "\tau"] & L \arrow[d] \\ BV \arrow[r, "\pi"] & B \arrow[r, "\bar{g}"] \arrow[ur, dashed, "\bar{\sigma}"] & \bigtimes_{i=1}^{a} BU(d_i) \arrow[r, "f"] & K \end{tikzcd}$$

where $\bar{g}$ classifies the descended bundles. By replacing f by $|G| \cdot f$ if need be we may assume that

$$\bar{g}^* \cdot f^* \colon \tilde{H}^*(K; \mathbf{Z}/q) \to \tilde{H}^*(B; \mathbf{Z}/q)$$

is the zero map for all primes $q \neq p$. This composite also vanishes for $q = p$ by construction and the fact that

$$H^*(B; \mathbf{F}_p) \underset{\simeq}{\overset{\pi^*}{\to}} H^*(BV; \mathbf{F}_p)^G.$$

Since also $\tilde{H}^*(B; \mathbf{Q}) = 0$, it follows that

$$\bar{g}^* \cdot f^* \colon \tilde{H}^*(K; \mathbf{Z}); \mathbf{Z}) \to \tilde{H}^*(B; \mathbf{Z})$$

is the zero map. Thus $\bar{g} \cdot f$ is null homotopic, and there exists a lift $\bar{\sigma}$ as indicated.

Consider the diagram

$$\begin{tikzcd} & & PH^*(\Omega K; \mathbf{F}_p) \arrow[dll, dashed] \\ H^*(BV; \mathbf{F}_p) & H^*(B; \mathbf{F}_p) \arrow[l, "\pi^*"'] & M \arrow[l, "\bar{\sigma}^*"'] \arrow[u, hook] \end{tikzcd}$$

By the dual of a theorem of Carlsson [**4**] and Miller [**10**] [**23**] $H^*(BV; \mathbf{F}_p)$ is an injective object in the category of unstable modules over the Steenrod algebra. Since $|G| \not\equiv O(p)$ we may form the averaging operator

$$A := \frac{1}{|G|} \sum_{g \in G} g \colon H^*(Bv; \mathbf{F}_p) \to H^*(BV; \mathbf{F}_p)$$

whose image is $H^*(BV; \mathbf{F}_p)^G \simeq H^*(B; \mathbf{F}_p)$. Since A is $\mathscr{A}^*$ linear it follows that $H^*(B; \mathbf{F}_p)$ is an $\mathscr{A}^*$ direct summand in $H^*(BV; \mathbf{F}_p)$, and therefore $H^*(B; \mathbf{F}_p)$ is an injecitve object in the category of unstable modules over the Steenrod algebra. Therefore there is a dotted morphism as in the preceding diagram. Now recall that by construction

$$\Omega K = \times K(\mathbf{Z}, 2j - 1)$$

for suitable integers j. Therefore, $PH^*(\Omega K; \mathbf{F}_p)$ being generated as an $\mathscr{A}^*$-module by its fundamental classes, is generated by odd dimensional classes with vanishing Bocksteins. However, by [**1**; §3].

$$H^*(BV; \mathbf{F}_p)^G \simeq E[\tau_1, \ldots, \tau_n] \otimes P[\rho_1, \ldots, \rho_n]$$

where $\beta\tau_i = \tau_i$, $i = 1, \ldots, n$, so

$$H^*(B; \mathbf{F}_p) \simeq H^*(BV; \mathbf{F}_p)^G$$

contains no odd dimensional classes with vanishing Bocksteins. Therefore the dotted map is zero. Thus $\pi^*\bar{\sigma}^*(m) = 0$. The fibration

$$\Omega K \to X \to \mathop{\times}_{i=1}^{a} BU(d_i)$$

is a principal ΩK bundle. Thus any two lifts σ', σ'' of g over τ differ by a map $BV \to \Omega K$. But the preceding considerations with Bockstein shows any such map induces the zero map in $\mathbf{F}_p$ cohomology. Thus $\sigma^* = \pi^*\bar{\sigma}^*$ and so $\sigma^*(M) = 0$. □

Proof of (7). By Lemma 3 and [**14**; II.1.1]

$$f^*(i_{m_j}) \in H^*\left(\mathop{\times}_{i=1}^{a} BU(d_i); \mathbf{F}_p\right) \qquad j = 1, \ldots, c$$

is an ESP-sequence. Therefore

$$p, f^*(i_{m_1}), \ldots, f^*(i_{m_c}) \in H^*\left(\mathop{\times}_{i=1}^{a} BU(d_i); \mathbf{Z}\left[\frac{1}{p'}\right]\right)$$

is an ESP-sequence. A standard Eilenberg-Moore spectral sequence argument now completes the proof. □

Proof of (6). It only remains to verify the statement about Bocksteins. But this follows because Proposition 7 assures us that X contains no unwanted rational cohomology classes. □

We turn now to a discussion of examples of groups G satisfying the weak splitting principle.

EXAMPLE 1. Let $G = \mathrm{GL}(V)$. Then [7] [**18**]

$$P(V^*)^G = D^*(n) = P[y_1, \dots, y_n]$$

where

$$X^{p^n-1} + y_1 X^{p^{n-1}-1} + \cdots + y_n = \prod_{v \neq 0 \in V^*} (X + v).$$

This is perhaps the most basic example, but of course for $n > 1$ does not satisfy the non-modularity condition $|G| \not\equiv O(p)$. As previously noted, the groups $G < \mathrm{GL}(V)$ generated by pseudo reflections and satisfying $|G| \not\equiv O(p)$ are products of irreducible examples that have been listed by Clark and Ewing [**6**]. In this list there are three infinite families.

EXAMPLE 2. Let $G = \Sigma_n$ be the symmetric group on n letters acting on $V = \oplus_n \mathbf{F}_p$ by permutation of the basis vectors. Then

$$P(V^*)^{\Sigma_n} = P[\sigma_1, \dots, \sigma_n]$$

where

$$X^n + \sigma_1 X^{n-1} + \cdots + \sigma_n = \prod_{i=1}^{n} (X + t_i),$$

$\{t_1, \dots, t_n\}$ being the basis of V permuted by Σ_n. Thus Σ_n satisfies the splitting principle.

EXAMPLE 3. $G = D_m = \mathbf{Z}/m \ltimes \mathbf{Z}/2$, the dihedral group of order $|D_m| = 2m$, where m is a divisor of $p - 1$, acting on $V = \mathbf{F}_p \oplus \mathbf{F}_p$ via the matrices

$$\begin{pmatrix} \theta & 0 \\ 0 & \theta^{-1} \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where $\theta \in \mathbf{F}_p^*$ is a primitive mth root of unity. If we write $\{u, v\}$ for the dual basis in V^*, then

$$P(V^*)^{D_m} \simeq P[\rho_1, \rho_2]$$

where we may choose

$$\rho_1 = muv, \qquad \rho_2 = -(u^m + v^m),$$

and

$$X^m + \rho_1 X^{m-2} + \cdots + \rho_2 = \prod_{i=1}^{m-1} \left(X + (\theta^i u + \theta^{-i} v)\right).$$

Thus D_m satisfies the weak splitting principle.

EXAMPLE 4. $G(n,m,r)$. These are the groups of type $2b$ in the Clark-Ewing list. Here $p \equiv 1 \bmod m$, $p \nmid n!$, and $r|m$. If $\theta \in \mathbf{F}_p^*$ is a primitive mth root of unity, then $G(n,m,r)$ may be taken to be the subgroup of $\mathrm{GL}(V)$ generated by the permutation matrices and the matrices

$$\begin{pmatrix} \theta^{\nu_1} & & & 0 \\ & \theta^{\nu_2} & & \\ & & \ddots & \\ 0 & & & \theta^{\nu_n} \end{pmatrix}$$

with respect to some fixed basis of V, where $\nu_1 + \cdots + \nu_n \equiv 0 \bmod r$. One has

$$P(V)^G = P[\rho_1, \dots, \rho_n]$$

where

$$\rho_i = \begin{cases} \sigma_i(t_1^m, \dots, t_n^m): & 1 \le i \le n-1, \\ \sigma_n(t_1^{m/r}, \dots, t_n^{m/r}): & i = n. \end{cases}$$

Moreover

$$\prod_{\substack{1 \le i \le m \\ 1 \le j \le n}} \left(X + \theta^i t_j\right) = X^{nm} + \rho_1 X^{(n-1)m} + \cdots + \rho_n^r.$$

Let $w = t_1 + \cdots + t_n$. The orbit of w "sees" r; to wit

$$[w] = \{\theta^{\nu_1} t_1 + \cdots + \theta^{\nu_n} t_n \,|\, \nu_1 + \cdots + \nu_n \equiv 0 \bmod r\}.$$

This orbit has $m^n r$ elements. Let

$$s = \sum_{v \in [w]} v^{mn/r} = \sum_{\nu_1 + \cdots + \nu_n \equiv o(r)} (\theta^{\nu_1} t_1 + \cdots + \theta^{\nu_n} t_n)^{nm/r}.$$

Since s is a polynomial in the elementary symmetric functions, it lies in the subalgebra generated by the coefficients of $\varphi_{[w]}(X)$. Let us look at the coefficient of $\rho_n = t_1^{m/r} \cdots t_n^{m/r}$ in s. The sum defining s has nm/r terms, namely

$$\theta^{(\nu_1 + \cdots \nu_n)m/r} \frac{(nm/r)!}{[(m/r)!]^n} = \frac{(nm/r)!}{[(m/r)!]^n} \not\equiv O(p)$$

as $p > nm/r$ and $p \nmid n!$. Therefore the coefficient of $\rho_n = t_1^{m/r} \cdots t_n^{m/r}$ in s is

$$\frac{(nm/r)(nm/r)!}{[(m/r)!]^n} \equiv O(p),$$

so we conclude that

$$s \equiv \rho_n \in P(V)^{G(n,m,r)} \quad \text{mod decomposable.}$$

Thus for $p > nm/r$, $p \nmid n!$, $P(V)^{G(n,m,r)}$ satisfies the weak splitting principle.

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For a thorough discussion of splitting principles we refer to [**17**].

A Consequence of the Main Theorem

THEOREM. *Suppose $G < \mathrm{GL}(V)$ is generated by pseudo reflections, $|G| \neq 0 \bmod p$, and $P(V)^G$ satisfies the weak splitting principle. Let $B \subset V$ be an orbit with orbit Chern class $c(B) \in P(V^*)^G$. Then there exists a simply connected space X of finite type and a vector bundle $\xi_B \downarrow X$ such that*

$$H^*(X; \mathbf{F}_p) \simeq P(V^*)^G$$

and under this identification

$$c(B) = c(\xi_B)$$

where $c(\xi_B)$ is the total Chern class of the vector bundle ξ_B.

Proof. We modify ever so slightly the proof of the main theorem. Let

$$\mathbf{B} = \{B \subset V \mid B \text{ is an orbit of } G \text{ in } V\}.$$

For $B \in \mathbf{B}$ let $|B|$ be the cardinality of B. The orbit Chern classes then provide a map

$$H^*(BU(|B|); \mathbf{F}_p) \to P(V^*)^G$$

and the induced map

$$H^*\Big(\bigtimes_{b \in \mathbf{B}} BU(|B|); \mathbf{F}_p\Big)$$

is an epimorphism. We now start the construction of the tower in the main theorem with this product $\times_{b \in \mathbf{B}} BU(|B|)$ to construct the desired space X. The desired vector bundle ξ_B is then classified by the composite

$$X \to \bigtimes_{b \in \mathbf{B}} BU(|B|) \to BU(|B|)$$

where the last map is projection onto the factor corresponding to the orbit B. $\square$

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