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# CHARACTER IDENTITIES AND ASYMPTOTIC BEHAVIOR OF MATRIX COEFFICIENTS OF DISCRETE SERIES

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**Using Schmid's character identities we give a new proof of a result of Trombi and Varadarajan which describes a necessary condition for a discrete series representation of a semisimple Lie group to have a specific asymptotic decay.**

In this note we want to explain a simple relationship between Schmid's character identities [7] and the results of Trombi and Varadarajan on the rate of decay of the $K$-finite matrix coefficients of the discrete series [**8**].

Let $G$ be a connected semisimple Lie group with finite center. We assume that $G$ has nonempty discrete series, therefore by a result of Harish-Chandra it has a compact Cartan subgroup $H$. Let $K$ be a maximal compact subgroup of $G$ containing $H$. Let $\mathfrak{g}_0$, $\mathfrak{k}_0$ and $\mathfrak{h}_0$ be the Lie algebras of $G, K$ and $H$ respectively, and $\mathfrak{g}, \mathfrak{k}$ and $\mathfrak{h}$ their complexifications. Denote by $\Phi$ the root system of $(\mathfrak{g}, \mathfrak{h})$. A root $\alpha \in \Phi$ is called *compact* if its root subspace is contained in $\mathfrak{k}$ and *noncompact* otherwise. Let $W$ be the Weyl group of $(\mathfrak{g}, \mathfrak{h})$ and $W_k$ its subgroup generated by the reflections with respect to the compact roots. The Killing form of $\mathfrak{g}$ induces an inner product $(\,|\,)$ on $i\mathfrak{h}_0^*$, the space of all linear forms on $\mathfrak{h}$ which assume imaginary values on $\mathfrak{h}_0$. An element $\lambda$ of $i\mathfrak{h}_0^*$ is singular if it is orthogonal to at least one root of $\Phi$ and nonsingular otherwise. The differentials of the characters of $H$ form a lattice $\Lambda$ in $i\mathfrak{h}_0^*$. Let $\rho$ be the half-sum of positive roots in $\Phi$ with respect to some ordering on $i\mathfrak{h}_0^*$. Then $\Lambda + \rho$ is independent of the choice of this ordering.

Harish-Chandra has shown that to each nonsingular $\lambda \in \Lambda + \rho$ we can attach a discrete series representation $\pi_\lambda$, $\pi_\lambda$ is equal to $\pi_\mu$ if and only if $\lambda$ and $\mu$ are conjugate under $W_k$, and the discrete series are exhausted in this way ([3], [**9**, 10.2.4.1]).

Choose an Iwasawa decomposition $G = KAN$ of $G$. Let $\mathfrak{a}_0$ and $\mathfrak{n}_0$ be the Lie algebras of $A$ and $N$ respectively, and $\mathfrak{a}$ and $\mathfrak{n}$ their complexifications. Let $\Sigma$ be the root system of $(\mathfrak{g}, \mathfrak{a})$. Then $\mathfrak{n}$ determines a set of positive roots $\Sigma^+$ and its negative Weyl chamber $A^-$ in $A$. Let $\delta$ be the modular function of the group $AN$; it is evidently rapidly decreasing on $A^-$.

Put

$$k(\alpha)=\frac{1}{4}\sum_{\beta\in\Phi}|(\alpha|\beta)|,\qquad \alpha\in\Phi.$$

By the Cartan decomposition of $G$ we know that the growth of $K$-finite matrix coefficients of a representation at infinity on $G$ is completely determined by their behavior on $\mathrm{Cl}(A^-)$. Trombi and Varadarajan proved in **[8]**:

THEOREM. *For a nonsingular $\lambda\in\Lambda+\rho$ and $\kappa>0$, the assertion*

(i) *for every $K$-finite matrix coefficient $c$ of the discrete series representation $\pi_\lambda$ there exist $C, r\geq 0$ such that*

$$|c(a)|\leq C\delta^{(1+\kappa)/2}(a)(1+\|\log a\|)^r$$

*for all $a\in A^-$; implies*

(ii) $|(\lambda|\alpha)|\geq\kappa k(\alpha)$ *for every noncompact root $\alpha\in\Phi$.*

In **[5]** one of us proved that the converse holds, i.e. (i) and (ii) are equivalent.

The original argument of Trombi and Varadarajan is based on a long and complicated analysis of differential equations satisfied by the matrix coefficients and the argument in **[5]** to prove (i) $\Rightarrow$ (ii) is based on the Langlands classification of representations.

Therefore, it seems worthwhile to point out the real simplicity of this result—it is more or less just a formal consequence of the character identities. Roughly speaking, the growth estimate (i) is the best possible estimate which does not contradict the character identities involving the character of $\pi_\lambda$ (as was remarked in [7], these identities can be deduced from Harish-Chandra's matching conditions, which were also used in **[8]** to determine the character of $\pi_\lambda$ on Cartan subgroups with one dimensional split part). We hope that our argument clarifies the meaning of the condition (ii).

In the first section we interpret the character identities as a statement about an imbedding of $\pi_\lambda$ into a principal series representation corresponding to a maximal parabolic subgroup. Evidently this is just a part of information one can extract from the character identities, but we want to stress the fact that this part follows easily. In the second section we give a growth estimate for the matrix coefficients of this principal series representation and show, via an orthogonality argument, that it contradicts any growth condition better than (i).

Finally, let us remark that the companion result to the theorem (for $1 \leq p < 2$, $\pi_\lambda$ is $p$-integrable implies that $|(\lambda|\alpha)| > (2/p - 1)k(\alpha)$ for all noncompact roots $\alpha \in \Phi$ [**8**]) follows immediately from it using 8.11 and 8.14 of [**2**].

**1. Representation-theoretic interpretation of the character identities.** Let $\Psi$ be a system of positive roots in $\Phi$. In [7] Schmid constructed a "coherent" family of eigendistributions $\Theta(\Psi, \lambda)$, $\lambda \in \Lambda + \rho$, which "extends" characters of discrete series. More precisely:

(i) If $Z$ belongs to the center $\mathscr{Z}(\mathfrak{g})$ of the universal enveloping algebra $\mathscr{U}(\mathfrak{g})$ of $\mathfrak{g}$, then

$$Z\Theta(\Psi, \lambda) = \chi_\lambda(Z)\Theta(\Psi, \lambda)$$

where $\chi_\lambda$ is the character of $\mathscr{Z}(\mathfrak{g})$ corresponding to $\lambda \in \mathfrak{h}^*$ via the (twisted) Harish-Chandra isomorphism of $\mathscr{Z}(\mathfrak{g})$ onto the Weyl group invariants in $\mathscr{U}(\mathfrak{h})$.

(ii) Let $\varphi$ be the character of an irreducible finite-dimensional representation $\tau$ of $G$. Then

$$\varphi \cdot \Theta(\Psi, \lambda) = \sum_\nu \Theta(\Psi, \lambda + \nu)$$

where the sum is taken over the set of all weights of $\tau$, counted with multiplicites.

(iii) If $\lambda$ is nonsingular and dominant with respect to $\Psi$ then $\Theta(\Psi, \lambda)$ is the character of $\pi_\lambda$.

The character identities relate the eigendistributions $\Theta(\Psi, \lambda)$ corresponding to different positive root systems $\Psi$. Fix a simple noncompact root $\alpha \in \Psi$. We decompose $\mathfrak{h}$ as

$$\mathfrak{h} = \mathfrak{c} \oplus \mathbf{C} H_\alpha$$

where $\mathfrak{c} = \ker \alpha$ and $H_\alpha \in \mathfrak{h}$ is the dual root of $\alpha$. We attach to $\alpha$ the Cayley transform $c_\alpha$ [7]. It is an inner automorphism of $\mathfrak{g}$ which acts as an identity on $\mathfrak{c}$ and maps $H_\alpha$ into $\tilde{H}_\alpha \in \mathfrak{g}_0$. Thus

$$c_\alpha(\mathfrak{h}) = \mathfrak{c} \oplus \mathbf{C}\tilde{H}_\alpha$$

is the complexification of a real Cartan subalgebra of $\mathfrak{g}_0$, with one-dimensional split part. Let $P_\alpha$ be the corresponding maximal cuspidal parabolic subgroup of $G$ and

$$P_\alpha = M_\alpha A_\alpha N_\alpha$$

its Langlands decomposition. Denote by $\mathfrak{m}_\alpha$, $\mathfrak{a}_\alpha$, $\mathfrak{n}_\alpha$ the complexified Lie algebras of $M_\alpha$, $A_\alpha$, $N_\alpha$, respectively. Then $H \cap M_\alpha$ becomes a compact Cartan subgroup of $M_\alpha$ and $\mathfrak{c}$ is its complexified Lie algebra. The root system of $(\mathfrak{m}_\alpha, \mathfrak{c})$ can be identified with

$$\Phi_\alpha = \{\beta \in \Phi \mid (\alpha|\beta) = 0\}.$$

In particular, $\Psi_\alpha = \Psi \cap \Phi_\alpha$ is a system of positive roots in $\Phi_\alpha$. Let $\lambda_\alpha = \lambda|\mathfrak{c}$. Then $\Psi_\alpha$ and $\lambda_\alpha$ determine an eigendistribution $\Theta_{M_\alpha^0}(\Psi_\alpha, \lambda_\alpha)$ on the identity component $M_\alpha^0$ of $M_\alpha$, which in turn (as explained in [7]) determines an eigendistribution $\Theta_{M_\alpha}(\Psi_\alpha, \lambda_\alpha)$ on $M_\alpha$. On the other hand, $\lambda$ determines a linear form $\nu$ on $\mathfrak{a}_\alpha = \mathbf{C}\tilde{H}_\alpha$ by $\nu(\tilde{H}_\alpha) = \lambda(H_\alpha)$. Now, by the normalized induction, $\Theta_{M_\alpha}(\Psi_\alpha, \lambda_\alpha)$ and $\nu$ determine an eigendistribution $\Theta_{\Psi,\alpha,\lambda}$ on $G$.

The character identity we shall use asserts that

$$\Theta(\Psi, \lambda) + \Theta(s_\alpha\Psi, \lambda) = \Theta_{\Psi,\alpha,\lambda}. \tag{1.1}$$

The eigendistributions $\Theta(\Psi, \lambda)$ are, in general, only virtual characters; and, a priori, it is difficult to attach a representation-theoretic meaning to (1.1). However, in the special case we shall need, the situation turns out to be much nicer. Assume that $\lambda$ is nonsingular and dominant with respect to $\Psi$. Then $\lambda_\alpha$ is nonsingular and dominant with respect to $\Psi_\alpha$ and $\Theta(\Psi, \lambda)$, $\Theta_{M_\alpha}(\Psi_\alpha, \lambda_\alpha)$ are the characters of $\pi_\lambda$ and a discrete series representation $\sigma$ of $M_\alpha$, respectively. The eigendistribution $\Theta_{\Psi,\alpha,\lambda}$ becomes therefore the character of the induced representation $\operatorname{Ind}_{P_\alpha}^G(\sigma, \nu)$.

If

$$\rho_\alpha(H) = \frac{1}{2}\operatorname{tr}(\operatorname{ad} H|\mathfrak{n}_\alpha), \qquad H \in \mathfrak{a}_\alpha,$$

then

$$\rho_\alpha(\tilde{H}_\alpha) = \frac{1}{4}\sum_{\beta\in\Phi}|\beta(H_\alpha)| = \frac{1}{4}\sum_{\beta\in\Phi}\left|\frac{2(\alpha|\beta)}{(\alpha|\alpha)}\right| = \frac{2k(\alpha)}{(\alpha|\alpha)}.$$

Therefore, if we put

$$t = \frac{(\lambda|\alpha)}{k(\alpha)}$$

we have

$$\nu = t\rho_\alpha.$$

The interpretation of (1.1), we alluded to in the Introduction, is contained in the following proposition.

1.1. PROPOSITION. *Let $\lambda \in \Lambda + \rho$ be nonsingular and dominant with respect to $\Psi$. Then the discrete series representation $\pi_\lambda$ is infinitesimally equivalent to a subquotient of* $\operatorname{Ind}_{P_\alpha}^G(\sigma, t\rho_\alpha)$.

A more complete representation-theoretic explanation of the character identity (1.1) was given by Schmid [**6**]; however, 1.1 is all we need in this paper, and we give a simple proof of it below. Since in this case $\Theta(\Psi, \lambda)$ and $\Theta_{\Psi,\alpha,\lambda}$ are characters, it is enough to show the following lemma.

1.2. LEMMA. *Let $\lambda \in \Lambda + \rho$ be dominant and nonsingular with respect to $\Psi$. Then $\Theta(s_\alpha\Psi, \lambda)$ is a character.*

*Proof*. Let $w_0$ be the longest element in $W$. Let $\mu$ be the highest weight of an irreducible finite-dimensional representation $\tau_\mu$ of $G$ such that $s_\alpha\lambda + \mu$ is dominant. Then the contragredient representation $\tilde{\tau}_\mu$ of $\tau_\mu$ has the highest weight $-w_0\mu$. Let $\varphi$ be the character of $\tilde{\tau}_\mu$.

Now we can compute the character of the component of $\tilde{\tau}_\mu \otimes \pi_{\lambda + s_\alpha\mu}$ with generalized infinitesimal character $\chi_\lambda$; it is equal to

$$[\varphi \cdot \Theta(s_\alpha\Psi, \lambda + s_\alpha\mu)]_{(\lambda)} = \sum \Theta(s_\alpha\Psi, \lambda + s_\alpha\mu + \nu)$$

where the sum is taken over all weights $\nu$ of $\tilde{\tau}_\mu$ such that

$$\lambda + s_\alpha\mu + \nu = w\lambda$$

for some $w \in W$, counting multiplicities. This implies that

$$w\lambda - \lambda = s_\alpha(\mu + s_\alpha\nu)$$

and, $-\mu$ being the lowest weight of $\tilde{\tau}_\mu$,

$$s_\alpha(w\lambda - \lambda) = \mu + s_\alpha\nu$$

is a sum of positive roots. But $\lambda$ being dominant, $w\lambda - \lambda$ is a combination of simple roots with negative coefficients, therefore we conclude that

$$s_\alpha(w\lambda - \lambda) = \mu + s_\alpha\nu = n_\alpha\alpha$$

where $n_\alpha \in \mathbb{Z}_+$. Now,

$$\|\lambda\|^2 = \|s_\alpha w\lambda\|^2 = \|s_\alpha\lambda + n_\alpha\alpha\|^2 = \|\lambda\|^2 - 2n_\alpha(\lambda|\alpha) + n_\alpha^2\|\alpha\|^2,$$

which implies that either

$$n_\alpha = 0 \quad \text{or} \quad n_\alpha = \frac{2(\lambda|\alpha)}{(\alpha|\alpha)}.$$

Assume first that

$$\frac{2(\lambda|\alpha)}{(\alpha|\alpha)} \notin \mathbb{Z}.$$

Then the only possibility is $n_\alpha = 0$, which implies $\nu = -s_\alpha\mu$ and $w = 1$ by the nonsingularity of $\lambda$. Therefore

$$[\varphi \cdot \Theta(s_\alpha\Psi, \lambda + s_\alpha\mu)]_{(\lambda)} = \Theta(s_\alpha\Psi, \lambda)$$

and $\Theta(s_\alpha\Psi, \lambda)$ is a character in this case.

Now assume that

$$\frac{2(\lambda|\alpha)}{(\alpha|\alpha)} \in \mathbb{Z}.$$

Then $s_\alpha\lambda \in \Lambda + \rho$. For $n_\alpha \neq 0$, we have

$$s_\alpha w\lambda = s_\alpha\lambda + n_\alpha\alpha = \lambda,$$

so by the nonsingularity of $\lambda$, it follows that $w = s_\alpha$. Also

$$\nu = -s_\alpha\mu - n_\alpha\alpha = -\mu + \frac{2(\mu|\alpha)}{(\alpha|\alpha)}\alpha - \frac{2(\lambda|\alpha)}{(\alpha|\alpha)}\alpha$$

$$= -\mu + \frac{2(\mu - \lambda|\alpha)}{(\alpha|\alpha)}\alpha.$$

By our assumption $s_\alpha\lambda + \mu$ is dominant, therefore,

$$0 \leq (s_\alpha\lambda + \mu|\alpha) = -(\lambda|\alpha) + (\mu|\alpha) = (\mu - \lambda|\alpha) \leq (\mu|\alpha),$$

which implies that $\nu$ is a weight of $\tilde{\tau}_\mu$ lying between $-\mu$ and $-s_\alpha\mu$. Hence the multiplicity of $\nu$ is equal to one.

Therefore, in this case,

$$[\varphi \cdot \Theta(s_\alpha\Psi, \lambda + s_\alpha\mu)]_{(\lambda)} = \Theta(s_\alpha\Psi, \lambda) + \Theta(s_\alpha\Psi, s_\alpha\lambda).$$

Thus, $\Theta(s_\alpha\Psi, \lambda)$ is a difference of a character and a character of the discrete series representation $\pi_{s_\alpha\lambda}$. The character identity (1.1) implies that it is a difference of a character and the character of $\pi_\lambda$. The fact that $\pi_\lambda$ and $\pi_{s_\alpha\lambda}$ are not equivalent and the linear independence of irreducible characters show that $\Theta(s_\alpha\Psi, \lambda)$ is a character. $\square$

**2. Asymptotic behavior of discrete series representations.** Now we want to show how the results of the previous section imply the theorem we stated in the Introduction.

Let $\pi_\lambda$ be a discrete series representation, $\Psi$ a system of positive roots in $\Phi$ such that $\lambda$ is dominant with respect to it and $\alpha \in \Psi$ a simple noncompact root. By 1.1 we know that there exists a discrete series representation $\sigma$ of $M_\alpha$ such that $\pi_\lambda$ is infinitesimally equivalent to a subquotient of $\mathrm{Ind}_{P_\alpha}^G(\sigma, t\rho_\alpha)$.

First we need a rough estimate for the $K$-finite matrix coefficients of $\operatorname{Ind}_{P_\alpha}^G(\sigma, s\rho_\alpha)$, $s \geq 0$, which is analogous to 3.8 in [4]. We include a proof for the sake of completeness.

Let $K_\alpha = K \cap P_\alpha$ be a maximal compact subgroup of $M_\alpha$. The restriction to $K$ defines an isometric isomorphism of $\operatorname{Ind}_{P_\alpha}^G(\sigma, s\rho_\alpha)$ onto $\operatorname{Ind}_{K_\alpha}^K(\sigma|K_\alpha)$ as unitary representations of $K$ for all $s \in \mathbb{C}$, so we can view all representations $\operatorname{Ind}_{P_\alpha}^G(\sigma, s\rho_\alpha)$ as acting on the same space.

Let $P$ be a minimal parabolic subgroup of $G$ contained in $P_\alpha$ and $P = MAN$ its Langlands decomposition.

2.1. LEMMA. *Let $v$ be a $K$-finite vector in $\operatorname{Ind}_{P_\alpha}^G(\sigma, s\rho_\alpha)$, $\tilde{v}$ a $K$-finite linear form on $\operatorname{Ind}_{P_\alpha}^G(\sigma, s\rho_\alpha)$ and $c_{v,\tilde{v}}^s$ the corresponding matrix coefficient. Then,*

(i) *the function $(s, x) \to c_{v,\tilde{v}}^s(x)$ is continuous on $\mathbb{C} \times G$;*

(ii) *if $s_0 \geq 0$, then there exist constants $C, r \geq 0$, depending on $v, \tilde{v}$ and $s_0$ only, such that*

$$|c_{v,\tilde{v}}^s(a)| \leq C\delta^{(1-s_0)/2}(a)(1 + \|\log a\|)^r$$

*for all $a \in A^-$ and $0 \leq s \leq s_0$.*

*Proof*. By the Iwasawa decomposition, there exist unique smooth maps $\kappa\colon G \to K$ and $H\colon G \to \mathfrak{a}$ such that

$$x \in \kappa(x) \exp H(x) \cdot N$$

for all $x \in G$. The subgroup ${}^*P = M_\alpha \cap P$ is a minimal parabolic subgroup of $M_\alpha$. Let ${}^*P = M{}^*A{}^*N$ be its Langlands decomposition. Denote by ${}^*\mathfrak{a}$ the complexified Lie algebra of ${}^*A$. Then $\mathfrak{a} = {}^*\mathfrak{a} + \mathfrak{a}_\alpha$. Let ${}^*H\colon G \to {}^*\mathfrak{a}$ and $H_\alpha\colon G \to \mathfrak{a}_\alpha$ be such smooth maps that $H = {}^*H + H_\alpha$. The Iwasawa decomposition $M_\alpha = K_\alpha{}^*A{}^*N$ allows us to define a unique smooth map $\mu\colon G \to {}^*A{}^*N$ such that

$$x \in \kappa(x)\mu(x) \exp H_\alpha(x) \cdot N_\alpha$$

for all $x \in G$. Evidently, for $m \in M_\alpha$, we have

$$\mu(xm) = \mu(\mu(x)m) \tag{2.1}$$

and

$$H_\alpha(xm) = H_\alpha(x) \tag{2.2}$$

for all $x \in G$. Also, $\mu(x) \in \exp{}^*H(x){}^*N$, so

$${}^*H(\mu(x)) = {}^*H(x);$$

and by (2.1)

(2.3) $\quad {}^*H(\mu(x)m) = {}^*H(\mu(\mu(x)m)) = {}^*H(\mu(xm)) = {}^*H(xm)$

for all $x \in G$ and $m \in M_\alpha$.

Let $f, g \in \operatorname{Ind}_{K_\alpha}^K(\sigma|K_\alpha)$ be such that $f$ corresponds to $v$ and $g$ defines the linear form $\tilde{v}$. Then, $f$ and $g$ are $K$-finite. Assume that $(f_1, f_2, \ldots, f_p)$ and $(g_1, g_2, \ldots, g_q)$ are bases of the finite-dimensional subspaces spanned by $K$-orbits of $f$ and $g$ respectively. Then

$$f(kh) = \sum_{i=1}^{p} a_i(h) f_i(k), \qquad h, k \in K;$$

and

$$g(kh) = \sum_{j=1}^{q} b_j(h) g_j(k), \qquad h, k \in K;$$

where $a_i$, $1 \leq i \leq p$, and $b_j$, $1 \leq j \leq q$, are smooth functions on $K$. In particular,

$$f(k) = \sum_{i=1}^{p} a_i(k) f_i(1),$$
$$g(k) = \sum_{j=1}^{q} b_j(k) g_j(1),$$

for $k \in K$, and $f_i(1)$, $1 \leq i \leq p$, $g_j(1)$, $1 \leq j \leq q$, are $K_\alpha$-finite.

A short calculation shows that

$$c_{v,\tilde{v}}^s(x) = \int_K e^{-(s+1)\rho_\alpha(H_\alpha(x^{-1}k))} \Big(\sigma\big(\mu(x^{-1}k)^{-1}\big) f\big(\kappa(x^{-1}k)^{-1}\big) | g(k^{-1})\Big)\, dk$$

for all $x \in G$ and $s \in \mathbf{C}$, which immediately implies (i); and therefore for real $s$

$$|c_{v,\tilde{v}}^s(x)| \leq \sum_{i=1}^{p} \sum_{j=1}^{q} \left|a_i\big(\kappa(x^{-1}k)^{-1}\big)\right| \left|b_j(k^{-1})\right|$$

$$\cdot \int_K e^{-(s+1)\rho_\alpha(H_\alpha(x^{-1}k))} \left|\Big(\sigma\big(\mu(x^{-1}k)^{-1}\big) f_i(1) | g_j(1)\Big)\right| dk$$

for $x \in G$.

Let $\Xi$, $\Xi_\alpha$ be the zonal spherical functions on $G$, $M_\alpha$ corresponding to the zero linear forms on $\mathfrak{a}$, ${}^*\mathfrak{a}$, respectively. Then, by a result of Harish-Chandra ([**3**], [**9**, 9.3.1.5]; this also follows directly from [**2**, 7.7] and [**9**, 8.3.7.3]), any $K$-finite matrix coefficient of $\sigma$ is majorized by a multiple of $\Xi_\alpha$. Therefore

$$|c^s_{v,\bar v}(x)| \le c_1 \cdot \int_K e^{-(s+1)\rho_\alpha(H_\alpha(x^{-1}k))}\Xi_\alpha\left(\mu(x^{-1}k)^{-1}\right) dk$$

for all $x \in G$.

Let

$$^*\rho(H) = \tfrac{1}{2}\mathrm{tr}\left(\mathrm{ad}\, H|^*\mathfrak{n}\right), \qquad H \in {}^*\mathfrak{a};$$

and

$$\rho(H) = \tfrac{1}{2}\mathrm{tr}\left(\mathrm{ad}\, H|\mathfrak{n}\right), \qquad H \in \mathfrak{a};$$

where $^*\mathfrak{n}$ and $\mathfrak{n}$ are the complexified Lie algebras of $^*N$ and $N$, respectively. Considering $^*\rho$ and $\rho_\alpha$ as linear forms on $\mathfrak{a}$ equal to zero on $\mathfrak{a}_\alpha$ and $^*\mathfrak{a}$, respectively, we see that $\rho = {}^*\rho + \rho_\alpha$.

By [**9**, 6.2.2.1], we have

$$\Xi_\alpha(y^{-1}) = \Xi_\alpha(y) = \int_{K_\alpha} e^{-{}^*\rho({}^*H(yh))}\, dh;$$

and using (2.2) and (2.3) it follows that

$$|c^s_{v,\bar v}(x)| \le c_1 \int_K e^{-(s+1)\rho_\alpha(H_\alpha(x^{-1}k))}\left(\int_{K_\alpha} e^{-{}^*\rho({}^*H(\mu(x^{-1}k)h))}\, dh\right) dk$$

$$= c_1 \int_K e^{-(s\rho_\alpha+\rho)(H(x^{-1}k))}\, dk$$

for all $x \in G$.

Let $w_0$ be the longest element in the Weyl group of $(\mathfrak{g}, \mathfrak{a})$. Then, by [**9**, 3.3.2.3],

$$\rho(\log a) \le -(w_0\rho_\alpha)(\log a) \le \rho_\alpha\left(H(a^{-1}k)\right)$$

for all $a \in A^-$ and $k \in K$. Hence, for $0 \le s \le s_0$, we have

$$|c^s_{v,\bar v}(a)| \le c_1 e^{-s\rho(\log a)} \int_K e^{-\rho(H(a^{-1}k))}\, dk$$

$$= c_1 \delta^{-s_0/2}(a)\Xi(a),$$

for all $a \in A^-$. The assertion (ii) now follows from [**9**, 8.3.7.4]. □

Assume now that the $K$-finite matrix coefficients of $\pi_\lambda$ satisfy the condition (i) from Introduction for some $\kappa > 0$. Then by 2.1 and the well-known integral formula associated to the Cartan decomposition, the products of $K$-finite matrix coefficients of $\pi_\lambda$ with $K$-finite matrix coefficients of $\mathrm{Ind}^G_{P_\alpha}(\sigma, s\rho_\alpha)$ are integrable on $G$ for all $s < \kappa$.

Now we need a kind of "orthogonality relations" for $K$-finite matrix coefficients. The next lemma is an "infinitesimal" version of 5.21 in [**1**].

Let $(\pi_i, V_i)$, $1 \leq i \leq 2$, be two admissible representations of $(\mathfrak{g}, K)$; $(\tilde{\pi}_i, \tilde{V}_i)$, $1 \leq i \leq 2$, their contragredient representations and $c^{(i)}\colon \tilde{V}_i \otimes V_i \to C^\infty(G)$, $1 \leq i \leq 2$, their matrix coefficient maps (see §8 of [**2**]).

2.2. LEMMA. *Assume that the functions* $x \to c^{(1)}_{v,\tilde{v}}(x) c^{(2)}_{w,\tilde{w}}(x^{-1})$ *are integrable on* $G$ *for all* $v \in V_1$, $\tilde{v} \in \tilde{V}_1$, $w \in V_2$, $\tilde{w} \in \tilde{V}_2$; *and that* $\operatorname{Hom}_{(\mathfrak{g},K)}(V_1, V_2) = 0$. *Then*

$$\int_G c^{(1)}_{v,\tilde{v}}(x) c^{(2)}_{w,\tilde{w}}(x^{-1})\, dx = 0,$$

*for all* $v \in V_1$, $\tilde{v} \in \tilde{V}_1$, $w \in V_2$, $\tilde{w} \in \tilde{V}_2$.

*Proof*. Fix $\tilde{v} \in \tilde{V}_1$ and $w \in V_2$ and define

$$T(v, \tilde{w}) = \int_G c^{(1)}_{v,\tilde{v}}(x) c^{(2)}_{w,\tilde{w}}(x^{-1})\, dx,$$

for $v \in V_1$ and $\tilde{w} \in \tilde{V}_2$. We claim that this bilinear form on $V_1 \times \tilde{V}_2$ is $(\mathfrak{g}, K)$-invariant.

Denote by $R, L$ the right and left regular actions of $G$ and $\mathfrak{g}$ on $C^\infty(G)$ respectively.

Let $X \in \mathfrak{g}$. Then

$$\begin{aligned}
&\frac{d}{dt}\left(c^{(1)}_{v,\tilde{v}}(x \exp tX) c^{(2)}_{w,\tilde{w}}(\exp(-tX)x^{-1})\right) \\
&\quad= \left(R_X c^{(1)}_{v,\tilde{v}}\right)(x \exp tX) c^{(2)}_{w,\tilde{w}}(\exp(-tX)x^{-1}) \\
&\qquad+ c^{(1)}_{v,\tilde{v}}(x \exp tX)\left(L_X c^{(2)}_{w,\tilde{w}}\right)(\exp(-tX)x^{-1}) \\
&\quad= c^{(1)}_{\pi_1(X)v,\tilde{v}}(x \exp tX) c^{(2)}_{w,\tilde{w}}(\exp(-tX)x^{-1}) \\
&\qquad+ c^{(1)}_{v,\tilde{v}}(x \exp tX) c^{(2)}_{w,\tilde{\pi}_2(X)\tilde{w}}(\exp(-tX)x^{-1}),
\end{aligned}$$

so by the assumption,

$$\int_G \left|\frac{d}{dt}\left(c^{(1)}_{v,\tilde{v}}(x \exp tX) c^{(2)}_{w,\tilde{w}}(\exp(-tX)x^{-1})\right)\right| dx < +\infty;$$

and by the invariance of the Haar measure it is independent of $t \in \mathbb{R}$. By Fubini's theorem and the invariance of Haar measure

$$\begin{aligned}
0 &= \int_G \left[\int_0^1 \frac{d}{dt}\left(c^{(1)}_{v,\tilde{v}}(x \exp tX) c^{(2)}_{w,\tilde{w}}(\exp(-tX)x^{-1})\right) dt\right] dx \\
&= \int_0^1 \left[\int_G c^{(1)}_{\pi_1(X)v,\tilde{v}}(x \exp tX) c^{(2)}_{w,\tilde{w}}(\exp(-tX)x^{-1})\, dx\right. \\
&\qquad \left. + \int_G c^{(1)}_{v,\tilde{v}}(x \exp tX) c^{(2)}_{w,\tilde{\pi}_2(X)\tilde{w}}(\exp(-tX)x^{-1})\, dx\right] dt \\
&= T(\pi_1(X)v, \tilde{w}) + T(v, \tilde{\pi}_2(X)\tilde{w});
\end{aligned}$$

so

$$T(\pi_1(X)v, \tilde{w}) = -T(v, \tilde{\pi}_2(X)\tilde{w})$$

for all $X \in \mathfrak{g}$, $v \in V_1$ and $\tilde{w} \in \tilde{V}_2$; i.e. $T$ is $\mathfrak{g}$-invariant. Also

$$\begin{aligned} T(\pi_1(k)v, \tilde{w}) &= \int_G c^{(1)}_{\pi_1(k)v,\tilde{v}}(x) c^{(2)}_{w,\tilde{w}}(x^{-1})\, dx \\ &= \int_G c^{(1)}_{v,\tilde{v}}(xk) c^{(2)}_{w,\tilde{w}}(x^{-1})\, dx = \int_G c^{(1)}_{v,\tilde{v}}(x) c^{(2)}_{w,\tilde{w}}(kx^{-1})\, dx \\ &= \int_G c^{(1)}_{v,\tilde{v}}(x) c^{(2)}_{w,\tilde{\pi}_2(k^{-1})\tilde{w}}(x^{-1})\, dx = T(v, \tilde{\pi}_2(k^{-1})\tilde{w}), \end{aligned}$$

so

$$T(\pi_1(k)v, \tilde{w}) = T(v, \tilde{\pi}_2(k^{-1})\tilde{w})$$

for all $k \in K$, $v \in V_1$ and $\tilde{w} \in \tilde{V}_2$; i.e. $T$ is $K$-invariant. Therefore the linear map $S: V_1 \to V_2$ defined by

$$\langle S(v), \tilde{w} \rangle = T(v, \tilde{w})$$

for $v \in V_1$ and $\tilde{w} \in \tilde{V}_2$, is an element of $\operatorname{Hom}_{(\mathfrak{g},K)}(V_1, V_2)$, and by our assumption equal to zero. □

It is evident (for example, by considering the infinitesimal characters) that $\pi_\lambda$ is not infinitesimally equivalent to a subquotient of $\operatorname{Ind}^G_{P_\alpha}(\sigma, s\rho_\alpha)$ for "almost all" $s < \kappa$. Therefore, 2.1 and 2.2 imply that for any $K$-finite matrix coefficients $c$ of $\pi_\lambda$ and $d$ of $\operatorname{Ind}^G_{P_\alpha}(\sigma, s\rho_\alpha)$, $0 \leq s < \kappa$, we have

$$\int_G c(x) d(x^{-1})\, dx = 0.$$

Because of 1.1, this is evidently impossible for $s = t$, so $t \geq \kappa$.

Therefore, if $\pi_\lambda$ satisfies the condition (i) we have that

$$(\lambda | \alpha) \geq \kappa k(\alpha)$$

for all simple noncompact roots $\alpha \in \Psi$, what is apparently a condition weaker than (ii). To end the proof of the theorem we need only to establish the following result.

2.3. LEMMA. *Let $\lambda$ be dominant and regular. If the function*

$$\beta \to \frac{(\lambda | \beta)}{k(\beta)}$$

*on positive noncompact roots attains its minimum at $\alpha$, then $\alpha$ is a simple root.*

*Proof*. Firstly, it is evident that $k$ is a convex function constant on Weyl group orbits and $k(\gamma)$ depends only on the irreducible component of $\Phi$ in which $\gamma$ lies. This implies, in particular, that if $\gamma$ and $\delta$ are in the same irreducible component and $\|\gamma\| = \|\delta\|$, then $k(\gamma) = k(\delta)$.

Assume that $\alpha$ is not simple. Then there exist $\gamma, \delta \in \Psi$ such that $\alpha = \gamma + \delta$. We can assume that one of them is simple. Also they cannot both be either compact or noncompact, therefore we can assume that $\gamma$ is noncompact and $\delta$ compact. Evidently

$$(\lambda|\alpha) = (\lambda|\gamma) + (\lambda|\delta),$$

which implies

$$(\lambda|\gamma) < (\lambda|\alpha),$$

and by the choice of $\alpha$

$$\frac{(\lambda|\alpha)}{k(\alpha)} \leq \frac{(\lambda|\gamma)}{k(\gamma)}.$$

Therefore

$$\frac{k(\gamma)}{k(\alpha)} \leq \frac{(\lambda|\gamma)}{(\lambda|\alpha)} < 1,$$

i.e. $k(\gamma) < k(\alpha)$, so by the above remark $\|\gamma\| \neq \|\alpha\|$.

Let $S$ be the $\delta$-string containing $\gamma$ and $\alpha$. It cannot have length 2, so its length is either 3 or 4. The reflection $s_\delta$ maps $S$ into itself and

$$s_\delta\alpha = s_\delta\gamma - \delta;$$

therefore we have the following two possibilities: either the length of $S$ is 3 and

$$S = \{s_\delta\alpha, \gamma, \alpha\}$$

or the length of $S$ is 4 and

$$S = \{s_\delta\alpha, s_\delta\gamma, \gamma, \alpha\}.$$

Assume that $\delta$ is simple. Then $S$ consists of positive noncompact roots and

$$(\lambda|s_\delta\alpha) < (\lambda|\alpha);$$

also $k(s_\delta\alpha) = k(\alpha)$, so

$$\frac{(\lambda|s_\delta\alpha)}{k(s_\delta\alpha)} < \frac{(\lambda|\alpha)}{k(\alpha)}$$

contrary to our assumption.

Assume that $\gamma$ is simple. Then in both cases $\alpha - 2\delta \notin \Psi$. Therefore

$$(\lambda|\alpha) - 2(\lambda|\delta) < 0$$

and

$$(\lambda|\gamma) = (\lambda|\alpha) - (\lambda|\delta) < \tfrac{1}{2}(\lambda|\alpha).$$

Also, in both cases $\|\gamma\| = \|\delta\|$ and therefore $k(\gamma) = k(\delta)$, and

$$k(\alpha) \leq k(\gamma) + k(\delta) \leq 2k(\gamma).$$

Finally, this implies

$$\frac{(\lambda|\gamma)}{k(\gamma)} < \frac{(\lambda|\alpha)}{k(\alpha)},$$

which again contradicts our assumption. □

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