PACIFIC JOURNAL OF MATHEMATICS
Vol 121, No 1, 1986

A PROBLEM OF DOUGLAS AND RUDIN ON FACTORIZATION

J. BOURGAIN

If f is a bounded measurable function on the circle π, then $\int_\pi \log|f|\,dm > -\infty$ expresses the necessary and sufficient condition on $f \neq 0$ to be of the form $f = g \cdot \bar{h}$ where $g, h \in H^\infty$. This question was proposed by Douglas and Rudin in [1], where they approximate unimodular functions on π by quotients of Blaschke products.

Introduction. π stands for the circle group and m its normalized invariant measure. H^∞ will be considered as a (closed) subalgebra of $L^\infty = L^\infty(\pi)$. In [2], Douglas and Rudin consider the set $\tilde{Q}$ of these functions in L^∞ which are of the form $\phi\bar{\psi}$ with ϕ, ψ H^∞-functions. They noticed that then, by Jensen's inequality if $f = \phi\bar{\psi} \neq 0$

$$\int_\pi \log|f|\,dm = \int_\pi \left(\log|\phi| + \log|\psi|\right) dm > -\infty$$

has to be true and asked whether this property was also sufficient. If $\log|f|$ in $L^1(\pi)$, we may define the outer function

$$h(z) = \exp\left\{\int \log|f(e^{i\theta})| \frac{e^{i\theta}+z}{e^{i\theta}-z} m(d\theta)\right\}$$

for which $|h(z)| = |f(\theta)|$ if $z = e^{i\theta}$ a.e. Thus $h^{-1}(e^{i\theta})f(\theta)$ is an unimodular function on π and can be written as e^{ia} where a in $L^\infty(\pi)$ takes values in $[-\pi, \pi[$.

PROPOSITION. *There are Blaschke products B_1, B_2 such that*

$$\left\| \mathscr{H}\left[a - \operatorname{Arg}\frac{B_1}{B_2}\right]\right\|_\infty < c$$

where $\mathscr{H}$ denotes the Hilbert-transform and c is numerical. We consider Arg *as $[-\pi, \pi[$ valued.*

If $b = a - \operatorname{Arg} B_1/B_2$, we obtain the decomposition

$$e^{ia} = B_1 e^{iF}\overline{B_2 e^{-iF}}$$

taking F in H^∞ with $b = 2\operatorname{Re} F$. This will imply the result stated in the abstract.

Notice that as corollary $L^\infty = H^\infty \overline{H^\infty} + \mathbf{C}$ and f in $L^1(\pi)$ belongs to $H^2\overline{H^2}$ iff $\log|f|$ in L^1 (or $f = 0$).

To verify the first assertion, let $f \in L^\infty$ and define $g_\theta = f + e^{i\theta}$. Then, applying Jensen's inequality in the θ-variable

$$\int\left\{\int \log|g_\theta|\right\} m(d\theta) = \int\int \log|f(\psi)e^{-i\theta} + 1|m(d\theta)\, m(d\psi) \geq 0.$$

Hence $\int \log|g_\theta|\, m(d\theta) \geq 0$, $g_\theta \in H^\infty\overline{H^\infty}$ for some θ.

The second property is seen by writing

$$f = fe^{-a}F\bar{F}$$

where $a = \log|f|$ is in L^1, $F = \exp\frac{1}{2}(a + i\mathscr{H}[a])$ is the boundary value of an H^2-function. Finally fe^{-a} is again unimodular and hence in $H^\infty\overline{H^\infty}$.

Proof of Proposition. The argument is constructive. It will be based on L^1-approximation of L^∞_R functions by elements of $\operatorname{Re} H^\infty$ and the constructive proof given by P. Jones of the Douglas-Rudin approximation theorem (see [**3**]).

LEMMA 1. *If a in $L^\infty_R(\pi)$ and $\varepsilon > 0$, there is b in $L^\infty_R(\pi)$ satisfying $\|a - b\|_1 < \varepsilon\|a\|_\infty$ and $\|b\|_\infty + \|\mathscr{H}[b]\|_\infty < c_1 \log(1/\varepsilon)\|a\|_\infty$.*

Proof. We may clearly suppose $\|a\|_\infty \leq 1$. Define the BMOA function $A = a + i\mathscr{H}[a]$ and the outer function τ

$$\tau(z) = \exp\left\{\int_\pi \log\alpha(\theta)\frac{e^{i\theta} + z}{e^{i\theta} - z} m(d\theta)\right\} \quad \text{where } \alpha^{-1} = \max(1, \delta|A|).$$

Since $\tau = \alpha \exp i\mathscr{H}[\log\alpha]$ on π, it follows that for $|z| = 1$

$$|\tau A| = \alpha|A| \leq \delta^{-1}$$

and hence $b = \operatorname{Re}\tau A$ satisfies $\|b\|_\infty + \|\mathscr{H}[b]\|_\infty \leq \sqrt{2}\,\delta^{-1}$. Also

$$|a - b| \leq |1 - \operatorname{Re}\tau|\,|a| + |\operatorname{Im}\tau|\,|A|$$

implying

$$\begin{aligned}
\|a - b\|_2 &\leq \|1 - \alpha\cos\mathscr{H}[\log\alpha]\|_2 + \delta^{-1}\|\sin\mathscr{H}[\log\alpha]\|_2 \\
&\leq \|1 - \alpha\|_2 + 2\delta^{-1}\|\mathscr{H}[\log\alpha]\|_2 \\
&\leq m\big[|A| > \delta^{-1}\big]^{1/2} + 2\delta^{-1}\left\{\int_{[|A|>\delta^{-1}]}\big(\log\delta|A|\big)^2\right\}^{1/2}
\end{aligned}$$

Since $m[|A| > \lambda] \leq c^{-1}e^{-c\lambda}$ for numerical $c > 0$, the latter quantity is dominated by $c_2 e^{-c/3\delta}$. Taking $\delta \sim (\log 1/\varepsilon)^{-1}$, the lemma follows.

The next fact is a consequence of the proof of Th. 5.1 in [2].

LEMMA 2. *Assume a in $L^\infty(\pi)$ taking values in $[-\pi, \pi[$ and $a = 0$ outside some measurable set U of π. Then, for given $\varepsilon > 0$, there are Blaschke products B_1, B_2 fulfilling*

$$\left\| a - \operatorname{Arg}\frac{B_1}{B_2} \right\|_\infty < \varepsilon \quad \textit{and} \quad \sum_{B_1(z)B_2(z)=0} (1 - |z|) < \frac{c_2}{\varepsilon} m(U).$$

The method consists in covering U by a countable family of disjoint intervals which union has approximately the same measure as U and then starting consecutive approximations in L^1-norm using Lemma 5.5 of [2]. The construction yields moreover that the zeros of B_1, B_2 form an interpolating sequence, which will, however, not be used here.

Proof of the Proposition. We make an iteration construction which is again in the spirit of the proof of Theorem 5.1 of [2]. Starting from a in $L^\infty(\pi)$ with values in $[-\pi, \pi[$, a first application of Lemma 2 permits to replace a by a function of small L^∞-norm. We show now that if a in $L^\infty_R(\pi)$ satisfies $\|a\|_\infty < \gamma$ ($\gamma > 0$ sufficiently small), there is a decomposition

$$a = b + \operatorname{Arg}\frac{B_1}{B_2} + a_1 \qquad (a_1, b \text{ real})$$

where

(i) $\|b\|_\infty + \|\mathscr{H}[b]\|_\infty \le c_3(\log 1/\gamma)\gamma$
(ii) $\sum_{B_1(z)B_2(z)=0}(1 - |z|) \le c_3\gamma$
(iii) $\|a_1\|_\infty \le 2\gamma^2$.

Iteration provides then the required Blaschke products B_1', B_2' in the form $B_j' = \prod_s B_j^{(s)}$ ($j = 1, 2$), where (ii) bounds $\sum_{B_j'(z)=0}(1 - |z|)$ by the sum of an obviously converging series. The difference

$$a - \operatorname{Arg}\frac{B_1'}{B_2'} = \sum_{s=0}^{\infty}\left(a_s - \operatorname{Arg}\frac{B_1^{(s)}}{B_2^{(s)}} - a_{s+1}\right) = \sum b_s$$

then has a bounded Hilbert transform in view of (i).

To prove the decomposition stated above, apply first Lemma 1 with $\varepsilon = \gamma^4$ to obtain b satisfying

$$\|b\|_\infty + \|\mathscr{H}[b]\|_\infty < 4c_1\left(\log\frac{1}{\gamma}\right)\gamma \quad \text{and} \quad \|a - b\|_1 < \gamma^5.$$

For γ small, the function $a - b$ is still $]-\pi, \pi[$ valued. Moreover, the set $U = [|a - b| > \gamma^2]$ has measure less than γ^3. Application of Lemma 2 to the function $f = (a - b)\chi_U$ gives Blaschke products B_1 and B_2 so that

$$\left\| f - \operatorname{Arg}\frac{B_1}{B_2} \right\|_\infty < \gamma^2 \quad \text{and} \quad \sum_{B_1(z)B_2(z)=0} (1 - |z|) < c_2\gamma.$$

Put $a_1 = a - b - \operatorname{Arg}(B_1/B_2)$, then $\|a_1\|_\infty < 2\gamma^2$, completing the proof.

References

[1] R. G. Douglas and W. Rudin, *Approximation by inner function*, Pacific J. Math., **31**, No. 2, (1969), 313–320.

[2] J. Garnett, *Bounded Analytic Functions*, Academic Press, 1967.

[3] P. Jones, *Ratios of interpolating Blaschke products*, Pacific J. Math., in press.

Received April 10, 1984 and in revised form September 12, 1984.

Vrije Universiteit Brussel
Pleinlaan 2, 10F
1050 Brussels, Belgium