

A NORMAL FORM AND INTEGRATION IN FINITE TERMS FOR A CLASS OF ELEMENTARY FUNCTIONS

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Let C be the field of complex numbers, E the usual exponential on C . So (C, E) is an exponential field.

We define an exponential ring extension $C\{x\}^E$ of (C, E) and give a functional representation: $C\{x\}^E$ is isomorphic to the smallest exponential ring extension of (C, E) containing the functions x^s , x a real and positive variable, and $s \in C$.

Finally, we give a simple integration-in-finite-terms algorithm for elements of $C\{x\}^E$.

Introduction. An “exponential ring” (R, E) or “ E -ring” is an algebraic structure with R a commutative ring with 1 and $E: R \rightarrow R$ (the “exponential”) a map such that

$$\begin{aligned} E(a + b) &= E(a)E(b) \quad \text{for all } a, b \in R \\ E(0) &= 1. \end{aligned}$$

A “derivation” on (R, E) is a map $D: R \rightarrow R$ such that for all $a, b \in R$ we have:

$$\begin{aligned} D(a + b) &= Da + Db \\ Dab &= (Da)b + a(Db) \\ DE(a) &= E(a)Da. \end{aligned}$$

In [5] the concept of “ring of E -polynomials” in the indeterminate x is defined and denoted “ $R[x]^E$ ”. It is an exponential ring extension of R , generated as an exponential ring by the set $R \cup \{x\}$. A unique derivation D can be defined on $R[x]^E$ such that $Dr = 0$ for $r \in R$, and $Dx = 1$. (More generally, the ring $R[x_1, \dots, x_n]^E$ is defined in a similar way in [5] but we will not deal with it here.)

In §1, we define an exponential differential ring extension $R\{x\}^E$ of a given (R, E) . The ring $R[x]^E$ is the exponential subring of $R\{x\}^E$ generated by the set $R \cup \{x\}$. It should be remarked however that $R[x]^E$ contains an isomorphic copy of $R\{x\}^E$, namely $R[E(x)]^E$ which is the exponential subring generated by $R \cup \{E(x)\}$.

The main purpose of this article is to give an integration-in-finite-terms algorithm for elements of $\mathbf{C}\{x\}^E$, with (\mathbf{C}, E) the field of complex numbers with the usual exponential $E(z) = 1 + z/1! + \dots$. Thanks are due to Professor Maxwell Rosenlicht who suggested a study of §3 (3.1) which implies in particular that if an element of $\mathbf{C}[x]^E$ has an elementary integral this integral must be an element of $\mathbf{C}[x]^E$. \square

The general problem of integration in finite terms was first studied by J. Liouville [2] in the context of complex functions rather than abstract rings. There is a natural E -morphism $f \rightarrow \hat{f}$, $f \in R[x]^E$ onto the exponential ring $R[x]^E$ of functions from R to R , generated by the identity $x: R \rightarrow R$ and the constant functions $r \in R$ and such that $\hat{x} = x$ and $\hat{r} = r$. The following interesting assertions are proved in [5].

Let R be an integral domain of characteristic 0. Then

- (a) The ring of constants (i.e. the ring of elements having zero derivative) of $R[x]^E$ is exactly R .
- (b) The morphism “ $\hat{}$ ” is an isomorphism.

In fact, an n -dimensional version of this is proved in [5].

In a previous version of the present paper, we gave a natural isomorphic representation

$$\mu: \mathbf{C}\{x\}^E \rightarrow \mathbf{C}\{x\}^E.$$

With $\mathbf{C}\{x\}^E$ the ring of complex valued functions of a real and positive variable x , generated, as an exponential ring, by constant functions $r \in \mathbf{C}$ and the functions $x^s = E(s \lg x)$ with $s \in \mathbf{C}$. This result was established by completely different methods to those used in [5]. In fact, it is not difficult to see that the injectivity of μ implies (a), (b) and conversely (the surjectivity of μ is a trivial fact).

Finally, we remark that the problem of integration in finite terms, even in the particular case of exponential polynomials, is undecidable in a strict sense (see [3], [4]). This is because of the impossibility of deciding equality or not between two given numbers. Of course we have such a decision method if the numbers belong to a given algebraic extension of \mathbf{Q} .

Our algorithm reduces the integration problem to the question of deciding if certain given numerical polynomial equations are satisfied for given values of the indeterminates, and this is the best possible answer.

1. Throughout this section (R, E) is an exponential ring. To simplify the exposition it will be assumed from the beginning that R is an integral domain of characteristic 0.

We are going to define a differential exponential ring extension $R\{x\}^E$ of R .

Let S be a set of indeterminates, having the same cardinal as the set R , and let $x \in S$ be a fixed element.

Notation. The term “monomial” will denote an expression of type $c_1 x_1^{s_1} \cdots x_m^{s_m}$, $x_i \in S$, $c_i, s_i \in R$. We will write $x_i^0 = 1$. A “polynomial” is a sum of monomials. We will often use a notation of type “ $z^{\bar{s}}$ ”, with $z = x_1 \cdots x_m$, $\bar{s} = (s_1, \dots, s_m)$ to denote the monomial $x_1^{s_1} \cdots x_m^{s_m} = z^{\bar{s}}$. In particular we identify $z = z^{(1,1,\dots,1)}$.

Let S be partitioned as a disjoint union:

$$S = \{x\} \cup \bigcup_{n=1}^{\infty} S_n$$

where each S_n has the same cardinal number as the set R .

Define $R\{x\}_k^E$, $k = -1, 0, 1, 2, \dots$, as follows: $R\{x\}_{-1}^E = R$. $R\{x\}_0^E$ is the ring of polynomials in the indeterminate x . $R\{x\}_n^E$ is the ring of polynomials in indeterminates belonging to $\{x\} \cup S_1 \cup \cdots \cup S_n$.

Let $R\{x\}^E$ be the ring of all polynomials in indeterminates belonging to S , so we have

$$R\{x\}^E = \bigcup_{k=-1}^{\infty} R\{x\}_k^E.$$

For each $n \geq 0$, choose a one-to-one surjective function E_n from the set monomials with unit coefficient of type

$$M = x^s x_1^{s_1} \cdots x_m^{s_m} \in R\{x\}_n^E - R\{x\}_{n-1}^E$$

onto the set S_{n+1} , say: $E_n(M) = x_M$. (Note that these sets both have the same cardinal as the set R .) Then extend the functions E_n to a function $E: R\{x\}^E \rightarrow R\{x\}^E$ in such a way that

$$\begin{aligned} E(a + b) &= E(a)E(b) && \text{for all } a, b \in R\{x\}^E \\ E(0) &= 1 \\ E(sM) &= (x_M)^s && \text{for each monomial with unit} \\ &&& \text{coefficient } M, s \in R. \end{aligned}$$

So we have defined an exponential ring $R\{x\}^E$ which is an exponential ring extension of R , generated by the set $R \cup \{x\}$. □

Before we define a derivation on $R\{x\}^E$ we will state the following easy proposition and a definition:

1.1. PROPOSITION. $R\{x\}_n^E$ is a free $R\{x\}_{n-k}^E$ -module. The set of monomials in indeterminates belonging to $R\{x\}_n^E - R\{x\}_{n-k}^E$ is a free basis for $0 \leq k \leq n$.

In particular, we can write each $a \in R\{x\}_n^E$, $n \geq 0$, uniquely as follows:

$$a = c_1(x)E(a_1) + \cdots + c_q(x)E(a_q)$$

with $c_i(x) \in R\{x\}_0^E - \{0\}$; $a_i \in R\{x\}_{n-1}^E$ a polynomial having constant term 0, $a_i \neq a_j$ if $i \neq j$.

1.2. DEFINITION. The expression $a = c_1(x)E(a_1) + \cdots$ is the “normal form” of the element a .

Using the normal form we can define by induction on m a unique derivation D on $R\{x\}_n^E$, $n \geq 1$, and then on $R\{x\}^E$, having the properties

$$\begin{aligned} Dx^s &= sx^{s-1}, & s \in R, \\ Da &= 0, & \text{if } a \in R. \end{aligned}$$

This completes the definition of the differential exponential ring $R\{x\}^E$. Now we will define a “change of variable” to get an isomorphic representation:

$$\nu: R\{x\}^E \rightarrow R\{x_0\}^E$$

with

$$R\{x_0\}^E \subset R\{x\}^E$$

as follows:

Let x_0 be the indeterminate $E(x)$ and $R\{x_0\}^E$ the exponential subring of $R\{x\}^E$ generated by $R \cup \{x_0\}$. Define “ ν ” as the unique exponential ring isomorphism such that $\nu|_R = 1_R$, $\nu(cx^s) = cx_0^s$ for all $c, s \in R$.

It is often simpler (and equivalent) to deal with $R\{x_0\}^E$ instead of $R\{x\}^E$. This is because the normal form for an element $a \in R\{x_0\}_n^E$, $n \geq 0$, has simpler coefficients c_i and exponents a_i , namely:

$$a = c_1E_1(a_1) + \cdots + c_qE(a_q)$$

with $c_i \in R - \{0\}$; a_i a polynomial having constant term 0 of type

$$a_i \in R\{x_0\}_{n-1}^E + \{sX\}_{s \in R}$$

and $a_i \neq a_j$ if $i \neq j$.

REMARK. Note that ν is not a differential morphism. Moreover, we have

$$D\nu(a) = x_0\nu(D) \quad \text{for } a \in R\{x\}^E.$$

1.3. PROPOSITION. *The ring of constants of $R\{x\}^E$ is exactly R .*

Proof. Since $R\{x_0\}^E$ is a subring of $R[x]^E$ it follows from (a) in the introduction that the ring of constants of $R\{x_0\}^E$ is R . Using the Remark, the assertion follows at once.

2. We are going to establish a formula to calculate Df , $f \in R\{x_0\}^E$, (R, E) an exponential ring without zero divisors and characteristic 0.

For a given $h \in R\{x_0\}^E$ we can effectively determine the smallest set of indeterminates, say: $\{x_0, x_1, \dots, x_m\}$ such that, for some $s_{ij}, s'_{ki} \in R$ and $s_k \in \{0, 1\}$ ($s_k = 0$ iff $k \neq 0$) and $c_i \in R$, the following equalities hold:

$$h = \sum_i c_i x_0^{s_{i0}} x_1^{s_{i1}}, \dots, x_m^{s_{im}}$$

$$x_k = E(x_0^{s_k} x_0^{s'_{k0}}, \dots, x_m^{s'_{km}}) \quad \text{for } 0 \leq k \leq m.$$

2.1. DEFINITION. We will call x_0, \dots, x_m the indeterminates “associated to h ” and s_{i0}, s'_{ki}, s_k the elements “associated to h ”.

The proof of the following proposition is immediate.

2.2. PROPOSITION. *The relation “ x_i associated to x_j ” is a partial order on the set of indeterminates belonging to $R\{x_0\}^E$, and x_0 is the first element. The set of indeterminates associated to h is a finite set. In particular, for each x_i associated to h there exists a maximal x_j associated to h such that: x_i associated to x_j . If $x_i \in R\{x_0\}_n^E$ and x_i associated to x_j , $x_i \neq x_j$, then $x_j \in R\{x_0\}_{n+1}^E + R\{x_0\}_n^E$ for some $l \geq 1$. l is exactly 1 if x_j is an immediate successor of x_i .*

Notation. Let $z = x_0, x_1, \dots, x_m \in R\{x_0\}_n^E - R\{x_0\}_{n-1}^E$, $n \geq 0$, with x_0, \dots, x_m the indeterminates associated to z . Set $z = z_0 z_1 \cdots z_n$ with z_i the product of those indeterminates x_j such that $x_j \in R\{x_0\}_i^E - R\{x_0\}_{i-1}^E$. Let $R^{m+1} = V_0 \oplus \cdots \oplus V_n$ with $V_i = \{(\lambda_0, \dots, \lambda_m) \in R^{m+1}; \lambda_j = 0 \text{ if } x_j \text{ is not a factor of } z_i\}$, and let $P_i: R^{m+1} \rightarrow V_i$ be the projection.

By studying the expression of the derivative of an indeterminate: $x_k = E(x^{s_k} x_0^{s'_k} \cdots x_m^{s'_m})$ it follows that the derivative $Dz^{\bar{t}}$ (for a generic \bar{t}) has a uniquely determined expression of type:

$$Dz^{\bar{t}} = \sum_{j=1}^q d_j(\bar{t}) z^{\bar{t} + \bar{\omega}_j}.$$

The vectors $\bar{\omega}_j \in R^{m+1}$ and the R -linear functionals $d_j \neq 0$, $d_j(\bar{t}) = a_0 t_0 + \cdots + a_m t_m$ can be effectively determined for a given z . The components of the $\bar{\omega}_j$ are certain sums of the s'_{kj} , s_k , and the a_i are certain polynomials in the s'_{kj} , s_k .

It is clear that

$$\begin{aligned} Dz^{\bar{t}} &= \sum_{i=0}^n \left(\prod_{k \neq i} z^{P_k(\bar{t})} \right) \sum_{j=1}^q d_j(P_i(\bar{t})) z^{P_i(\bar{t}) + \bar{\omega}_j} \\ &= \sum_{j=1}^q \sum_{i=0}^n d_j(P_i(\bar{t})) z^{\bar{t} + \bar{\omega}_j}. \end{aligned}$$

Set $I = \{1, \dots, q\}$ and for $0 \leq i \leq n$, set $I_i = \{j \in I: d_j(P_i(\bar{t})) \neq 0\}$. Observe that

$$Dz_i^{P_i(\bar{t})} = \sum_{j \in I_i} z_i^{P_i(\bar{t}) + \bar{\omega}_j}.$$

We have the following (almost) obvious facts:

- (i) $I_i \neq \emptyset$ for $i = 0, \dots, n$ (this follows from $P_i(t) \neq 0$ and the fact that the ring of constants, i.e. the ring of those elements having zero derivative = R). In particular, I_0 contains exactly one element, say $j \in I_0$ and we have: $\bar{\omega}_j = 0$.
- (ii) For $1 \leq i \leq n$, $j \in I_i$, we have $P_l(\bar{\omega}_j) = 0$ for $l \geq i$, $P_{l-1}(\bar{\omega}_j) \neq 0$. (To conclude this, you only have to observe the expression of the derivative of an indeterminate.)
- (iii) $I = \bigcup_{i=0}^n I_i$, disjoint union (use (ii)).
- (iv) If $j \in I_i$ then $d_j(\bar{t}) = d_j(P_i(\bar{t}))$ (see the above expressions of $Dz^{\bar{t}}$).
- (v) If x_k is a maximal indeterminate in the set $\{x_0, \dots, x_m\}$ ordered under "associated to" and if we write $\bar{\omega}_j = (\omega_{j0}, \omega_{j1}, \dots, \omega_{jm})$, then $\omega_{jk} = 0$ for $j = 1, \dots, q$.
- (vi) x_k is associated to $z^{\bar{t}}$ if and only if x_k is associated to $Dz^{\bar{t}}$. (It suffices to prove the case in which x_k is a maximal indeterminate and this follows at once, using (v).)

The following proposition is the main result in this section. It can be easily established using the above results.

2.2. PROPOSITION. Let $f \in R\{x_0\}^E$, $f = \sum_{i=1}^n c_i z^{i\bar{\omega}_i}$, $z = x_0 \cdots x_m$ with x_0, \dots, x_m the indeterminates associated to f . Then

$$Df = \sum_{i=1}^n \sum_{j=1}^q c_i d_j(\bar{\omega}_i) z^{i+\bar{\omega}_j}.$$

The $d_j, \bar{\omega}_j$ have the properties (i) to (vi).

3. This section deals with integration in finite terms of elements of $R\{x\}^E$ in the particular case $R = \mathbf{C}$, E being an arbitrary exponential on \mathbf{C} (for example: $E(z) = 1 + z/1! + \cdots$).

The general case of an arbitrary (R, E) with R an integral domain of characteristic 0 is a consequence. In fact, only finitely generated field extensions of \mathbf{Q} (the prime ring of R) are essentially involved in proofs and these are isomorphic to subfields of \mathbf{C} .

3.1. PROPOSITION. Let $f \in \mathbf{C}\{x\}^E$, $f = \sum_{i=1}^l c_i x^{s_i} x_1^{s_{i1}} \cdots x_m^{s_{im}} + cx^{-1}$ with $(s_i, s_{i1}, \dots, s_{im}) \neq (-1, 0, \dots, 0)$ for $i = 1, \dots, l$. Then $\int f$ is elementary iff $\int (f - cx^{-1}) \in \mathbf{C}\{x\}^E$.

Proof. Using the “change of variable” ν we can see that the statement is equivalent to the following:

$$(*) \quad \begin{aligned} &\text{Let } f \in \mathbf{C}\{x_0\}^E, \quad f = \sum_i c_i x^{s_{i0}} \cdots x_m^{s_{im}} + c \quad \text{with} \\ &(s_{i0}, \dots, s_{im}) \neq (0, \dots, 0). \quad \text{Then } \int f \text{ is elementary iff} \\ &\int (f - c) \in \mathbf{C}\{x_0\}^E. \end{aligned}$$

To prove (*), let x_0, \dots, x_m be the indeterminates associated to f ; $b_1, \dots, b_q \in \mathbf{R}$ a basis of the free \mathbf{Z} -module generated by the real and imaginary parts of the numbers associated to f , and set:

$$\{U_1, \dots, U_r\} = \{x_i^{b_j}, x_k^{\sqrt{-1}b_l}\}.$$

The U_i are algebraically independent over \mathbf{C} , and f can be written

$$f = \sum_i a_i U^{n_i} \cdots U_r^{n_r}, \quad a_i \in \mathbf{C} - \{0\}, \quad n_{ij} \in \mathbf{Z}$$

so $f \in \mathbf{C}(U_1, \dots, U_r)$.

$\mathbf{C}(U_1, \dots, U_r)$ is a differential subfield of the field of quotients of $\mathbf{C}\{x_0\}^E$. By Liouville's Theorem (see [3], [4]) we have that, for some $Q \in \mathbf{C}(U_1, \dots, U_r)$, $P_k \in \mathbf{C}[U_1, \dots, U_r]$, P_k irreducible, $c_k \in \mathbf{C}$, we have that

$$f = DQ + \sum_k c_k \frac{DP_k}{P_k}.$$

Now (as in [4, Thm. 2]) we can see that if $P \in \mathbf{C}[U_1, \dots, U_r]$ is not a monomial (recall our definition of “monomial” in §1), then P does not divide DP . In fact, let $P = \sum_j g_j U^{n_j} \cdots U_p^{n_{jp}} \in \mathbf{C}\{x_0\}_n^E$, with $U_1, \dots, U_r \in \mathbf{C}\{x_0\}_n^E - \mathbf{C}\{x_0\}_{n-1}^E$, $U_{p+1}, \dots, U_r, g_j \in \mathbf{C}\{x_0\}_{n-1}^E + \{0\}$; $(n_{j_1}, \dots, n_{j_p}) \neq (n_{i_1}, \dots, n_{i_p})$ if $i \neq j$. Then we have

$$DP = \sum_j g_j U^{n_{j_1}} \cdots U_p^{n_{jp}} \left(\frac{Dg_j}{g_j} + n_{j_1} \frac{DU_1}{U_1} + \cdots + n_{jp} \frac{DU_p}{U_p} \right).$$

If DP is a multiple of P then

$$\frac{Dg_j}{g_j} + n_{j_1} \frac{DU_1}{U_1} + \cdots + n_{jp} \frac{DU_p}{U_p}$$

is independent of j_1 , giving

$$D(g_i U_1^{n_{i_1}} \cdots U_p^{n_{ip}} / g_j U_1^{n_{j_1}} \cdots U_p^{n_{jp}}) = 0$$

which is not true for $i \neq j$.

It follows by simple reasoning (for example, using simple partial fraction decomposition) that the P_k are monomials, say $P_k = E(N_k)$, with $N_k = M_k + s_k x$, for some $M_k \in \mathbf{C}\{x_0\}_n^E$, $s_k \in \mathbf{C}$; and that Q is a sum of monomials. We can conclude that if $c = \sum_k s_k$, then

$$\int (f - c) \in \mathbf{C}\{x_0\}_n^E.$$

This finishes the proof.

3.2. Description of the algorithm. We will look at an algorithm to decide, for a given $f \in \mathbf{C}\{x\}_n^E$, if $\int f$ is elementary, and if this is the case, to find it.

As we know, we can reduce the above problem to the following: For a given $f \in \mathbf{C}\{x_0\}_n^E$ without constant term, decide if there is $g \in \mathbf{C}\{x_0\}_n^E$ such that $Dg = f$, and if this is the case, find it.

Let $f = \sum_{i=1}^M c_i z^{\tilde{i}}$ be given with $\tilde{i}_i \neq 0$; $z = x_0 x_1 \cdots x_m = z_0 z_1 \cdots z_n$, x_0, \dots, x_m the indeterminates associated to f and z_i the product of the indeterminates $x_j \in \mathbf{C}\{x_0\}_i^E - \mathbf{C}\{x_0\}_{i-1}^E$, as in §2.

If there exists $g \in \mathbf{C}\{x_0\}_n^E$ such that $Dg = f$ we can deduce from §2 (vi) that it must be of type

$$g = \sum_{l=1}^N b_l z^{\bar{s}_l}, \quad \bar{s}_l \neq \bar{s}_k \text{ if } l \neq k, \quad b_l \in \mathbf{C} - \{0\}.$$

So the problem finally reduces to the following:

To find N , b_l , \bar{s}_l , $l = 1, \dots, N$, such that

$$(**) \quad \sum_{i=1}^M c_i z^{\bar{t}_i} = \sum_{l=1}^N \sum_{j=1}^q d_j(\bar{s}_l) b_l z^{\bar{s}_l + \bar{\omega}_j}$$

or to establish that they do not exist.

The case $n = 0$, i.e.: $f = \sum_{i=1}^M c_i z_0^{\bar{t}_i} = \sum_{i=1}^M c_i x_0^{\bar{t}_i}$, $t_i \neq 0$ is easy. In fact, we have

$$g = \sum_{i=1}^N \left(\frac{c_i}{t_i} \right) \cdot x_0^{\bar{t}_i}.$$

Assume from now on that $n \geq 1$. Besides, since $P_n(\bar{\omega}_j) = 0$ for $j \in I_l$ (§2 (ii)), we can assume that $0 \neq P_n(\bar{t}_i)$ is independent of i and that $P_n(\bar{t}_i) = P_n(\bar{s}_l)$ for all i, l (because we can arrange terms and decompose (**)) into several equalities having this property). It follows under this assumption that if $j \in I_n$, then $d_j(\bar{t}_i) = d_j(P_n(\bar{t}_i)) = d_j(P_n(\bar{s}_l)) = d_j(\bar{s}_l) = d_j$ does not depend on l, i .

The general idea of the algorithm is as follows:

Choose a set of $2(m+1)$ linearly independent \mathbf{R} -linear functionals $\phi_i: \mathbf{C}^{m+1} \rightarrow \mathbf{R}$ and define a linear order “ $<$ ” on \mathbf{C}^{m+1} by means of:

$\bar{a} < \bar{b}$ iff for some $k = 1, 2, \dots, 2(m+1)$ we have:

$$\phi_k(\bar{a}) < \phi_k(\bar{b}) \text{ and } \phi_l(\bar{a}) = \phi_l(\bar{b}) \text{ for } l < k.$$

Changing the indices if necessary, we have:

$$\bar{t}_1 < \bar{t}_2 < \dots; \quad \omega_1 < \omega_2 < \dots; \quad \bar{s}_1 < \bar{s}_2 < \dots.$$

Assume that the ϕ_i are chosen in such a way that, for some $j_0 \in I_n$ we have:

$$d_{j_0} \neq 0 \quad \text{and} \quad d_j = 0 \quad \text{if } j < j_0.$$

The non-zero element having lowest exponent in the matrix of elements:

$$(***) \quad b_l d_j(\bar{s}_l) z^{\bar{s}_l + \bar{\omega}_j}$$

is

$$b_1 d_{j_0}(\bar{s}_1) z^{\bar{s}_1 + \omega_{j_0}}.$$

From this and (**)) we can deduce that: $b_1 = c_1/d_{j_0}$; $\bar{s}_1 = \bar{t}_1 - \bar{\omega}_{j_0}$.

To calculate b_2, \bar{s}_2 , set

$$R^1 f = D(g - b_1 z^{\bar{s}_1}) = f - D \frac{c_1}{d_{j_0}} z^{\bar{t}_1 - \bar{\omega}_{j_0}} = \sum_i a_{1,i} z^{\bar{u}_{1,i}}$$

with $a_{1,i} \in \mathbf{C} - \{0\}$, $\bar{U}_{1,1} < \bar{U}_{1,2} < \bar{U}_{1,3} < \dots$. Since

$$R^1 f = D \left(\sum_{i=2}^N b_i z^{\bar{s}_i} \right) = \sum_{i=2}^N \sum_{j=j_0}^q b_i d_j(\bar{s}_i) z^{\bar{s}_i + \bar{\omega}_j}$$

and since terms of lowest exponents must be equal, we have

$$b_2 = \frac{a_{11}}{d_{j_0}}; \quad \bar{s}_2 = \bar{U}_{11} - \bar{\omega}_{j_0}.$$

More generally, define

$$R^l f = \sum_i a_{li} z^{\bar{U}_l}$$

by induction as follows:

- (1) $R^0 f = f$
- (2) If $R^l f \neq 0$ then

$$R^{l+1} f = R^l f - D \left[\frac{a_{l1}}{d_{j_0}} z^{\bar{U}_l - \bar{\omega}_{j_0}} \right],$$

and, if $R^l f = 0$ then $R^{l+1} f = 0$.

We can see that, if $R^l f \neq 0$ then $\bar{U}_l < \bar{U}_{l+1,1}$ and $\bar{s}_{l+1} = \bar{U}_l - \bar{\omega}_{j_0}$; $b_{l+1} = a_{l1}/d_{j_0}$. We have:

$$R^k f = D \left(g - \sum_{l=1}^k b_l z^{\bar{s}_l} \right) = f - \sum_{l,j} b_l d_j(\bar{s}_l) z^{\bar{s}_l + \bar{\omega}_j}.$$

A useful picture of the situation is the following: $R^k f$ is the difference of f minus the elements belonging to the k first columns of the matrix of elements (***) . Observe that only the elements belonging to the same ascending diagonal may have equal exponents. In particular, since $\bar{s}_N + \bar{\omega}_q > \bar{s}_l + \bar{\omega}_j$ if $j < q$ or $l < N$, we have that $b_N d_q(\bar{s}_N) z^{\bar{s}_N + \bar{\omega}_1}$ could be cancelled only with some term $c_l z^{\bar{s}_l}$ of f and not with some $b_l d_j(\bar{s}_l) z^{\bar{s}_l + \bar{\omega}_j}$.

It is clear that if we repeatedly apply the algorithm, we will finally reach a step, say N , such that one and only one of the following assertions holds:

- (A) $R^{N-1} f \neq 0$, $R^N f = 0$
- (B) $R^l f \neq 0$ for all $l \geq N$.

If (A) holds, then $g = \sum_{l=1}^N b_l z^{\bar{s}_l}$ is a solution of $Dg = f$. If (B) holds, the problem $Dg = f$ has no solution.

Now we will see how to calculate a bound \tilde{N} for N . \tilde{N} has the property that if we reach the step \tilde{N} and $R^{\tilde{N}} f \neq 0$, then we have: $R^l f \neq 0$ for $l \geq \tilde{N}$, which implies that we are in case (B). So, if there is a solution it must be found at a certain step $N \leq \tilde{N}$.

We will introduce the following notation:

$$j_2(\bar{a}) = \max\{j: d_j(\bar{a}) \neq 0\}$$

so in the expression

$$Dg = \sum_{i,j} d_j(\bar{s}_i) z^{\bar{s}_i + \bar{\omega}_j}$$

the sum is extended to $1 \leq j \leq j_2(\bar{s}_i)$. Besides, we define

$$J = J_1 \cup (I - I_n), \quad \text{with } J_1 = \{j \in I_n: d_j \neq 0\}$$

(observe that $J_1 \neq 0$).

There are two cases to be considered regarding the position of the vectors $\bar{\omega}_j$ in the \mathbf{C}^{m+1} space. The idea is as follows: Given f we have a z determined as explained before and, hence, the $\bar{\omega}_j$ are also determined. Then we must check if we are in Case 1 or in Case 2, which will depend on the position of the vectors $\bar{\omega}_j$ in $\mathbf{C}^{m+1} \cong \mathbf{R}^{2(m+1)}$ space. Then we have to choose the functionals $\phi_1, \dots, \phi_{2(m+1)}$ to fill the condition assumed for the case we are in. All this is a matter of elementary geometry in \mathbf{C}^{m+1} . To easily understand the description of the algorithm, it will be useful to take into account a few remarks.

Let $\mathbf{C}^{m+1} = v_0 \oplus \dots \oplus v_n$, $P_i: \mathbf{C}^{m+1} \rightarrow V_i$ be the projection as in §2. A given \mathbf{R} -linear function $\phi: \mathbf{R}^{2(m+1)} \rightarrow \mathbf{R}$ can always be written: $\phi(\bar{x}) = (\bar{x}, \bar{a})$, “ $(\ , \)$ ” meaning standard inner product in $\mathbf{R}^{2(m+1)}$. For given vectors $\bar{b}_1, \bar{b}_2, \dots$ we can choose \bar{a} and the indices “ j ” of the \bar{b}_j in such a way that $\phi(\bar{b}_1) < \phi(\bar{b}_2) < \dots$.

The following observations will be especially useful in Case 2. Assume that $\phi_1(x) = (x, -v)$, and that $\phi_1(\bar{\omega}_1) \leq \phi_1(\bar{\omega}_2) \leq \dots$. We can find an arbitrarily small vector $\bar{\varepsilon}$ such that

$$\text{If } \phi_2(x) = (\bar{x}, -\bar{v} + \bar{\varepsilon}) \text{ then } \phi_2(\bar{\omega}_1) < \phi_2(\bar{\omega}_2) \dots$$

Moreover, if we know that $\bar{v} \in V_{n-1}$ and $\phi_1(\bar{\omega}_j) \leq 0$, $j \in J$, we can deduce (using §2 (ii)) that $\bar{\varepsilon}$ can be chosen in such a manner that: $\bar{\varepsilon} \in V_0 \oplus \dots \oplus V_{n-2}$. If, in addition, we fix $j_1 \in I_{n-1}$, we can choose $\bar{\varepsilon}$ in such a manner that $j \geq j_1$ implies $j \in I_{n-1}$. (If $n > 2$ it is enough to choose $\bar{\varepsilon} = \bar{\varepsilon}' + \bar{\varepsilon}''$ with $\bar{\varepsilon}'$ having the same direction and sense as $P_{n-2}(\bar{\omega}_{j_1})$ and $\bar{\varepsilon}'' \in V_0 \oplus \dots \oplus V_{n-3}$ small. If $n = 2$, $I_{n-2} = \{j\} = I_0$.)

Case 1. In this case we assume that there are at least two indices $j, l \in I_n$ such that $P_{n-1}(\bar{\omega}_j), P_{n-1}(\bar{\omega}_l) \neq 0$, do not have the same direction and sense. This implies that we can choose an \mathbf{R} -linear function such that

(changing indices if necessary),

$$(1) \quad \phi_1(\bar{t}_1) < \phi_1(\bar{t}_2) < \cdots; \phi_1(\bar{\omega}_1) < \phi_1(\bar{\omega}_2) < \cdots$$

(2) For some $j_0, j_1 \in J_1$ we have:

$$(i) \quad j_0 < j_1$$

$$(ii) \quad j < j_0 \text{ or } j > j_1 \text{ implies } j \in I_n \text{ and } d_j = 0.$$

Now we choose $\phi_2, \dots, \phi_{2(m+1)}$ in such a manner that $\{\phi_1, \dots, \phi_{2(m+1)}\}$ is a linearly independent set and we define the linear order $<$ as we have already explained.

By looking at the definition of $R^l f$ it can be shown that $\bar{U}_{l,1} = \bar{t}_{i_l} + \bar{\omega}'_l$ with $\bar{t}_{i_l} \in \{\bar{t}_1, \dots, \bar{t}_n\}$ and $\bar{\omega}'_l$ a sum having L'_l terms of type $\bar{\omega}_j - \bar{\omega}_{j_0}$, $j \in J$, $j > j_0$. We can therefore deduce that, for a given L , there exists a number $l = l(L)$ such that: If $l > l(L)$ and $R^{l-1}f \neq 0$ then $L'_l > L$. (Because, if for each given l_0 , there is an $l > l_0$ such that $R^{l-1}f \neq 0$ and $L'_l < L$, then the number of terms in the sequence $\bar{U}_{1,1} < \bar{U}_{2,1} < \cdots$ remains bounded, a contradiction, since it is an increasing sequence.)

It follows from the previous facts that

$$\phi_1(\bar{U}_l) \geq \min_i \phi_1(t_i) + L'_l \phi_1(\bar{\omega}_{j_0+1} - \bar{\omega}_{j_0})$$

and then we do reach a step, say \tilde{N} , such that either (A) holds, or

$$\phi_1(\bar{U}_{\tilde{N},1} - \bar{\omega}_{j_0} + \bar{\omega}_{j_1}) > \max_i \phi_1(\bar{t}_i).$$

The last possibility implies that the nonzero term:

$$b_{\tilde{N}+1} d_j(\bar{s}_{\tilde{N}}) z^{\bar{s}_{\tilde{N}+1} + \bar{\omega}_{j_1}}$$

of highest exponent in the expression of $R^{\tilde{N}} f$ cannot be cancelled neither with a term of type $b_l d_j(\bar{s}_l) z^{\bar{s}_l + \bar{\omega}_j} \neq 0$, because $\bar{s}_l + \bar{\omega}_j < \bar{s}_{\tilde{N}+1} + \bar{\omega}_{j_1}$ nor with a term $c_l z^{\bar{t}_i}$. We can conclude that $R^{\tilde{N}} f \neq 0$. Moreover, we have

$$\phi_1(\bar{U}_{\tilde{N}+1,1} - \bar{\omega}_{j_0} + \bar{\omega}_{j_1}) \geq \phi_1(\bar{U}_{\tilde{N},1} - \bar{\omega}_{j_0} + \bar{\omega}_{j_1}) > \max_i \phi_1(\bar{t}_i).$$

As before, we can deduce from this that $R^{\tilde{N}+1} f \neq 0$. It is clear that we could repeat the process, and then, prove that

$$\phi_l(\bar{U}_l - \bar{\omega}_{j_0} + \bar{\omega}_{j_1}) > \max_i \phi_1(\bar{t}_i) \quad \text{for } l > \tilde{N}$$

and finally, we can conclude that $R^l f \neq 0$ for every l , so (B) holds. So \tilde{N} is the bound we were looking for.

Case 2. For some $\bar{v} \in V_{n-1}$, $|\bar{v}| = 1$, holds: $P_{n-l}(\bar{\omega}_j) = \lambda_j \bar{v}$, we have $\lambda_j \geq 0$ for $j \in J$ and $\lambda_j > 0$ iff $j \in J_1$. Thus this case is opposite to the previous one.

The general idea and notation are as in Case 1; to find \tilde{N} is a bit more delicate. We must first choose carefully ϕ_1, ϕ_2 and then get (by any procedure) a linearly independent set: $\{\phi_1, \phi_2, \dots, \phi_2(m+1)\}$.

First of all, we choose $\phi_1(\bar{x}) = (\bar{x}, -\bar{v})$. Now we choose $\phi_2(\bar{x}) = (\bar{x}, -\bar{v} + \bar{\epsilon})$ such that (changing the indices if necessary) the following assertions hold:

$$(1) \quad \begin{aligned} & \phi_2(\bar{\omega}_1) < \phi_2(\bar{\omega}_2) < \dots; \quad \phi_1(\bar{\omega}_1) \leq \phi_1(\bar{\omega}_1) \leq \dots; \\ & \phi_2(\bar{t}_1) < \phi_2(\bar{t}_2) < \dots. \text{ As a consequence we have } j \in J_1, i \notin I_n \text{ implies } j < i. \text{ (Observe that } \phi_1(\bar{\omega}_j) < 0 \text{ and } \phi_1(\bar{\omega}_i) = 0.) \end{aligned}$$

$$(1') \quad \begin{aligned} & j_0 = \min_j \{j \in J\} \text{ (definition of } j_0\text{). Then we have } j_0 \in J_1 \text{ (observe that } \phi_1(\bar{\omega}_j) = -\lambda_j < 0, j \in J_1\text{). } j'_0 = \max_j \{j \in J: \phi_1(\bar{\omega}_j) = \phi_1(\bar{\omega}_{j_0})\} \text{ (definition of } j'_0\text{).} \\ & \text{Then } j'_0 \in J_1. j''_0 = \min_j \{j \in J: \phi_1(\bar{\omega}_{j_0}) < \phi_1(\bar{\omega}_j)\} \text{ (definition of } j''_0\text{). } j_1 = \max_j \{j \in I: d_j(\bar{v}) \neq 0\} \text{ (definition of } j_1\text{). We have: } j_1 \in I_{n-1} \text{ (in fact, } d_j(\bar{v}) = 0 \text{ for } j \notin I_{n-1} \text{ since } \bar{v} \in V_{n-1}\text{).} \end{aligned}$$

From the above definitions and remarks we can deduce:

$$(1'') \quad \begin{aligned} & j_0 \leq j'_0 < j_1; \quad j \leq j'_0 \text{ implies } j \in I_n. \text{ Besides, we can} \\ & \text{choose } \phi_2 \text{ in such a way that } j \in J \text{ and } j \geq j_1 \text{ implies} \\ & j \in I_{n-1}. \end{aligned}$$

It can be shown, by using the definition of $R^l f$, that: $\bar{U}_{l,1} = \bar{t}_{l,1} + \bar{\omega}'_l + \bar{\omega}''_l$, with $\bar{\omega}'_l$ a sum of a number, say L'_l of terms $\bar{\omega}_j - \bar{\omega}_{j_0}$, $j \in J$, $j > j'_0$; $\bar{\omega}''_l$ a sum of a certain number, say L''_l of terms of type $\bar{\omega}_j - \bar{\omega}_{j_0}$, $j \in J$, $j_0 < j \leq j'_0$, and $\bar{t}_{l,1} \in \{\bar{t}_1, \dots, \bar{t}_n\}$.

We can see that the following facts hold:

(2) For a given $L > 0$, there is an $l(L)$ such that if $l > l(L)$ and $R^{l-1}f \neq 0$ then $L'_l + L''_l > L$ (proof as in Case 1).

(3) If $R^{l-1}f \neq 0$, then

$$\begin{aligned} & \max_i \phi_1(\bar{t}_i) + L'_l \phi_1(\bar{\omega}_q - \bar{\omega}_{j_0}) \\ & \geq \phi_1(\bar{U}_{l,1}) \geq \min_i \phi_1(\bar{t}_i) + L'_l \phi_1(\bar{\omega}_{j'_0} - \bar{\omega}_{j_0}). \end{aligned}$$

(To prove this, observe that $\phi_1(\bar{\omega}''_l) = 0$.)

(4) If $R^{l-1}f \neq 0$, then

$$\begin{aligned} & \max_i \phi_2(\bar{t}_i) + |(L'_l + L''_l)| |\phi_2(\bar{\omega}_q - \bar{\omega}_{j_0})| \\ & \geq \phi_2(\bar{U}_l) \geq \min_i \phi_2(\bar{t}_i) + |(L'_l + L''_l)| |\phi_2(\bar{\omega}_{j_0+1} - \bar{\omega}_{j_0})|; \end{aligned}$$

(5) If $R'^{-1}f \neq 0$, then:

$$\left| d_{j_1}(\bar{U}_{l,1} - \bar{\omega}_{j_0}) \right| \geq - \max_i \left| d_{j_1}(\bar{t}_i - \bar{\omega}_{j_0}) \right| + L'_l |\lambda_{j_0''} - \lambda_{j_0}| \cdot |f_{j_1}(\bar{v})|.$$

(Note that for $j \in J$ holds):

$$d_{j_1}(P_{n-1}(\bar{\omega}_j - \bar{\omega}_{j_0})) = d_{j_1}((\lambda_j - \lambda_{j_0})(\bar{v})) = (\lambda_j - \lambda_{j_0}) \cdot d_{j_1}(\bar{v}),$$

and also recall from §2(iv) that

$$d_{j_1}(P_{n-1}(\bar{\omega}_j - \bar{\omega}_{j_0})) = d_{j_1}(\bar{\omega}_j - \bar{\omega}_{j_0}).$$

Now we will use these facts to establish our algorithm in Case 2. First of all, assume $j_0 = j_0'$. Then $\bar{\omega}'_l + \bar{\omega}''_l = \bar{\omega}'_l$, $L''_l + L'_l = L'_l$. Using (2), (3) we can show that we reach a step, say \tilde{N} , such that either (A) holds or $\phi_1(\bar{U}_{\tilde{N},1} - \bar{\omega}_{j_0} + \bar{\omega}_{j_1}) > \phi$, with

$$\begin{aligned} \phi &= \max_i \phi_1(\bar{t}_i) + \left(\max_i \frac{d_{j_1}(\bar{t}_i - \bar{\omega}_{j_0})}{(\lambda_{j_0''} - \lambda_{j_0}) d_{j_1}(\bar{v})} \right) \phi_1(\bar{\omega}_q - \bar{\omega}_{j_0}) \\ &\quad + \phi_1(\bar{\omega}_{j_1} - \bar{\omega}_{j_0}). \end{aligned}$$

Now we will see that if this last possibility holds, then (B) holds.

First we will see that

$$0 \neq R^{\tilde{N}+1}f = f - \sum_{l,j} b_l d_j(\bar{s}_l) Z^{\bar{s}_l + \bar{\omega}_j}, \quad 1 \leq l \leq \tilde{N} + 1; j_0 \leq j \leq j_2(\bar{s}_l).$$

Using (3) and the relation $\phi(\bar{U}_{\tilde{N},1} - \bar{\omega}_{j_0} + \bar{\omega}_{j_1}) > \phi$ we can deduce that $d_{j_1}(\bar{U}_{\tilde{N},1} - \bar{\omega}_{j_0}) = d_{j_1}(\bar{s}_{\tilde{N}+1}) \neq 0$. So, if we set $j_2 = j_2(\bar{s}_{\tilde{N}+1})$, we have $j_2 \geq j_1$ and then $\phi_1(\bar{s}_{\tilde{N}+1} + \bar{\omega}_{j_2}) \geq \phi > \max_i \phi_1(\bar{t}_i)$. Therefore: $\bar{s}_{\tilde{N}+1} + \bar{\omega}_{j_2} > \bar{t}_i$.

Besides, if $l < \tilde{N} + 1$, $j > j_2$, $j \in J$, $d_j(\bar{s}_l) \neq 0$ and $\bar{s}_l + \bar{\omega}_j = \bar{s}_{\tilde{N}+1} + \bar{\omega}_{j_2}$, then (by (1'')), $j \in I_{n-1}$ and therefore

$$d_j(\bar{s}_{\tilde{N}+1}) = d_j(P_{n-1}(\bar{s}_l + \bar{\omega}_j - \bar{\omega}_{j_2})) = d_j(P_{n-1}(\bar{s}_l)) = d_j(\bar{s}_l) \neq 0,$$

a contradiction (see the definition of $j_2(\bar{x})$). We have proved that the term $b_{\tilde{N}+1} d_{j_2}(\bar{s}_{\tilde{N}+1}) Z^{\bar{s}_{\tilde{N}+1} + \bar{\omega}_{j_2}}$ of $R^{\tilde{N}+1}f$ cannot be cancelled, so $R^{\tilde{N}+1}f \neq 0$. Besides since $\bar{U}_{\tilde{N}+1,1} > \bar{U}_{\tilde{N},1}$ we have

$$\phi_1(\bar{U}_{\tilde{N}+1,1} - \bar{\omega}_{j_0} + \bar{\omega}_{j_1}) \geq \phi_1(\bar{U}_{\tilde{N},1} - \bar{\omega}_{j_0} + \bar{\omega}_{j_1}) > \phi,$$

and therefore we can iterate and finally we can conclude that (B) holds.

We have established the case $j_0 = j_0'$.

Now assume that $j'_0 > j_0$. We will see the following assertion: a step, say \tilde{N} , is reached such that one (and only one) of the following conditions holds:

- (a) $R^{\tilde{N}-1}f \neq 0$ and $R^{\tilde{N}}f = 0$ (condition (A))
- (b) $R^{\tilde{N}}f \neq 0$ and $\phi_1(\bar{U}_{\tilde{N},1} - \bar{\omega}_{j_0} + \bar{\omega}_{j_1}) > \phi$
(ϕ as in the case $j'_0 = j_0$).
- (c) $R^{\tilde{N}}f \neq 0$ and for some $N > \tilde{N}$ we have

$$\phi_1(\bar{U}_{\tilde{N},1}) = \phi_1(\bar{U}_{\tilde{N}+1,1}) = \dots = \phi_1(\bar{U}_{N,1})$$

and

$$\phi_2(\bar{U}_{N,1} - \bar{\omega}_{j_0} + \bar{\omega}_{j'_0}) > \psi_{\tilde{N}},$$

with $\psi_{\tilde{N}} = \max\{\phi_2(\bar{U}_{\tilde{N},l}): l \in B_{\tilde{N}}\}$ with $B_{\tilde{N}} = \{l: \phi_1(\bar{U}_{\tilde{N},l}) = \phi_1(\bar{U}_{\tilde{N},1})\}$.

To prove this, assume that there is not a number \tilde{N} such that (a) or (b) holds, and therefore, that $R^l f$ remains nonero and $\phi_1(\bar{U}_{L,1})$ remains bounded. Then (by (3)), L'_l remains bounded. On the other hand, we have (since $\phi_1(\bar{\omega}'') = 0$), $\phi_1(\bar{U}_{l,1}) = \phi_1(\bar{t}_{il}) + \phi_1(\bar{\omega}'_i)$. We can deduce from these facts that $\phi_1(\bar{U}_{l,1})$ takes only a finite number of values and then that there is a number \tilde{N} such that the sequence $\phi_1(\bar{U}_{0,1}) \leq \phi_1(\bar{U}_{1,1}) \leq \dots \leq \phi_1(\bar{U}_{l,1}) \leq \dots$ remains constant for $l \geq \tilde{N}$, and besides (by (2) and (4)) we reach a step, say N , such that $N > \tilde{N}$ and $\phi_2(\bar{U}_{N,1} - \bar{\omega}_{j_0} + \bar{\omega}_{j'_0}) > \psi_{\tilde{N}}$. We have proved the assertion.

If (b) holds, we can prove as in case $j_0 = j'_0$ that (B) holds.

Now assume that (C) holds. We therefore have

$$R^{N+1}f = R^{\tilde{N}}f - \sum_{l,j} b_l d_j(\bar{s}_l) Z^{\bar{s}_l + \bar{\omega}_j},$$

$$\tilde{N} + 1 \leq l \leq N + 1, j_0 \leq j \leq j_2(\bar{s}_l).$$

If $\bar{s}_{N+1} + \bar{\omega}_{j'_0} = \bar{U}_{\tilde{N},l}$, we can deduce: $\phi_1(\bar{U}_{\tilde{N},l}) = \phi_1(\bar{U}_{N,1}) = \phi_1(\bar{U}_{\tilde{N},1})$, so $l \in B_{\tilde{N}}$. It follows that $\phi_2(\bar{s}_{N+1} + \bar{\omega}_{j'_0}) > \psi_{\tilde{N}} \geq \phi_2(\bar{U}_{\tilde{N},l})$, a contradiction. Now assume that, for some l, j ($j \in J$) such that $\tilde{N} + 1 \leq l < N + 1, j'_0 < j \leq j_2(\bar{s}_l)$, we have that $\bar{s}_l + \bar{\omega}_j = \bar{s}_{N+1} + \bar{\omega}_{j'_0}$. Then

$$\phi_1(\bar{U}_{l-1,1} - \bar{\omega}_{j_0} + \bar{\omega}_j) = \phi_1(\bar{U}_{N,1} - \bar{\omega}_{j_0} + \bar{\omega}_{j'_0}) = \phi_1(\bar{U}_{N,1}).$$

Now, since $\phi_1(\bar{\omega}_j - \bar{\omega}_{j_0}) > 0$ we can deduce that $\phi_1(\bar{U}_{l-1,1} - \bar{\omega}_{j_0} + \bar{\omega}_j) > \phi_1(\bar{U}_{l-1,1}) = \phi_1(\bar{U}_{N,1})$, a contradiction. So we have proved that $R^{N+1}f \neq 0$.

Now for some $l \geq 1$, say $l = l_0$, we have that $\bar{s}_{N+1} + \bar{\omega}_{j'_0} = \bar{U}_{N+1,l_0}$, so we can deduce that $\phi_1(\bar{U}_{N+1,1}) \leq \phi_1(\bar{U}_{N,1})$. We can deduce from this (since $\bar{U}_{N,1} < \bar{U}_{N+1,1}$) that: $\phi_1(\bar{U}_{N+1,1}) = \phi_1(\bar{U}_{N,1})$ and $\phi_2(\bar{U}_{N,1}) \leq \phi_2(\bar{U}_{N+1,1})$ so we can conclude that $\phi_2(\bar{U}_{N+1,1} - \bar{\omega}_{j_0} + \bar{\omega}_{j'_0}) > \psi_{\tilde{N}}$. It is clear that we can iterate and prove that $R^{\tilde{N}+l}f \neq 0$ for every integer l , so (B) holds.

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Received July 21, 1982 and in revised form April, 4, 1985.

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