EIGENVALUES OF SUMS OF HERMITIAN MATRICES

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Let $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\beta = (\beta_1, \cdots, \beta_n)$ be arbitrary nonincreasing sequences of real numbers. We consider the question: for which nonincreasing sequences $\gamma = (\gamma_1, \cdots, \gamma_n)$ do there exist Hermitian matrices A and B such that A, B and $A + B$ have α, β and γ respectively as their sequences of eigenvalues. Necessary conditions have been obtained by several authors including Weyl [4], Lidskii [3], Wielandt [5], and Amir-Moez [1]. Besides the obvious condition

$$(1) \qquad \gamma_1 + \cdots + \gamma_n = \alpha_1 + \cdots + \alpha_n + \beta_1 + \cdots + \beta_n ,$$

these conditions are linear inequalities of the form

$$(2) \qquad \gamma_{k_1} + \cdots + \gamma_{k_r} \leqq \alpha_{i_1} + \cdots + \alpha_{i_r} + \beta_{j_1} + \cdots \beta_{j_r} ,$$

where i, j and k are increasing sequences of integers. As far as I know all other known necessary conditions are consequences of these inequalities. It is therefore natural to conjecture that the set E of all possible γ forms a convex subset of the hyperplane (1). The set E has hitherto not been determined except in the simple cases $n = 1, 2$, and will not be determined in general here.

In § 2, which is independent of § 1, we are going to give a method of finding conditions of the form (2) which will yield many new ones. We shall find all possible inequalities (2) for $r = 1, 2$, and arbitrary n, and establish a large class of such inequalities for $r = 3$. In § 1, we use Lidskii's method to find a necessary condition on the boundary points of a subset E' of E. These results are used in § 3 to determine the set E for $n = 3, 4$. In addition a conjecture is given for E in general.

If x is a sequence, x_p denotes the pth component of x. If A is a matrix, A^* and A' denote the conjugate transpose and transpose of A. If i is a sequence of integers such that $1 \leqq i_1 < \cdots < i_r \leqq n$, by the complement of i with respect to n we mean the sequence obtained by deleting the terms of i from the sequence $1, 2, \cdots, n$. If α is a sequence of numbers, diag $(\alpha_1, \cdots, \alpha_n)$ denotes the diagonal matrix with diagonal α. If M and N are matrices, diag (M, N) denotes the direct sum matrix

$$\begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix} .$$

The inner product of the vectors x and y is written (x, y). I_r is the

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unit matrix of order r. Finally exp B denotes the sum $\sum_{n=0}^{\infty} B^n/n!$.

1. Boundary points of E'. In this section we are going to use methods introduced by Lidskii [3]. Lidskii gave sketchy proofs of his results and it is not obvious how to reconstruct his argument, see [5]. I will therefore derive the results of Lidskii which are needed. These are Theorem 1 and formula (18) below.

The set E referred to in the introduction is the set of points γ such that $\gamma_1 \geqq \cdots \geqq \gamma_n$ and γ is the sequence of eigenvalues of diag $(\alpha_1, \cdots, \alpha_n) + U^*$ diag $(\beta_1, \cdots, \beta_n)$ U, where U ranges over all unitary matrices. Fix α, β, with $\alpha_1 > \cdots > \alpha_n$ and $\beta_1 > \cdots > \beta_n$, and let E' be the subset of E obtained by letting U range over real orthogonal matrices. To indicate the dependence of E' on α and β we write E' $(\alpha_1, \cdots, \alpha_n; \beta_1, \cdots, \beta_n)$. Boundary points and interior points of E' are always taken with respect to the relative topology of the hyperplane (1).

THEOREM 1. *If γ is a boundary point of E' with distinct coordinates then there exist a positive integer $r < n$ and increasing sequences i, j, and k of order r such that*

$$(\gamma_{k_1}, \cdots, \gamma_{k_r}) \in E'(\alpha_{i_1}, \cdots, \alpha_{i_r}; \beta_{j_1}, \cdots, \beta_{j_r})$$

and

$$(\gamma_{k'_1}, \cdots, \gamma_{k'_{n-r}}) \in E'(\alpha_{i'_1}, \cdots, \alpha_{i'_{n-r}}; \beta_{j'_1}, \cdots, \beta_{j'_{n-r}}) ,$$

where i', j' and k' are the complements of i, j and k with respect to n.

Proof. Let U_0 be a real orthogonal matrix such that diag $(\alpha_1, \cdots, \alpha_n) + U_0'$ diag $(\beta_1, \cdots, \beta_n) U_0$ has eigenvalues γ. If $T=(t_{pq})$ is a real anti-symmetric matrix, exp T is orthogonal. For sufficiently small values of t_{pq}, the eigenvalues $\lambda_1 > \cdots > \lambda_n$ of

$$\operatorname{diag}(\alpha_1, \cdots, \alpha_n) + U_0' \exp(-T) B \exp(T) U_0 ,$$

where $B = \operatorname{diag}(\beta_1, \cdots, \beta_n)$, are distinct and determine a point of E'. Let $A = U_0 \operatorname{diag}(\alpha_1, \cdots, \alpha_n) U_0'$, and let x_l be a unit eigenvector of $A + \exp(-T) B \exp T$ corresponding to the eigenvector λ_l which varies continuously with T. We have

$$Ax_l + \exp(-T) B \exp(T) x_l = \lambda_l x_l . \tag{3}$$

Using superscripts to denote derivatives with respect to t_{pq}, $p < q$, it follows that

$$Ax_l^{pq} + \exp(-T) B \exp(T) x_l^{pq} + (\exp(-T) B \exp(T))^{pq} x_l = \lambda_l^{pq} x_l + \lambda_l x_l^{pq} . \tag{4}$$

It is easily seen that $(\exp T)^{pq}$ reduces to T^{pq} when $T = 0$. Hence when $T = 0$, $(\exp(-T)B \exp T)^{pq}$, $= (\beta_p - \beta_q)Z^{pq}$, where Z^{pq} is the matrix whose (p, q) and (q, p) entries are 1 and whose other entries are 0. Since x_l is a unit vector, $(x_l, x_l^{pq}) = 0$. Therefore by (3),

$$(5) \qquad (Ax_l, x_l^{pq}) + (\exp(-T)B\exp(T)x_l, x_l^{pq}) = 0 .$$

Taking the inner product of (4) with x_l we find by (5) and the symmetry of A and B,

$$\lambda_l^{pq} = ((\exp(-T)B\exp T)^{pq}x_l, x_l) .$$

Setting $T = 0$,

$$(6) \qquad \gamma_l^{pq} = 2(\beta_p - \beta_q)w_{lp}w_{lq} ,$$

where w_l and γ_l^{pq} denote the values of x_l and λ_l^{pq} when $T = 0$. If γ is not an interior point of E' the rank of the n by $n(n-1)/2$ matrix $G = (\gamma_l^{pq})$ must be less than $n - 1$. Now[1] let $D = (w_{lp}w_{lq})$ be the n by $n(n-1)$ matrix whose rows are indexed by l, where $1 \leqq l \leqq n$, and whose columns are indexed by (p, q), where p and q vary over the range $1 \leqq p \leqq n$, $1 \leqq q \leqq n$, and $p \neq q$ rather than $p < q$. Clearly D, and hence DD', has the same rank as G. If F is the square matrix (w_{lm}^2) of order n, then $DD' = I - FF'$. Thus if rank $D < n - 1$, FF' has 1 as a multiple eigenvalue. Since FF' is stochastic, it follows that FF' is decomposable [2, pp. 47 and 73]. That is to say, $FF' = P \operatorname{diag}(M, N)P'$, where M and N are square matrices and P is a permutation matrix. Let

$$F = P\begin{pmatrix} G & H \\ J & K \end{pmatrix}P'$$

be the decomposition of F corresponding to that of FF'. Then $GJ' + HK' = 0$. Since the entries of F are nonnegative, we have $GJ' = HK' = 0$. It follows that if a column of G contains a nonzero term then all terms of the corresponding column of J vanish, and similarly for H and K. Moving all nonzero columns of G and H to the left, we find

$$F = P\begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}R ,$$

where R is another permutation matrix. Since F is doubly stochastic, S_1 and S_2 must be square matrices. If $W = (w_{lm})$, then $W = P\begin{pmatrix} W_1 & 0 \\ 0 & W_2 \end{pmatrix}R$, where W_1 and W_2 are square. Setting $\Gamma = \operatorname{diag}(\gamma_1, \cdots, \gamma_n)$, we have $A + B = W'\Gamma W$. Therefore $RAR' + RBR' = C$, where $C =$

[1] The following argument is due to Robert Steinberg.

diag $(W_1', W_2')P'\Gamma P$ diag (W_1, W_2). Let j and k be such that $RBR' =$ diag $(\beta_{j_1}, \cdots, \beta_{j_n})$ and $P'\Gamma P =$ diag $(\gamma_{k_1}, \cdots, \gamma_{k_n})$. If W_1 is of order r, then $C =$ diag (C_1, C_2), where C_1 has eigenvalues $\gamma_{k_1}, \cdots, \gamma_{k_r}$ and C_2 has eigenvalues $\gamma_{k_{r+1}}, \cdots, \gamma_{k_n}$. Therefore $RAR' =$ diag (A_1, A_2), where $A_1 +$ diag $(\beta_{j_1}, \cdots, \beta_{j_r}) = C_1$ and $A_2 +$ diag $(\beta_{j_{r+1}}, \cdots, \beta_{j_n}) = C_2$. This completes the proof.

If $M = (m_{ij}), 1 \leqq i \leqq r, 1 \leqq j \leqq r$ is a matrix of order r and $N = (n_{kl}), r + 1 \leqq k \leqq n, r + 1 \leqq l \leqq n$ is a matrix of order $n - r$, we define $M \times N$ to be the matrix $(m_{ij}n_{kl})$ of order $r(n - r)$ whose rows are indexed by pairs (i, k) and whose columns are indexed by pairs (j, l). This product is left and right distributive and $(M \times N)' = M' \times N'$. Also $(M_1 \times N_1)(M_2 \times N_2) = (M_1M_2 \times N_1N_2)$. We set $M \ominus N = (M \times I_{n-r}) - (I_r \times N)$. It follows from these remarks that if W_1 and W_2 are orthogonal then so is $W_1 \times W_2$ and

$$(7) \qquad (W_1'MW_1) \ominus (W_2'NW_2) = (W_1 \times W_2)'(M \ominus N)(W_1 \times W_2) .$$

The index of a real symmetric matrix is the number of its positive eigenvalues.

LEMMA 1. *If M, N, and $M + N$ are nonsingular real symmetric matrices then index M + index N = index $(M + N)$ + index $(M^{-1} + N^{-1})$.*

Proof.[2] We have $M^{-1} + N^{-1} = N^{-1}(N + M)M^{-1}$, so that $M^{-1} + N^{-1}$ is nonsingular. Also

$$\begin{pmatrix} I & I \\ M^{-1} & -N^{-1} \end{pmatrix}\begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}\begin{pmatrix} I & M^{-1} \\ I & -N^{-1} \end{pmatrix}\begin{pmatrix} M + N & 0 \\ 0 & M^{-1} + N^{-1} \end{pmatrix} .$$

The result now follows by the Law of Inertia.

THEOREM 2. *Let γ be a boundary point of E' with distinct coordinates. Then there exist sequences i, j and k satisfying the conclusion of Theorem 1 and such that*

$$i_1 + \cdots + i_r + j_1 + \cdots + j_r = k_1 + \cdots + k_r + r(r + 1)/2 .$$

Proof. Using a slight change of notation, we have seen that there exist permutations i, j and k of $(1, \cdots, n)$ and real symmetric matrices $A_1, A_2, B_1, B_2, C_1, C_2$ such that A_1 has eigenvalues $\alpha_{i_1}, \cdots, \alpha_{i_r}$, A_2 has eigenvalues $\alpha_{i_{r+1}}, \cdots, \alpha_{i_n}$, $B_1 =$ diag $(\beta_{j_1}, \cdots, \beta_{j_r})$, $B_2 =$ diag $(\beta_{j_{r+1}}, \cdots, \beta_{j_n})$, C_1 has eigenvalues $\gamma_{k_1}, \cdots, \gamma_{k_r}$, C_2 has eigenvalues $\gamma_{k_{r+1}}, \cdots, \gamma_{k_n}$, and $A + B = C$, where $A =$ diag (A_1, A_2), $B =$ diag (B_1, B_2) $C =$ diag (C_1, C_2). We also assume $i_1 < \cdots < i_r$ and $i_{r+1} < \cdots < i_n$, and similarly for the j's and k's. We set $\bar{\alpha}_l = \alpha_{i_l}$, $\bar{\beta}_l = \beta_{j_l}$ and $\bar{\gamma}_l = \gamma_{k_l}$, $1 \leqq l \leqq n$. Let

[2] This simple proof is due to Robert Steinberg.

$T=(t_{pq})$ be a real anti-symmetric matrix and let $\lambda_1 > \cdots > \lambda_n$ be the eigenvalues of $A+\exp(-T)B\exp T$. If $x_1, \cdots, x_n$ is a real orthonormal system of corresponding eigenvectors, we let w_l and w_l^{pq} be the values of x_{k_l} and $x_{k_l}^{pq}$ when $T=0$, where x_l^{pq} denotes the derivative of x_l with respect to t_{pq}, $p<q$. If W is the matrix whose rows are $w_1, \cdots, w_n$, then $W=\operatorname{diag}(W_1, W_2)$ and $C_1 = W_1' \Gamma_1 W_1$, $C_2 = W_2' \Gamma_2 W_2$, where $\Gamma_1 = \operatorname{diag}(\gamma_{k_1}, \cdots, \gamma_{k_r})$, $\Gamma_2 = \operatorname{diag}(\gamma_{k_{r+1}}, \cdots, \gamma_{k_n})$, Clearly λ_{k_l} reduces to $\bar{\gamma}_l$ when $T=0$, and we let $\bar{\gamma}_l^{pq}$ be the value of $\lambda_{k_l}^{pq} = \partial\lambda_{k_l}/\partial t_{pq}$ when $T=0$.

Starting from the equation

$$(8) \qquad Ax_{k_l} + (\exp(-t)B\exp T)x_{kl} = \lambda_{kl}x_{kl}$$

we find

$$(9) \qquad Ax_{k_l}^{pq} + (\exp(-T)B\exp T)x_{k_l}^{pq} + (\exp(-T)B\exp T)^{pq}x_{k_l} = \lambda_{k_l}^{pq}x_{k_l} + \lambda_{k_l}x_{k_l}^{pq}\ .$$

As in Theorem 1 it follows that

$$(10) \qquad \lambda_{k_l}^{pq} = ((\exp(-T)B\exp T)^{pq}x_{k_l}, x_{k_l})$$

and therefore

$$(11) \qquad \bar{\gamma}_l^{pq} = 2(\bar{\beta}_p - \bar{\beta}_q)w_{lp}w_{lq}\ .$$

We are going to test $\sigma = \lambda_{k_1} + \cdots + \lambda_{k_r}$ for a local extreme at $T=0$. If p and q are $\leqq r$, then $\exp T$ has the form diag $(\exp T_1, 0)$ when $t_{uv}=0$ for $(u,v)\neq(p,q)$, and hence σ remains constant for t_{pq} in a neighborhood of 0. Therefore all partial derivatives of σ with respect to t_{pq} vanish at the origin when $p<q\leqq r$, and similarly when $r<p<q$. By (11), $\sigma^{pq}=0$ at $T=0$ when $p\leqq r<q$, since the last $n-r$ components of w_l are 0 when $1\leqq l\leqq r$. We now calculate $\lambda_{k_l}^{pq,uv}$ at $T=0$ when

$$(12) \qquad 1\leqq p\leqq r<q\leqq n\ , \qquad 1\leqq u\leqq r<v\leqq n\ , \qquad 1\leqq l\leqq r\ .$$

Differentiation of (10) yields

$$(13) \qquad \lambda_{k_l}^{pq,uv} = ((\exp(-T)B\exp T)^{pq,uv}x_{k_l}, x_{k_l}) + 2((\exp(-T)B\exp T)^{pq}x_{k_l}^{uv}, x_{k_l})\ .$$

It is easily seen that when $T=0$

$$(\exp(-T)B\exp T)^{pq,uv} = -(T^{pq}BT^{uv} + T^{uv}BT^{pq}) + \frac{1}{2}B(T^{pq}T^{uv} + T^{uv}T^{pq}) + \frac{1}{2}(T^{pq}T^{uv} + T^{uv}T^{pq})B\ .$$

Considering only the cases (12), a straightforward calculation shows that when $T=0$,

$$\begin{aligned}((\exp(-T)B\exp T)^{pq,uv}x_{k_l}, x_{k_l}) &= 0 && \text{for } p\neq u,\ q\neq v\\ &= (2\bar\beta_q-\bar\beta_p-\bar\beta_u)w_{lp}w_{lu} && \text{for } p\neq u,\ q=v\\ &= (2\bar\beta_p-\bar\beta_q-\bar\beta_v)w_{lq}w_{lv} && \text{for } p=u,\ q\neq v\\ &= -2(\bar\beta_p-\bar\beta_q)(w_{lp}^2-w_{lp}^2) && \text{for } p=u,\ q=v\,.\end{aligned}$$

Recalling that $w_{lq}=0$ for $l\leqq r<q$, we find that when $T=0$,

$$\sum_{l=1}^{r}((\exp(-T)B\exp T)^{pq,uv}x_{k_l}, x_{k_l}) = -2(\bar\beta_p-\bar\beta_q) \text{ for } p=u, q=v \qquad (14)$$
$$= 0 \quad \text{otherwise.}$$

The second term on the right of (13) reduces when $T=0$ to

$$2(\bar\beta_p-\bar\beta_q)w_{lq}^{uv}w_{lp}\,. \qquad (15)$$

To compute w_{lq}^{uv}, rewrite (9) in the form

$$\begin{aligned}(A+\exp(-T)B\exp T-\lambda_{k_l}I_n)x_{k_l}^{uv}\\ = -(\exp(-T)B\exp T)^{uv}x_{k_l}+\lambda_{k_l}^{uv}x_{k_l}\,.\end{aligned}$$

Setting $T=0$ and using (11), we find, since $w_{lv}=0$,

$$(C-\bar\gamma_l I_n)w_l^{uv} = -(\bar\beta_u-\bar\beta_v)y\,,$$

where y is the vector such that $y_u=w_{lv}=0, y_v=w_{lu}$ and $y_m=0$ for $m\neq u, m\neq v$. Therefore

$$w_{lq}^{uv} = (\bar\beta_u-\bar\beta_v)((\bar\gamma_l I_n-C)^{-1}y)_q\,.$$

Since $q>r$, and $C=\operatorname{diag}(C_1, C_2)$, we may replace C by C_2 and I_n by I_{n-r}. Thus

$$w_{lq}^{uv} = (\bar\beta_u-\bar\beta_v)d_{qv}w_{lu}\,, \qquad (16)$$

where d_{qv} is the (q, v) entry of $(\bar\gamma_l I_{n-r}-C_2)^{-1}$. Now

$$(\bar\gamma_l I_{n-r}-C_2)^{-1} = (W_2'(\bar\gamma_l I_{n-r}-\Gamma_2)W_2)^{-1}\,.$$

Therefore

$$d_{qv} = \sum_{m=r+1}^{n}\frac{w_{mq}w_{mv}}{\bar\gamma_l-\bar\gamma_m}\,. \qquad (17)$$

Combining (13), (14), (15), (16), and (17), we find at $T=0$

$$\begin{aligned}\sigma^{pq,uv} = 2(\bar\beta_p-\bar\beta_q)(\bar\beta_u-\bar\beta_v)\sum_{l=1}^{r}\sum_{m=r+1}^{n}\frac{w_{lp}w_{lu}w_{mq}w_{mv}}{\bar\gamma_l-\bar\gamma_m}\\ -2\delta_{uv}^{pq}(\bar\beta_p-\bar\beta_q)\,,\end{aligned} \qquad (18)$$

where $\delta_{uv}^{pq} = 1$ when $(p, q) = (u, v)$, and $= 0$ otherwise.

We must now determine the index of the matrix $G = (\sigma^{pq,uv})_{T=0}$ of order $r(n-r)$ whose rows and columns are indexed by pairs (p, q) and (u, v) satisfying (12).

The double sum on the right of (18) is the (pq, uv) entry of

$$(W_1 \times W_2)'(\Gamma_1 \ominus \Gamma_2)^{-1}(W_1 \times W_2) = ((W_1 \times W_2)'(\Gamma_1 \ominus \Gamma_2)(W_1 \times W_2))^{-1} .$$

By (7) this reduces to

$$(C_1 \ominus C_2)^{-1} = ((A_1 + B_1) \ominus (A_2 + B_2))^{-1} = ((A_1 \ominus A_2) + (B_1 \ominus B_2))^{-1} .$$

Therefore by (18)

$$\begin{aligned}\frac{1}{2}G &= (B_1 \ominus B_2)((A_1 \ominus A_2) + (B_1 \ominus B_2))^{-1}(B_1 - B_2) \ominus (B_1 \ominus B_2) \\ &= (B_1 \ominus B_2(((A_1 \ominus A_2) + (B_1 \ominus B_2))^{-1} - (B_1 \ominus B_2)^{-1})(B_1 \ominus B_2) .\end{aligned}$$

Thus G has the same index as $((A_1 \ominus A_2) + (B_1 \ominus B_2))^{-1} - (B_1 \ominus B_2)^{-1}$. Applying Lemma 1 with $M = ((A_1 \ominus A_2) + (B_1 \ominus B_2))$, $N = -(B_1 \ominus B_2)$,

$$\begin{aligned}\operatorname{index} G &= \operatorname{index}((C_1 \ominus C_2)^{-1} - (B_1 \ominus B_2)^{-1}) \\ &= \operatorname{index}(C_1 \ominus C_2) + \operatorname{index} -(B_1 \ominus B_2) - \operatorname{index}(A_1 \ominus A_2) \\ &= r(n-r) + \operatorname{index}(C_1 \ominus C_2) - \operatorname{index}(B_1 \ominus B_2) \\ &\qquad - \operatorname{index}(A_1 \ominus A_2) .\end{aligned}$$

Thus G is positive definite if and only index $(C_1 \ominus C_2) =$ index $(A_1 \ominus A_2) +$ index $(B_1 \ominus B_2)$, and G is negative definite if and only if neg $(C_1 \ominus C_2) =$ neg $(A_1 \ominus A_2) +$ neg $(B_1 \ominus B_2)$, where neg H is the number of negative eigenvalues of H. Next we determine

$$\operatorname{neg}(B_1 \ominus B_2) = \operatorname{neg} \operatorname{diag}(\beta_{j_1} - \beta_{j_{r+1}}, \cdots, \beta_{j_1} - \beta_{j_n}, \cdots, \beta_{j_r} - \beta_{j_n}) .$$

Among the numbers $j_{r+1}, \cdots, j_n$, there are $j_1 - 1$ terms $< j_1$, $j_2 - 2$ terms $< j_2$, etc. Hence neg $(B_1 \ominus B_2) = j_1 + \cdots + j_r - r(r+1)/2$. Similarly neg $(A_1 \ominus A_2) = i_1 + \cdots + i_r - r(r+1)/2$, and neg $(C_1 \ominus C_2) = k_1 + \cdots + k_r - r(r+1)/2$. Thus G is negative definite if and only if

$$i_1 + \cdots + i_r + j_1 + \cdots + j_r = k_1 + \cdots + k_r + r(r+1)/2 , \tag{19}$$

and G is positive definite if and only if

$$\begin{aligned}i_{r+1} + \cdots + i_n + i_{r+1} + \cdots + j_n &= k_{r+1} + \cdots + k_n \\ &\quad + (n-r)(n-r+1)/2 .\end{aligned} \tag{20}$$

By Theorem 1 the boundary points of E' lie on a finite number of hyperplanes of the form

$$\gamma_{k_1} + \cdots + \gamma_{k_r} = \alpha_{i_1} + \cdots + \alpha_{i_r} + \beta_{j_1} + \cdots + \beta_{j_r} . \tag{21}$$

The hyperplane

$$\gamma_{k'_1} + \cdots + \gamma_{k'_{n-r}} = \alpha_{i'_1}, + \cdots + \alpha_{i'_{n-r}} + \beta_{j'_1} + \cdots \beta_{j'_{n-r}}$$

intersects the hyperplane (1) in the same set. If γ lies on only one of these hyperplanes (21) and does not satisfy (19) or (20), then in every small sphere about γ there exist points of E' on both sides of the hyperplane (21). Therefore E' must fill the sphere, for otherwise there would be boundary points of E' inside the sphere and off the hyperplane (21). This being impossible, λ must satisfy (19) or (20). Now suppose γ lies on several hyperplanes (21), and (19) and (20) both fail for each of these hyperplanes. By continuity the quadratic form G is not definite for all points near γ which satisfy the conclusion of Theorem 1. Therefore in a neighborhood of γ all points of E' lying on only one hyperplane (21) are interior points of E'. Therefore γ cannot be a boundary point of E', since E' is the closure of its interior, and a finite union of linear varieties of deficiency $\geqq 2$ cannot separate the interior of a sphere. The proof is complete.

2. Inequalities. This section is independent of § 1. If i, j and k are increasing sequences of integers of order r and (2) holds for the eigenvalues of $A + B$ for any Hermitian A, B with arbitrary eigenvalues $\alpha_1 \geqq \cdots \geqq \alpha_n$ and $\beta_1 \geqq \cdots \geqq \beta_n$, we write $(i; j; k) \in S_r^n$. If

$$\gamma_{k_1} + \cdots + \gamma_{k_r} \geqq \alpha_{i_1} + \cdots + \alpha_{i_n} + \beta_{j_1} + \cdots + \beta_{j_r}$$

for any such A, B we write $(i; j; k) \in \tilde{S}_r^n$.

THEOREM 3. *The following conditions are equivalent*:
(i) $(i; j; k) \in S_r^n$
(ii) $(n - i_r + 1, \cdots, n - i_1 + 1; n - j_r + 1, \cdots, n - j_1 + 1; n - k_r + 1, \cdots, n - k_1 + 1) \in \tilde{S}_r^n$
(iii) $(k_1, \cdots, k_r; n - j_r + 1, \cdots, n - j_1 + 1; i_1, \cdots, i_r) \in \tilde{S}_r^n$
(iv) $(i'; j'; k') \in \tilde{S}_{n-r}^n$, *where* i', j', k' *are the complements of* i, j, k *with respect to* n.

Proof. The equation $A + B = C$ may be written $-A - B = -C$ or $A = C - B$. This proves the equivalence of (i) with (ii) and (iii). The equivalence of (i) and (iv) is immediate by the trace Condition (1).

If A is a Hermitian matrix with eigenvalues $\alpha_1 \geqq \cdots \geqq \alpha_n$ and M is a linear subspace of dimension $n - 1$, let A_M be the transformation PA with domain restricted to M, where P is the orthogonal projection on M. A_M is a Hermitian transformation on M to M and $(A + B)_M = A_M + B_M$. It is well known that the eigenvalues α'_p of A_M separate those of A, that is $\alpha_{p+1} \leqq \alpha'_p \leqq \alpha_p$ for $1 \leqq p \leqq n - 1$. If (x_p) is an orthonormal sequence of eigenvectors corresponding to (α_p) and if M contains $x_1, \cdots, x_m$, then $\alpha'_p = \alpha_p$ for $1 \leqq p \leqq m$. This is an immediate consequence of the minimax principle, since $(A_M x, x) = (Ax, x)$ for $x \in M$. Dually if M contains $x_{m+1}, \cdots, x_n$, then $\alpha'_p = \alpha_{p+1}$ for $m \leqq p \leqq n-1$. The next theorem shows that S is essentially independent of n.

THEOREM 4. *If* $(i; j; k) \in S_r^n$ *for some* n, *then* $i_p \leqq k_p$ *and* $j_p \leqq k_p$ *for all* p, *and* $(i; j; k) \in S_r^n$ *for all* $n \geqq k_r$.

Proof. Suppose $(i; j; k) \in S_r^n$ for some n. Considering the case $\beta = 0$, it is clear that $i_p \leqq k_p$ and $j_p \leqq k_p$ for all p. If A and B are of order k_r, the identity $\operatorname{diag}(A, -\lambda I) + \operatorname{diag}(B, -\lambda I) = \operatorname{diag}(A + B, -2\lambda I)$ for large λ shows that $(i; j; k) \in S_r^{k_r}$. It remains to prove $(i; j; k) \in S_r^{n+1}$. Let A and B be of order $n+1$ with eigenvalues (α_p), (β_p), and let (z_p) be an orthonormal sequence of eigenvectors of $A + B$ corresponding to the eigenvalues (γ_p). Let M be the subspace spanned by $z_1, \cdots, z_n$. Letting (α'_p), (β'_p) and (γ'_p) be the eigenvalues of A_M, B_M and $(A + B)_M$, we have by hypothesis

$$\gamma'_{j_1} + \cdots + \gamma'_{k_r} \leqq \alpha'_{i_1} + \cdots + \alpha'_{i_r} + \beta'_{j_1} + \cdots + \beta'_{j_r} .$$

But $\gamma'_{k_p} = \gamma_{k_p}$, $\alpha'_{i_p} \leqq \alpha_{i_p}$ and $\beta'_{j_p} \leqq \beta_{j_p}$ for $1 \leqq p \leqq r$. Therefore $(1; j; k) \in S_r^{n+1}$.

THEOREM 5. *If* $(i; j; k) \in S_r^n$ *and* u, v *and* w *are integers such that* $r+1 \geqq u \geqq 1$, $r+1 \geqq v \geqq 1$ *and* $r \geqq w \geqq 1$, *and if* $i_u + j_v \geqq k_{w-1} + k_r + 2$ *then* $(i_1, \cdots, i_{u-1}, i_u + 1, \cdots, i_r + 1; j_1 \cdots, j_{v-1}, j_v + 1, \cdots, j_r + 1; k_1, \cdots, k_{w-1}, k_w + 1, \cdots, k_r + 1) \in S_r^{n+1}$. *Here* $k_0 = 0$ *and* $i_{r+1} = j_{r+1} = k_r + 1$ *by definition.*[3] *In particular,* $(i_1 + 1, \cdots, i_r + 1; j_1, \cdots, j_r; k_1 + 1, \cdots, k_r + 1) \in S_r^{n+1}$.

Proof. By Theorem 4 we may assume $n = k_r$. Let (x_p), (y_p) and (z_p), $1 \leqq p \leqq n+1$, be orthonormal sequences of eigenvectors corresponding to the eigenvalues (α_p), (β_p) and (γ_p) of A, B and $A + B$. Since $i_u + j_v \geqq k_{w-1} + n + 2$, there exists an n dimensional subspace M containing the vectors x_p, $i_u + 1 \leqq p \leqq n+1$, the vectors y_p, $j_v + 1 \leqq p \leqq n+1$, and the vectors z_p, $1 \leqq p \leqq k_{w-1}$. Let (α'_p), (β'_p), and (γ'_p) be the eigenvalues of A_M, B_M, and $(A + B)_M$. By hypothesis

$$\gamma'_{k_1} + \cdots + \gamma'_{k_r} \leqq \alpha'_{i_1} + \cdots + \alpha'_{i_r} + \beta'_{j_1} + \cdots + \beta'_{j_r} .$$

The theorem now follows because $\gamma'_p = \gamma_p$ for $1 \leqq p \leqq k_{w-1}$, $\gamma_{p+1} \leqq \gamma'_p$ for $k_w \leqq p \leqq n$, $\alpha'_p \leqq \alpha_p$ for $1 \leqq p \leqq i_{u-1}$, $\alpha'_p = \alpha_{p+1}$ for $i_u \leqq p \leqq n$, $\beta'_p \leqq \beta_p$ for $1 \leqq p \leqq j_{v-1}$ and $\beta'_p = \beta_{p+1}$ for $j_v \leqq p \leqq n$.

[3] This theorem was suggested by a special case which was pointed out to me by Alan J. Hoffman.

Theorem 5 yields a simple proof of the following theorem due to Lidskii.

THEOREM 6 [3]. *If* $1 \leqq p_1 < \cdots < p_r \leqq n$, *then* $(p_1, \cdots, p_r; 1, \cdots, r; p_1, \cdots, p_r) \in S_r^n$.

Proof. Obviously $(1, \cdots, r; 1, \cdots, r; 1, \cdots, r) \in S_r^n$. Using Theorem 5 $p_1 - 1$ times with $u = w = 1$, $v = r + 1$, we find $(p_1, p_1 + 1, \cdots, p_1 + r - 1,; 1, \cdots, r: p_1, p_1 + 1, \cdots, p_1 + r - 1) \in S_r^{p_1+r-1}$. Such use of Theorem 5 is justified since $i_1 + j_{r+1} = i_1 + k_r + 1 \geqq k_r + 2 = k_0 + k_r + 2$ at each stage. We may now apply Theorem 5 $p_2 - (p_1 + 1)$ times with $u = w = 2$, $v = r + 1$ since at each stage $i_2 + j_{r+1} = i_2 + k_r + 1 \geqq i_1 + k_r + 2 = k_1 + k_r + 2$. The result is

$$(p_1, p_2, p_2 + 1, \cdots, p_2 + r - 2; 1, \cdots, r; p_1, p_2, p_2 + 1, \cdots, p_2 + r - 2) \in S_r^{p_2+r-2} .$$

Continuing in this way we find

$$(p_1, \cdots, p_r; 1, \cdots, r; p_1, \cdots, p_r) \in S_r^{p_r} .$$

By Theorem 4 the proof is complete.

THEOREM 7. $(i_1; j_1; k_1) \in S_1^n$ *for* $n \geqq k_1$ *if and only if* $1 \leqq i_1 \leqq k_1$, $1 \leqq j_1 \leqq k_1$, *and* $i_1 + j_1 = k_1 + 1$.

Proof. The sufficiency of the conditions, due to Weyl, is usually proved by the minimax principle. It can also be proved using Theorem 5. We have already seen the necessity of $i_1 \leqq k_1$ and $j_1 \leqq k_1$ in the proof of Theorem 4. Now suppose $i_1 + j_1 \geqq k_1 + 2$. Let $A = \operatorname{diag}(1, \cdots, 1, 0, \cdots, 0)$ with $i_1 - 1$ ones, and $B = \operatorname{diag}(0, \cdots, 0, 1, \cdots, 1)$ with $j_1 - 1$ ones, where the orders of A and B are k_1. Since $k_1 - j_1 + 1 \leqq i_1 - 1$, all the eigenvalues of $A + B$ are $\geqq 1$. Therefore $\gamma_{k_1} \geqq 1$, while $\alpha_{i_1} = \beta_{j_1} = 0$, contradicting $(i_1; j_1; k_1) \in S_1^k$.

THEOREM 8. *If* i, j *and* k *are ordered pairs of integers satisfying*

$$1 \leqq i_1 < i_2 \leqq n, \qquad 1 \leqq j_1 < j_2 \leqq n, \qquad 1 < k_1 \leqq k_2 \leqq n \tag{22}$$

$$i_1 + j_1 \leqq k_1 + 1 \tag{23}$$

$$\left.\begin{matrix} i_1 + j_2 \\ i_2 + j_1 \end{matrix}\right\} \leqq k_2 + 1 \tag{24}$$

$$i_1 + i_2 + j_1 + j_2 = k_1 + k_2 + 3 , \tag{25}$$

then $(i; j; k) \in S_2^n$.

Proof. By Theorem 4 we may assume $n = k_2$. We proceed by induction on n. If $n = 2$ the theorem follows from (1). Suppose the theorem holds for all $n < N$, where $N > 2$. By (22), (23) and (24), $i_p \leqq k_p$ and $j_p \leqq k_p$, $p = 1, 2$. Suppose $i_1 > 1$. Then the pairs $(i_1 - 1, i_2 - 1)$, (j_1, j_2) and $(k_1 - 1, k_2 - 1)$ satisfy (22)–(25). Therefore by the induction hypothesis $(i_1 - 1, i_2 - 1; j_1, j_2; k_1 - 1, k_2 - 1) \in S_2^{N-1}$. If we apply Theorem 5 with $u = w = 1$, $v = 3$, we find $(i; j; k) \in S_2^N$. A similar method takes care of the case $j_1 > 1$. Therefore we may assume

$$i_1 = j_1 = 1 \ . \tag{26}$$

If

$$(i_1, i_2 - 1; j_1, j_2 - 1; k_1 - 1, k_2 - 1) \in S_2^{N-1} \tag{27}$$

and if

$$i_2 + j_2 \geqq 3 + k_2 \ , \tag{28}$$

then Theorem 5 with $u = v = 2$, $w = 1$ allows us to conclude $(i; j; k) \in S_2^N$. But the Condition (28) which is needed for the application of Theorem 5 will also guarantee (27). To see this, first note that (27) can fail only when

$$i_2 = i_1 + 1 = 2 \tag{i}$$

or

$$j_2 = j_1 + 1 = 2 \tag{ii}$$

or

$$k_1 = 1 \tag{iii}$$

or

$$i_1 + j_1 = k_1 + 1 \ . \tag{iv}$$

If (i) holds then $i_2 + j_2 = 2 + j_2 \leqq 2 + k_2$, contradicting (28). Similarly (ii) cannot hold. If (iii) holds, then by (26), $i_1 + i_2 + j_1 + j_2 = 2 + i_2 + j_2 = k_1 + k_2 + 3 = k_2 + 4$, or $i_2 + j_2 = k_2 + 2$, contradicting (28). Condition (iv) implies (iii) by (26). Therefore we may assume

$$i_2 + j_2 \leqq 2 + k_2 \ . \tag{29}$$

If $i_2 \geqq k_1 + 2$, it is easy to show by the induction hypothesis that $(i_1, i_2 - 1; j_1, j_2; k_1, k_2 - 1) \in S_2^{N-1}$ and Theorem 5 with $u = w = 2$, $v = 3$ implies $(i; j; k) \in S_2^N$. Hence we assume

$$i_2 \leqq k_1 + 1 \text{ and } j_2 \leqq k_1 + 1 \ . \tag{30}$$

Now (25) and (26) imply $i_2 + j_2 = k_1 + k_2 + 1$, which with (29) implies

$k_1 = 1$. Therefore by (30) and (22), $i_2 = j_2 = 2$ and hence $i_1 = j_1 = 1$. Using (25) we find $k_2 = 2$, contradicting $N > 2$. The proof is complete.

If in (25) we replace the equality sign by $\leqq$, Theorem 8 remains true. For if i, j and k satisfy (22)–(24) and the modified (25), there exists a pair $k' = (k_1', k_2')$ such that $k_1' \leqq k_1, k_2' \leqq k_2$ and i, j, k' satisfy (22)–(25). However Theorem 2 suggests that we consider only cases where (19) holds. Conditions (23) and (24) combined may be expressed as follows:

$i_u + j_v \leqq k_w + 1$ whenever $1 \leqq u \leqq 2$, $1 \leqq v \leqq 2$, $1 \leqq w \leqq 2$, and $u + v = w + 1$. This suggests the following conjecture. Let us define inductively the following sequence of sets of triples of sequences of integers: Let $(i_1; j_1; k_1) \in T_1^n$ if $1 \leqq i_1 \leqq n, 1 \leqq j_1 \leqq n, 1 \leqq k_1 \leqq n$, and $i_1 + j_1 = k_1 + 1$, and let $(i_1, \cdots, i_r; j_1, \cdots, j_r; k_1, \cdots, k_r) \in T_r^n$ if $1 \leqq i_1 < \cdots < i_r \leqq n, 1 \leqq j_1 < \cdots < j_r \leqq n, 1 \leqq k_1 < \cdots < k_r \leqq n$, and

(31) $$i_1 + \cdots + i_r + j_1 + \cdots + j_r = k_1 + \cdots + k_r + r(r+1)2,$$

and

(32) $$i_{u_1} + \cdots + i_{u_s} + j_{v_1} + \cdots + j_{v_s} \leqq k_{w_1} + \cdots + k_{w_s} + s(s+1)/2$$

whenever

$$(u; v; w) \in T_s^r, 1 \leqq s \leqq r - 1 \; .$$

Theorem 7 and 8 show that $T_r^n \subset S_r^n$ for $r = 1, 2$. It seems reasonable to conjecture $T_r^n \subset S_r^n$ for all r. I cannot prove this in general and I know no counterexamples. The case $r = 3$ is the following.

THEOREM 9. *If i, j and k are ordered triples of integers such that*

(33) $$1 \leqq i_1 < i_2 < i_3 \leqq n, 1 \leqq j_1 < j_2 < j_3 \leqq n, 1 \leqq k_1 < k_2 < k_3 \leqq n$$

(34) $$i_1 + j_1 \leqq k_1 + 1$$

(35) $$\left.\begin{matrix} i_1 + j_2 \\ i_2 + j_1 \end{matrix}\right\} \leqq k_2 + 1$$

(36) $$\left.\begin{matrix} i_1 + i_3 \\ i_2 + j_2 \\ i_3 + j_1 \end{matrix}\right\} \leqq k_3 + 1$$

(37) $$i_1 + i_2 + j_1 + j_2 \leqq k_1 + k_2 + 3$$

(38) $$\left.\begin{matrix} i_1 + i_2 + j_1 + j_3 \\ i_1 + i_3 + j_1 + j_2 \end{matrix}\right\} \leqq k_1 + k_3 + 3$$

(39) $$\left.\begin{matrix} i_1 + i_2 + j_2 + j_3 \\ i_2 + i_3 + j_1 + j_2 \\ i_1 + i_3 + j_1 + j_3 \end{matrix}\right\} \leqq k_2 + k_3 + 3$$

$$i_1 + i_2 + i_3 + j_1 + j_2 + j_3 = k_1 + k_2 + k_3 + 6 \ , \tag{40}$$

then $(i; j; k) \in S_3^n$.

Proof. The proof begins along the same lines as the proof of Theorem 8 and will only be sketched. We may assume $n = k_3$, and proceed by induction on n. When $n = 3$, $i_1 = j_1 = k_1 = 1$, $i_2 = j_2 = k_2 = 2$, $i_3 = j_3 = k_3 = 3$, and the result follows from (1). Assume the theorem for all $n < N$, where $N > 3$. As in Theorem 8, we may assume

$$i_1 = j_1 = 1 \ . \tag{41}$$

If

$$(i_1, i_2 - 1, i_3 - 1), (j_1 j_2, j_3 - 1), (k_1 - 1, k_2 - 1, k_3 - 1) \tag{42}$$

satisfies (33)–(40) and if

$$i_2 + j_3 \geqq k_3 + 3 \tag{43}$$

then the induction hypothesis and Theorem 5 with $u = 2$, $v = 3$, $w = 1$ yield the theorem. Again the condition (43) which is needed for the application of Theorem 5 will guarantee (42). For example $k_1 - 1 \geqq 1$, because if $k_1 = 1$, then by (38) and (41), $i_2 + j_3 \leqq k_3 + 2$, contradicting (43). The second inequality of (36) together with (43) and $j_3 \leqq k_3$ (which follows from (36)) ensure $j_3 - 1 > j_2$. We may therefore assume

$$\left.\begin{matrix} i_2 + j_3 \\ i_3 + j_2 \end{matrix}\right\} \leqq k_3 + 2 \ . \tag{44}$$

Next we show that we may assume

$$i_2 \leqq k_1 + 1 \quad \text{and} \quad j_2 \leqq k_1 + 1 \tag{45}$$

by showing that if $i_2 \geqq k_1 + 2$, then $(i_1, i_2 - 1, i_3 - 1; j_1, j_2, j_3; k_1, k_2 - 1, k_3 - 1) \in S_3^{N-1}$ and Theorem 5 with $u = 2$, $v = 3$, $w = 2$ gives $(i; j; k) \in S_3^N$. In a similar manner we may assume

$$i_3 + j_3 \leqq k_1 + k_3 + 2 \tag{46}$$

$$i_3 \leqq k_2 + 1, j_3 \leqq k_2 + 1 \ . \tag{47}$$

Now (33)–(41) together with (44)–(47) are easily seen to imply $k_1 + k_2 = k_3$, $i_2 = j_2 = k_3 + 1$, $i_3 = j_3 = k_2 + 1$ and $k_1 + 1 \leqq k_2 \leqq 2k_1$. Therefore the theorem will be proved if we can show that

$$(1, p + 1, p + q + 1; 1, p + 1, p + q + 1; p, p + q, 2p + q) \in S_3^n$$

whenever $1 \leqq q \leqq p$ and $2p + q = n$.

Let A, B and $A + B$ be of order n with eigenvalues (α_p), (β_p) and

(γ_p). We have $q\gamma_p \leqq \gamma_p + \gamma_{p-1} + \cdots + \gamma_{p-r+1}$, $q\gamma_{p+q} \leqq \gamma_{p+q} + \cdots + \gamma_{p+1}$, and $q\gamma_{2p+q} \leqq \gamma_{2p+q} + \cdots + \gamma_{2p+1}$. Hence

$$q(\gamma_p + \gamma_{p+q} + \gamma_{2p+q}) \leqq \operatorname{trace}(A + B) - (\gamma_1 + \cdots + \gamma_{p-q} + \gamma_{p+q+1} + \cdots + \gamma_{2p}) .$$

Similarly

$$q(\alpha_1 + \alpha_{p+1} + \alpha_{p+q+1}) \geqq \operatorname{trace} A - (\alpha_{q+1} + \cdots + \alpha_p + \alpha_{p+2q+1} + \cdots + \alpha_{2p+q})$$

and we have a similar statement for the β's. Therefore we need only prove

$$(q + 1, \cdots, p, p + 2p + 1, \cdots, 2p + q; q + 1, \cdots, p, p + 2q + 1, \cdots, 2p + q; 1, \cdots, p - q, p + q + 1, \cdots, 2p) \in \tilde{S}^n_{2p-2q} .$$

This will follows from Theorem 3 (ii) if we can show

$$(48) \qquad (1, \cdots, p - q, p + q + 1, \cdots, 2p; 1, \cdots, p - q, p + q + 1, \cdots, 2p; q + 1, \cdots, p, p + 2q + 1, \cdots, 2p + q) \in S^n_{2p-2q} .$$

By Theorem 6 we have

$$(1, \cdots, 2p - 2q; 1, \cdots, p - q, p + 1, \cdots, 2p - q; 1, \cdots, p - q, p + 1, \cdots, 2p - q) \in S^{2p-q}_{2p-2q} .$$

We may apply Theorem 5 q times with $u = w = p - q + 1$, $v = 2p - 2q + 1$ to obtain

$$(1, \cdots, p - q, p + 1, \cdots, 2p - q; 1, \cdots, p - q, p + 1, \cdots, 2p - q; 1, \cdots, p - q, p + q + 1, \cdots, 2p) \in S^{2p}_{2p-2q} .$$

Theorem 5 applied q times with $u = v = p - q + 1$, $w = 1$ yields (48). The proof is now complete.

A proof of $T^n_4 \subset S^n_4$ along the same lines runs into the following difficulty. The first half of the proof, that is, the application of Theorem 5 in all possible ways, carries through. However the cases left untouched turn out to be too numerous to handle by the methods of the second half of the proof of Theorem 9. I have verified $T^n_4 \subset S^n_4$ for $n \leqq 8$.

As for the statement $S^n_r \subset T^n_r$, it is possible to show by a consideration of diagonal matrices that if $(i; j; k) \in S^n_r$ then (32) holds for $s = 1, 2$. This together with the remark following Theorem 8 determines S^n_3. But the general statement $S^n_r \subset T^n_r$ is false even if we weaken the definition of T^n_r by replacing the equality sign in (31) by $\leqq$. For example a consideration of the trace condition shows that $(1, 5, 9, 12; 1, 5, 9, 12; 4, 8, 12, 16) \in S^{16}_4$.

Guided by Theorem 3 (ii), the dual set $\tilde{T}^n_r$ may be defined inductively as follows: $(i_1, j_1, k_1) \in \tilde{T}_1^n$ if $i_1 + j_1 = k_1 + n$, and $(i; j; k) \in \tilde{T}_r^n$ if $i_1 + \cdots + i_r + j_1 + \cdots + j_r = k_1 + \cdots + k_r + nr - r(r-1)/2$ and

$$i_{u_1} + \cdots + i_{u_s} + j_{v_1} + \cdots + j_{v_s} \geqq k_{w_1} + \cdots + k_{w_s} + ns - s(s-1)/2$$

whenever $(u; v; w) \in \tilde{T}_s^r$. It is easily seen that $(i; j; k) \in T_r^n$ if and only if $(n - i_r + 1, \cdots, n - i_1 + 1; n - j_r + 1, \cdots, n - j_1 + 1; n - k_r + 1, \cdots, n - k_1 + 1) \in \tilde{T}_r^n$. Hence by Theorem 3, $T_r^n \subset S_r^n$ is equivalent to $\tilde{T}_r^n \subset \tilde{S}_r^n$. I have been unable to prove the analogue of the last transformation rule of Theorem 3. However I can prove that if $(i; j; k) \in \tilde{T}_1^n$, then $(i'; j'; k') \in T_{n-1}^n$, where i', j', and k' are the complements with respect to n.

3. The set E. We return to the problem of determining the set E defined in the introduction. Let F be the set of points γ defined by $\gamma_1 \geqq \cdots \geqq \gamma_n$,

$$\gamma_1 + \cdots + \gamma_n = \alpha_1 + \cdots + \alpha_n + \beta_1 + \cdots \beta_n,$$

and

$$\gamma_{k_1} + \cdots + \gamma_{k_r} \leqq \alpha_{i_1} + \cdots + \alpha_{i_r} + \beta_j + \cdots + \beta_{j_r}$$

whenever

$$(i; j; k) \in T_r^n, 1 \leqq r \leqq n - 1 .$$

In § 2 we have shown that $E \subset F$ for $n \leqq 4$. In this section we will prove that $E = F$ for $n \leqq 4$.

There is no loss of generality in assuming $\alpha_1 > \cdots > \alpha_n$ and $\beta_1 > \cdots > \beta_n$. The set E' defined in § 1 is a closed subset of E. Since F is closed and convex, it will follow that $E' = F$, and therefore $E = F$, if the boundary of E' is contained in the boundary of F. To see this, let γ be an interior point of E' and suppose γ' is any point of F. If γ' is not in E' there must be a boundary point of E' in the open segment joining γ and γ'. But all points of this open segment are interior points of F.

A boundary points of E' with at least two equal coordinates is obviously a boundary point of F. If γ is a boundary point of E' with distinct coordinates, there is associated with γ a triple $(i; j; k)$ satisfying the conditions of Theorem 2. All that remains to prove is that $(i; j; k) \in T_r^n$. To this end we first prove the following theorem.

THEOREM 10. *If γ is a boundary point of E' with associated sequences $(i; j; k)$ of order r, then for any $(x; y; z) \in \tilde{S}_m^r$, there cannot exist a triple $(u; v; w) \in S_m^{n-r}$ such that $i_{x_p} \leqq x_p + u_p - 1$, $j_{y_p} \leqq y_p + v_p - 1$, and $k_{z_p} \geqq z_p + w_p$, for $1 \leqq p \leqq m$.*

Proof. For convenience, we write $\alpha(p)$ instead of α_p. By hypothesis there exist Hermitian matrices A_1, B_1, and $A_1 + B_1$ with eigenvalues $(\alpha(i_p))$, $(\beta(j_p))$, and $(\gamma(k_p))$, $p = 1, \cdots, r$, and Hermitian matrices A_2, B_2, and $A_2 + B_2$ with eigenvalues $(\alpha(i'_p))$, $(\beta(j'_p))$, and $(\gamma(k'_p))$, $p = 1, \cdots, n - r$, where i' is complement of i with respect to n. If there exists a triple $(u; v, w) \in S_m^{n-r}$ such that $i_{x_p} < i'_{u_p}$, $j_{y_p} < j'_{v_p}$, and $k'_{z_p} > k'_{w_p}$, $1 \leqq p \leqq m$, then we have

$$\sum_{p=1}^{m} \alpha(i_{x_p}) + \sum_{p=1}^{m} \beta(j_{y_p}) \leqq \sum_{p=1}^{m} \gamma(k_{z_p}) < \sum_{p=1}^{m} \gamma(k'_{w_p}) \leqq \sum_{p=1}^{m} \alpha(i'_u) + \sum_{p=1}^{m} \beta(j'_v) .$$

This is impossible since $\alpha(i_{x_p}) > \alpha(i'_{u_p})$ and $\beta(j_{y_p}) < \beta(j'_{v_p})$. Therefore it remains only to show that $i_p < i'_q$ is implied by $i_p \leqq p + q - 1$. If $i_p \leqq p + q - 1$, then at least p terms of the sequence i are $\leqq p + q - 1$. Therefore at most $q - 1$ positive integers $\leqq p + q - 1$ are not in i. Hence $i'_q > p + q - 1 \geqq i_p$.

THEOREM 11. *If γ is a boundary point of E' with associated sequences i, j, k of order r, then $i_x + j_y \geqq k_z + r$ whenever $(x, y, z) \in \tilde{T}_1^r$. More generally, if $x + y \geqq z + r$, the $i_x - x + j_y - y \geqq k_z - z$.*

Proof. We have $n \geqq r + 1 \geqq 2$. Since $(x; y; x + y - r) \in \tilde{T}_1^r \subset \tilde{S}_1^r$, it follows that $(x; y; z) \in \tilde{S}_1^r$. Let $u = i_x - x + 1$, $v = j_y - y + 1$, and $w = k_z - z$. Clearly, $u \geqq 1$, $v \geqq 1$, and $w \leqq n - r$ since $k_1 - 1 \leqq k_2 - 2 \leqq \cdots \leqq k_r - r \leqq n - r$. We must prove $u + v \geqq w + 2$. If $u + v \leqq w + 1$, then $w \geqq 1$, $u \leqq w$, and $v \leqq w$. Therefore $(u; v; w) \in T_1^{n-r}$. This contradicts Theorem 10.

THEOREM 12. *Under the same hypothesis as Theorem 11, if $n \geqq r + 2$, then $i_{x_1} + i_{x_2} + j_{y_1} + j_{y_2} \geqq k_{z_1} + k_{z_2} + 2r - 1$ whenever $(x, y, z) \in \tilde{T}_2^r$.*

Proof. We are given $x_1 + y_2 \geqq z_1 + r$, $x_2 + y_1 \geqq z_1 + r$, $x_2 + y_2 \geqq z_2 + r$, and $x_1 + x_2 + y_1 + y_2 = z_1 + z_2 + 2r - 1$. Let $a_p = i_{x_p} - x_p + 1$, $b_p = j_{y_p} - y_p + 1$, and $c_p = w_{z_p} - z_p$, $p = 1, 2$. By Theorem 11, $a_1 + b_2 \geqq c_1 + 2$, $a_2 + b_1 \geqq c_1 + 2$, and $a_2 + b_2 \geqq c_2 + 2$. Suppose the theorem fails. Then $a_1 + a_2 + b_1 + b_2 \leqq c_1 + c_2 + 3$. Therefore

$$a_1 + b_1 \leqq c_1 + 1 \tag{49}$$

$$a_1 + b_2 \leqq c_2 + 1 \tag{50}$$

$$a_2 + b_1 \leqq c_2 + 1 . \tag{51}$$

Also $1 \leqq a_1 \leqq a_2$, $1 \leqq b_1 \leqq b_2$, and $c_2 \leqq n - r$. By (49), $c_1 \geqq 1$. Moreover $c_1 + 2 \leqq a_1 + b_2 \leqq c_2 + 1$, so that $c_1 + 1 \leqq c_2$. Now let $u_1 = a_1$, $u_2 = \max(a_2, a_1 + 1)$, $v_1 = b_1$, $v_2 = \max(b_2, b_1 + 1)$, $w_1 = c_1$, and $w_2 = c_2$. It is easy to see that $u_1 + v_1 \leqq w_1 + 1$, $u_1 + v_2 \leqq w_2 + 1$, $u_2 + v_1 \leqq w_2 + 1$, and

$u_1 + u_2 + v_1 + v_2 \leqq w_1 + w_2 + 3$. As previously remarked there exists a pair (w_1', w_2') such that $w_1' \leqq w_1, w_2' \leqq w_2$, and $(u; v; w) \in T_2^{n-r}$. This contradicts Theorem 10.

Using a generalized version of Theorem 12, it is possible to show that

$$i_{x_1} + i_{x_2} + i_{x_3} + j_{y_1} + j_{y_2} + j_{y_3} \geqq k_{z_1} + k_{z_2} + k_{z_3} + r + r - 1 + r - 2$$

whenever $(x; y; z) \in \tilde{T}_3^r$, $n \geqq r + 2$.

THEOREM 13. *If γ is a boundary point of E' with associated sequences i, j, k of order $r = 1, 2, 3$ or $n - 1$, then $(i; j; k) \in T_r^n$.*

Proof. For $r = 1$ this is obvious. For $r = n - 1$, the complementary sequences with respect to n are of order 1 and satisfy $i_1' + j_1' = k_1' + n$. Therefore $(i'; j'; k') \in \tilde{T}_1^n$. By the last sentence of § 2, it follows that $(i; j; k) \in T_{n-1}^n$. For the cases $n = 3, 4$ this can be easily verified by listing cases. Now suppose $r = 2$. We must prove that (23) and (24) hold. In view of (25), this means we must show that $i_x + j_y \geqq k_z + 2$ whenever $(x; y; z) \in \tilde{T}_1^2$. But this follows from Theorem 11. Suppose $r = 3$. We may assume $n \geqq 5$. By (40) and Theorems 11 and 12 we have (34)–(39), since if $(x; y; z) \in \tilde{T}_p^3$ then $(x'; y'; z') \in T_{3-q}^3$, $p = 1, 2$.

Theorem 13 completes the proof that $E = F$ for $n \leqq 4$. It is possible to extend the proof to $n \leqq 8$. But the general case remains open.

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