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UTTERLY INTEGER VALUED ENTIRE FUNCTIONS (I)

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In Loving Memory of Ernst G. Straus

An entire function $f(z)$ is called *utterly integer valued* if $f(x)$ and all its derivatives assume integer values for all integer z. A historical survey of the theory of such functions is given, and a new class of them is constructed. There is no utterly integer valued entire functions of finite order except polynomials.

1. Introduction. G. Pólya [**21–23**] studied entire functions which take integral values at all nonnegative integral points. This is generally considered to be the origin of the research on arithmetic properties of analytic functions. Unless otherwise stated, a "function" in this paper means an analytic (entire) functin of a complex variable.

A function $w = f(z)$ is called *an integer valued function* if $f(l) =$ integer for $l = 0, 1, 2, \ldots$ [**21**]. A function $f(z)$ is called a *completely* integer valued function if $f(l) =$ integer for $l = 0, \pm 1, \pm 2, \ldots$ [**9**, **23**]. A function $f(z)$ is called a *q-fold integer valued function* if $f(z)$ and its derivatives $f'(z), f''(z), \ldots, f^{(q-1)}(z)$ are all integer valued [**28**, **30**]. On the other hand, a *Hurwitz function* is defined to be a function $f(z)$ which together with *all* its derivatives assumes integral values at the origin $z = 0$, [**32–35**]. If the function $f(z)$ and all its derivatives assume integral values at k consecutive integral points, say $z = 0, 1, 2, \ldots, k-1$, then $f(z)$ is called a *k-point Hurwitz function* [**36–37**, **40–44**]. These concepts of integral valued functions lead to the following

DEFINITION. Given two sets S and T of complex numbers, we say that a function $f(z)$ is *infinitely T-valued at S*, if the function and all its derivatives assume values in T at all points of the set S.

Here S is called *the set of interpolation* and T is called *the value set*. An infinitely T-valued function is called *an utterly T-valued function* if the set of interpolation S is *unbounded*. In particular, if S and T are the set of all rational integers, or the sets of all algebraic integers in a fixed imaginary quadratic number field K, then we say that an utterly T-valued function at S is an *utterly K-integer valued function*.

In this paper, the set of all utterly integer valued functions is denoted by $\mathcal{U}$.

Since Pólya's original paper of 1915, properties of (finitely) integer valued functions have been studied to a considerable extent. However, research on infinitely integer valued functions such as k-point Hurwitz functions with $k \geq 2$ was originated by E. G. Straus in 1950 **[41]**. After his paper on the arithmetic properties of k-point Hurwitz functions appeared, many generalizations and refinements followed. The later papers deal with infinitely T-valued functions with various different sets S and T. Most of this research was, however, restricted to those cases where the set of interpolation S is a finite (hence bounded) set of points. The study of the set of utterly integer valued functions other than polynomials seems to be much less complete.

This paper is the first of several reports on the set of utterly integer valued *transcendental* functions. We start by giving simple examples of such transcendental functions and discuss their significance with respect to the analytic-arithmetic properties which are similar to those already obtained for the various integer valued transcendental functions. The constructioin of such functions in §§3 and 4 answers affirmatively the question posed by E. G. Straus in 1951 **[6]**.

2. Survey of present knowledge. We deal with an entire function $w = f(z)$; let $M(r)$ be the maximum modulus of $f(z)$ on the circle $|z| = r$. The order ρ of $f(z)$ is defined by

$$\rho = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r}, \tag{1}$$

and when $0 < \rho < \infty$, the type σ of $f(z)$ is defined to be

$$\sigma = \limsup_{r\to\infty} \frac{\log M(r)}{r^\rho}. \tag{2}$$

The order ρ and the type σ are analogues of the concept of the degree

$$N = \limsup_{r\to\infty} (\log M(r))/(\log r)$$

and the size of the leading coefficient $L = \limsup_{r\to\infty} M(r)/r^N$ of a polynomial. These quantities N, L, ρ and σ measure the rate of growth of the function $f(z)$ when $|z| \to \infty$ **[46]**. If the order ρ is finite, then we say that the function is of *finite order*, and in particular when the order is $\rho = 1, f(z)$ is said to be of *exponential order*.

The original result of G. Pólya **[21]** states that among all integer valued transcendental functions, that of the least rate of growth is given by the exponential function $f(z) = 2^z$. More precisely, an integer valued function $f(z)$ is a polynomial if its order is $\rho < 1$ or if it is has order $\rho = 1$ and type $\sigma < \log 2$. This theorem has inspired many further results. Although the number 2 here cannot be replaced by anything larger, a slightly larger rate of growth for $f(z)$ still allows only a very restricted class of integer valued functions [**9**, **24**–**26**]. The corresponding growth rate for completely integer valued transcendental functions is attained by the function

$$f(z) = \frac{1}{\sqrt{5}}\left\{\left(\frac{3+\sqrt{5}}{2}\right)^{z} - \left(\frac{3+\sqrt{5}}{2}\right)^{-z}\right\}, \tag{3}$$

instead of 2^z. The completely integer valued transcendental function (3) is of exponential order with type $\sigma = \log((3 + \sqrt{5})/2)$, below which one finds no such transcendental functions. The transcendental Hurwitz function of the least possible order and type is given by $f(z) = e^z$, which is of order $\rho = 1$ and tye $\sigma = 1$ [**32**, **34**, **35**].

The construction of various integer valued transcendental function of least possible order and type and the determination of the least upper bound for the order and/or type of such functions below which one finds only polynomials is one of the fundamental studies of arithmetic properties of analytic functions. Gelfond [**28**] has shown that the q-fold integer valued transcendental functions must have order $\rho \geq 1$, and if $\rho = 1$, then the type must be at least $\sigma = q\log(1 + \exp((1 - q)/q))$. In the case $q = 1$, the example $f(z) = 2^z$ shows that this bound for the type σ is best possible [**30**].

The set of k-point Hurwitz functions for $k \geq 2$ is a more restrictive set of functions, because the values of all higher derivatives of the function $f(z)$ at any one point of the set of interpolation uniquely determine the values of the entire function $f(z)$ and all its higher derivatives $f^{(m)}(z)$ at any other point of the complex plane. There is *no* k-point Hurwitz function of exponential order if $k \geq 2$ [**35**–**42**]. A theorem of Straus [**41**] states that every k-point Hurwitz function must be either a polynomial, or of order $\rho \geq k$. If the order is $\rho = k$, then its type σ must be $\geq 1/((k-1)!)^2$. The example of the k-point transcendental Hurwitz function $f(z) = \exp(z(z-1)(z-2)\cdots(z-k+1))$, having order $\rho = k$ and type $\sigma = 1$, shows that the bound for the order $\rho \geq k$ given by Straus is best possible. Some improvement of the bound for the type σ was obtained by D. Sato [**40**]. The basis of the differential ring of utterly integer valued polynomials [**6**] gives a simple example of k-point Hurwitz transcendental functions which have order $\rho = k$ and type

$$\sigma_k = \prod_{p=\text{prime}} p^{[k/p]} \Big/ k!.$$

This is attained by the function

$$(4)\quad f(z) = \exp\left(\prod_{p=\text{prime}} p^{[k/p]} z(z-1)(z-2)\cdots(z-k+1)\Big/ k! \right).$$

Note that $\sigma_k < 1$ for $k = 4, 5, 6, \ldots$ [**44**]. Less simple, but arithmetically more significant constructions are given in [**38**–**40**].

One of the immediate but important consequences of the theorems on k-point Hurwitz functions is the following.

COROLLARY. *There is no utterly integer valued transcendental entire function of finite order.*

This is probably the reason Straus conjectured once that there would be *no* utterly integer valued transcendental functions at all [**6**].

It is to be noted that $\mathscr{U}$, the family of all utterly integer valued functions, forms a ring which is not only closed under differentiation but also closed under composition. Does there exist a function $f(z) \in \mathscr{U}$ which is *not* a polynomial? Straus asked this question just after the complete characterization of the composition-closed differential ring $\mathscr{U}_p$ of utterly integer valued polynomials was established. If $\mathscr{U}_p = \mathscr{U}$, then the structure of $\mathscr{U}$ is now known. If $\mathscr{U}_p \neq \mathscr{U}$, then what is the analytic-arithmetic structures of $\mathscr{U}$? This is an interesting and challenging problem.

We now know, and Straus himself knew, that $\mathscr{U} - \mathscr{U}_p \neq \varnothing$ and that a nondenumerable subset of transcendental functions in $\mathscr{U}$ exists. It can be constructed using the method of generalized interpolation by analytic functions developed by Straus and Sato [**47**, **48**]. However, to the best of the present author's knowledge, *no simple concrete examples* of such transcendental analytic functions $f(z) \in \mathscr{U}$ have appeared in the literature. A construction of such concrete examples can be made by the composition of an ordinarily interpolated function of exponential order and the *periodic integer valued function* $w = \sin(2\pi z)$ or $w = \sin(\pi z)$, both of which have zeros for *all* integral values of z.

3. Utterly integer valued entire functions. The following construction probably gives the simplest transcendental functions in $\mathscr{U}$.

THEOREM. *There exists a non-denumerable set of utterly integer valued transcendental entire functions.*

Proof. Let

$$(5)\qquad f(z) = \sum_{n=0}^{\infty} a_n \sin^n(2\pi z).$$

We want to determine the coefficients $a_n \neq 0$ in (5) in such a way that $f(z) \in \mathscr{U}$. Let $a_0 = 1$; then $f(l) = 1$ for all integral values of l. We determine a_n successively by induction. Since $\sin(2\pi l) = 0$ for all integers l, we have

$$f^{(n)}(l) = a_n n! \cos^n(2\pi l) \cdot (2\pi)^n + R(n), \tag{6}$$

where $R(n)$ is the sum of the terms which involve $a_0, a_1, \ldots, a_{n-1}$. Note that $R(n)$ is independent of the particular choices of the integer l. In order to make $|a_n| \neq 0$, but small enough so that the function (5) becomes entire, we take the value $f^{(n)}(l)$ close to $R(n)$, but not equal to it. For example, with the fixed number $B \geq 3$, we may choose $f^{(n)}(l)$ to be the integer $f^{(n)}(l) = [R(n)] + b$, where b is any integer such that $2 \leq b \leq B$. Here $[x]$ is the largest integer not exceeding x. For any choice of the value of $f^{(n)}(l)$ in such a manner, we have $1 < |f^{(n)}(l) - R(n)| \leq b \leq B$, and we get the estimate

$$\frac{1}{n!(2\pi)^n} \leq |a_n| \leq \frac{B}{n!(2\pi)^n}. \tag{7}$$

Now the series (5) having the coefficients $|a_n|$ in (7) converges for all values of $|z| < \infty$, since

$$\begin{aligned} |f(z)| &= \left| \sum_{n=0}^{\infty} a_n \sin^n(2\pi z) \right| \leq \sum_{n=0}^{\infty} |a_n| \cdot |\sin(2\pi z)|^n \\ &\leq \sum_{n=0}^{\infty} B \left| \frac{\sin(2\pi z)}{2\pi} \right|^n \Big/ n! = B \exp\left(\left| \frac{\sin(2\pi z)}{2\pi} \right| \right). \end{aligned} \tag{8}$$

The function (5) cannot be a polynomial since the $|a_n|$ were chosen to be nonzero, and it is clear from the construction that $f(z) \in \mathscr{U}$. In the process of the construction we have allowed some freedom so that there are at least two possible choices for a_n at each step of the selection of $f^{(n)}(l)$. We conclude therefore, that the cardinality of the subset of such functions in $\mathscr{U}$ has the power of the continuum.

Combining this theorem with the corollary given in §2, we get the following:

THEOREM. *All utterly integer valued transcendental entire functions must be of infinite order.*

Several refinements and generalizations of this theorem will be discussed in the later reports.

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