SOME THEOREMS IN PROBABILITY THEORY

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In memory of Ernst Straus

The following miscellany deals with questions of stochastic indepen-
dence, normality, and "quantum mechanical" probability. The author was
a close and admiring friend of Professor Straus from the time in 1941,
when he first came to the United States (on the last ship from Palestine),
until his untimely and tragic death in 1983.

0. Introduction. In the modern definitions [6, 7] of Gaussian
processes the model is that of a real Hilbert space $\mathcal{H}$, a (probability)
measure situation $(S, \mathcal{S}, P)$ and a linear isometry $T: \mathcal{H} \mapsto L^2(S)$ such
that for all f in $\mathcal{H}$, $\int_S (Tf)(x) \, dP = 0$, such that orthogonal elements f, g
in $\mathcal{H}$ map into (stochastically) independent random variables $T(f)$, $T(g)$,
and such that each nonzero Tf is normally distributed, i.e.,

$$P\{x: (T(f))(x) \le t\} = \frac{1}{\sqrt{2\pi}\,\|Tf\|_2} \int_{-\infty}^{t} \exp\left(-y^2/2\|Tf\|_2^2\right) dy.$$

Among the consequences of the results that follow is the fact that the last
two requirements stated above are mutually redundant (see §3).

The underlying phenomenon is found in the following:

THEOREM 1. *If $X, Y \in L^2(S)$, if X and Y are independent and noncon-
stant and if $X \pm Y$ are independent then X and Y are normally distributed
(and hence so are $X \pm Y$).*

The works of Skitovich and Darmois [1, 8, 9] imply this theorem but
it and some of its natural generalizations are derived below by methods
somewhat different from those used in [1, 8, 9]. Along the way there are
proved some analytical lemmas of independent interest.

1. The proof of Theorem 1. Since $X, Y \in L^2(S)$ it follows that
$X, Y \in L^1(S)$ and thus they may be normalized according to the transfor-
mation:

$$X \mapsto \left(X - \int_S X(x) \, dP\right) \Big/ \left\|X - \int_S X(x) \, dP\right\|_2 = \tilde{X}.$$

Then $\tilde{X}$, $\tilde{Y}$ are independent and nonconstant and the hypothesis *re* $X \pm Y$ implies $\tilde{X}/\sqrt{2} \pm \tilde{Y}/\sqrt{2}$ are independent. If φ and ψ are the respective characteristic functions of $\tilde{X}$ and $\tilde{Y}$, e.g., $\varphi(t) = \int_S e^{it\tilde{X}} dP$, and if

$$U = \frac{\tilde{X}}{\sqrt{2}} + \frac{\tilde{Y}}{\sqrt{2}}, \qquad V = \frac{\tilde{X}}{\sqrt{2}} - \frac{\tilde{Y}}{\sqrt{2}},$$

then U and V are independent,

$$\tilde{X} = \frac{U}{\sqrt{2}} + \frac{V}{\sqrt{2}}, \qquad \tilde{Y} = \frac{U}{\sqrt{2}} - \frac{V}{\sqrt{2}},$$

and if ξ and μ are the respective characteristic functions of U and V, the conditions of independence imply:

(1)
$$\xi(t) = \varphi(t/\sqrt{2})\psi(t/\sqrt{2})$$
$$\mu(t) = \varphi(t/\sqrt{2})\overline{\psi(t/\sqrt{2})}$$

(2)
$$\varphi(t) = \xi(t/\sqrt{2})\mu(t/\sqrt{2})$$
$$\psi(t) = \xi(t/\sqrt{2})\overline{\mu(t/\sqrt{2})}$$

Replacing ξ and μ in (2) by their formulae given by (1) yields:

$$\varphi(t) = \varphi\big(t/(\sqrt{2})^2\big)\psi\big(t/(\sqrt{2})^2\big)\varphi\big(t/(\sqrt{2})^2\big)\overline{\psi\big(t/(\sqrt{2})^2\big)}$$

$$\psi(t) = \varphi\big(t/(\sqrt{2})^2\big)\psi\big(t/(\sqrt{2})^2\big)\overline{\varphi\big(t/(\sqrt{2})^2\big)}\psi\big(t/(\sqrt{2})^2\big)$$

or

(3)
$$\varphi(t) = \Big[\varphi\big(t/(\sqrt{2})^2\big)\Big]^2\Big|\psi\big(t/(\sqrt{2})^2\big)\Big|^2$$
$$\psi(t) = \Big|\varphi\big(t/(\sqrt{2})^2\big)\Big|^2\Big[\psi\big(t/(\sqrt{2})^2\big)\Big]^2.$$

Mathematical induction applied to the process of replacing φ, ψ in the right members in (3) by the formulae provided by (3) yields the following general formula:

$$\varphi(t) = \Big[\varphi\big(t/(\sqrt{2})^{2^n}\big)/\big|\varphi\big(t/(\sqrt{2})^{2^n}\big)\big|\Big]^{2^{2n-1}}$$
$$\cdot\Big|\varphi\big(t/(\sqrt{2})^{2^n}\big)\Big|^{2^{2n-1}} \cdot \Big|\psi\big(t/(\sqrt{2})^{2^n}\big)\Big|^{2^{2n-1}}, \qquad n = 1, 2, \ldots,$$

and a similar formula for ψ.

Since

$$\varphi(t), \psi(t) = \left(1 - \frac{\|\tilde{X}\|_2^2 t^2}{2} + o(t^2)\right)$$

it follows, since also $\|\tilde{X}\|_2 = 1$, that $\varphi(t) = e^{-t^2/2} = \psi(t)$. And so the normality of X (and similarly of Y) is proved.

2. **Generalization.** Although Theorem 1 appears to be quite special, the line of its proof may be imitated in a far more general setting. If X, Y are in $L^2(S)$ and independent and if $U = aX + bY$, $V = cX + dY$ are also independent then X and Y may be normalized (so that $\int_S X \, dP = \int_S Y \, dP = 0$ and $\|X\|_2 = \|Y\|_2 = 1$) without disturbing the conclusion. Then $\int_S U \, dP = \int_S V \, dP = 0$ and so after multiplication by suitable constants the original hypothesis may be rephrased: X, Y are independent, U, V are independent and $a^2 + b^2 = c^2 + d^2 = 1$, $ac + bd = 0$. As given in the preceding sentence the hypothesis does *not* imply that X or Y or U or V is normal (e.g., $a = d = 1$, $b = c = 0$, X, Y arbitrary and independent). However, if the hypothesis, which already states that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is an orthogonal matrix, also requires that it represents a rotation through an angle θ that is not an integral multiple of $\pi/2$, then the conclusion that X, Y, U, V are all normal is forced. The supplementary requirement avoids triviality and without loss of generality is simply a normalization.

THEOREM 2. *If X, Y are in $L^2(S)$ and independent, $\int_S X \, dP = \int_S Y \, dP = 0$, $\|X\|_2 = \|Y\|_2 = 1$, if θ is such that $\cos \theta \cdot \sin \theta \neq 0$, and if $U = X \cos \theta + Y \sin \theta$, $V = -X \sin \theta + Y \cos \theta$ are independent, then U, V, X, and Y are all normally distributed.*

Proof. As in the proof of Theorem 1, it is readily established that (with the notations, c, s for $\cos \theta$, $\sin \theta$)

$$\xi(t) = \varphi(ct)\psi(st)$$

$$\mu(t) = \overline{\varphi(st)}\psi(ct)$$

$$\varphi(t) = \xi(ct)\overline{\mu(st)}$$

$$\psi(t) = \xi(st)\mu(ct).$$

Then the substitution procedure leads to

$$\varphi(t) = \varphi(c^2t)\psi(cst) \cdot \varphi(s^2t)\overline{\psi(cst)} = \varphi(c^2t)|\psi(cst)|^2\varphi(s^2t)$$

$$\psi(t) = \varphi(cst)\psi(s^2t) \cdot \overline{\varphi(cst)}\psi(c^2t) = \psi(c^2t)|\varphi(cst)|^2\psi(s^2t).$$

This time, the repeated process of replacement does not yield the simple formulas found earlier. But the fact that $\varphi(t)$, $\psi(t) = (1 - t^2/2 + o(t^2))$

allows the conclusion (established by induction)

$$\varphi(t) = \prod_{k=0}^{n} \left(1 - \binom{n}{k}(c^2)^{n-k}(s^2)^k \frac{t^2}{2} \right) + E_n(t^2), \qquad n = 1, 2, \ldots,$$

$$E_n(t^2) \to 0 \quad \text{as } n \to \infty.$$

A similar formula obtains for ψ.

At this point the following general lemmas are useful.

LEMMA 1. *If (t_{nk}) is a Toeplitz matrix ($\sum_{k=1}^{\infty} t_{nk} \to 1$ as $n \to \infty$, $t_{nk} \to 0$ as $n \to \infty$, $\sum_{k=1}^{\infty} |t_{nk}| \leq M < \infty$) and if furthermore $\sum_{k=1}^{\infty} t_{nk}^2 \to 0$ as $n \to \infty$ then for all complex z, $\prod_{k=1}^{\infty}(1 + t_{nk}z) \to e^z$ as $n \to \infty$.*

REMARKS. The condition that $\sum_{k=1}^{\infty} t_{nk}^2 \to 0$ as $n \to \infty$ cannot be dropped, e.g., if $t_{nk} = \delta_{nk}$ the conclusion is false. The special case $t_{nk} = 1/n$, $k = 1, 2, \ldots, n$, $t_{nk} = 0$, $k > n$, should be noted as related to the formula $e^z = \lim_{n \to \infty}(1 + z/n)^n$.

Proof. Since $\sum_{k=1}^{\infty} |t_{nk}| \leq M < \infty$, the infinite product $\prod_{k=1}^{\infty}(1 + t_{nk}z)$ converges for all z in $\mathbf{C}$ and represents an entire function $f_n(z)$. Furthermore, if $0 < \delta < 1/M$ there are functions $\alpha_n(z)$ such that for all n and all z such that $|z| \leq \delta, |\alpha_n(z)| \leq K < \infty$ and

$$\exp\left(t_{nk}z - (t_{nk}^2/2)z^2\alpha_n(z) \right) = (1 + t_{nk}z).$$

Hence if $|z| \leq \delta$ and $s_n = \sum_{k=1}^{\infty} t_{nk}^2$ then

$$f_n(z) = \prod_{k=1}^{\infty}(1 + t_{nk}z) = \exp\left(\sum_{k=1}^{\infty} t_{nk}z - \frac{s_n\alpha_n(z)}{2} \right) \to e^z \quad \text{as } n \to \infty.$$

Since the $f_n(z)$ and e^z are entire, $f_n(z) \to e^z$ as $n \to \infty$ for all z.

LEMMA 2. *If $0 < a, b, a + b = 1$, and if $t_{nk} = \binom{n}{k}a^{n-k}b^k$ then (t_{nk}) is a Toeplitz matrix such that $\sum_{k=1}^{\infty} t_{nk}^2 \to 0$ as $n \to \infty$.*

Proof. That (t_{nk}) is a Toeplitz matrix is immediate. The proof that $\sum_{k=1}^{\infty} t_{nk}^2 \to 0$ as $n \to \infty$ is divided into two cases.

Case 1. $a = b = \frac{1}{2}$. Then $t_{nk}^2 = \binom{n}{k}^2 2^{-2n}$. If n is even, $n = 2m$ then $\max_k \binom{n}{k}2^{-2n}$ is by Stirling's formula

$$\binom{2m}{m}2^{-4m} \doteq (1 + \delta_m)\frac{2^{-2m}}{\sqrt{\pi m}}, \qquad \delta_m \to 0 \text{ as } m \to \infty.$$

Thus since $\sum_{k=1}^{\infty}\binom{n}{k}/2^n = 1$ and $\binom{n}{k}/2^n > 0$, it follows that

$$\sum_{k=0}^{n} \frac{\binom{n}{k}^2 2^{-2n}}{2^n} \le \frac{2^{-2m}}{\sqrt{\pi m}}(1 + \delta_m)$$

and so

$$0 < \sum_{k=0}^{n} \binom{n}{k}^2 2^{-2n} \le \frac{1 + \delta_m}{\sqrt{\pi m}} \to 0 \quad \text{as } m \to \infty.$$

If n is odd a similar calculation yields the same conclusion.

Case 2. $0 < a < b, c = b/a > 1$. Then

$$\sum_{k=0}^{n} \binom{n}{k}^2 a^{2n-2k}b^{2k} = a^{2n}\sum_{k=0}^{n} \binom{n}{k}^2 c^{2k}.$$

The last term of the sum is c^{2n}, the first term of the sum is 1 and if $0 < k < n$ the ratio

$$c^{2n} \Big/ \binom{n}{k}^2 c^{2k}$$

is, again by Stirling's formula

$$(1 + \mu_{kn})k^{2k+1}\left(1 - \frac{k}{n}\right)^{2n} \cdot 2\pi \cdot (1 - k/n) \cdot \frac{1}{(n-k)^{2k}}c^{2n-2k}$$

($\mu_{kn} \to 0$ as $n \to \infty$) which is unbounded as $n \to \infty$. Hence for large n, c^{2n} is the largest term of the sum and so the sum is not more than $(n + 1)c^{2n}$. Consequently, since $0 < b < 1$,

$$0 \le a^{2n}\sum_{k=0}^{n} \binom{n}{k}^2 c^{2k} \le (ac)^{2n}(n + 1) = b^{2n}(n + 1) \to 0$$

as $n \to \infty$.

In the proof of the theorem itself the lemmas above show that (with $a = c^2, b = s^2$)

$$\varphi(t) = e^{-t^2/2}$$

as required.

3. **Gaussian processes.** In [6, 7] a Gaussian process is defined as a (linear) isometry $T: \mathcal{H} \to L^2_{\mathbf{R}}(S)$ of a (real) Hilbert space $\mathcal{H}$ into the (equivalence) classes of square-integrable **R**-valued functions on a probability sample space S. Furthermore, it is required that if x, y in $\mathcal{H}$ are

orthogonal then $T(x)$, $T(y)$ are independent and that each $T(x)$ is normally distributed with mean 0 and variance $(\mathrm{var}(T(x)))\|x\|^2$. Theorem 1 shows that the requirement that each $T(x)$ be normally distributed is redundant.

Indeed, if x and y are orthogonal so are $x \pm y$ whence $T(x) = X$ and $T(y) = Y$ are independent as are $X \pm Y$ and Theorem 1 implies they are normally distributed. The fact that T is an isometry implies that $\mathrm{var}(T(x)) = \|x\|^2$. On the other hand, the requirement $\mathrm{var}(T(x)) = \|x\|^2$ implies that T is an isometry and the additional requirement that $T(x)$ be distributed normally implies that orthogonal x, y go into orthogonal $T(x)$, $T(y)$, i.e., orthogonal normally distributed random variables having covariance 0, hence independent.

The (finitely additive) Gaussian cylinder measure induced on $\mathcal{H}$ by T is not extendable to a countably additive measure on the σ-algebra $Z_{\mathcal{H}}$ generated by the cylinders of $\mathcal{H}$ if $\mathcal{H}$ is infinite-dimensional. On the other hand, if S is a domain in $\mathbf{C}$, if the two-dimensional Lebesgue measure of S is finite (and normalized so that it is 1), if $\mathcal{H}$ is the set of $\mathbf{R}$-valued square-integrable harmonic functions on S, and if T is any bounded endomorphism of $\mathcal{H}$, then [4] T induces a *countably* additive measure on $Z_{\mathcal{H}}$. Thus no such $T\colon \mathcal{H} \to \mathcal{H}$ can satisfy either of the following requirements:

(a)   if $(x, y) = 0$ then $T(x)$, $T(y)$ are independent;

(b)   each $T(x)$ is normally distributed and
$\mathrm{var}(T(x)) = k\|x\|^2$ (k independent of x).

It should be noted also that *if* $\{X_\lambda\} \subset L^2(S)$ (*S arbitrary*), *if* $L^2(S)$ *is infinite-dimensional, and if all (finite-dimensional) joint distribution functions for the* X_λ *are (multidimensional) normal, then* $[\mathrm{span}\{X_\lambda\}]^\perp$ *is infinite-dimensional.* The proof follows.

If $\{Y_n\}_{n=1}^N$ is an orthonormal subset of $\overline{\mathrm{span}\{X_\lambda\}}$ then each Y_n is a limit of finite linear combinations of the X_λ and the joint distribution function of the Y_n is also multidimensional normal while all covariances are 0. Thus the Y_n are independent. Since there is an orthonormal subset $\{Y_\mu\}$ of $\overline{\mathrm{span}\{X_\lambda\}}$ and for which $\overline{\mathrm{span}\{Y_\mu\}} = \overline{\mathrm{span}\{X_\lambda\}}$ it follows [2] that $[\mathrm{span}\{X_\lambda\}]^\perp$ is infinite-dimensional.

The last result gives rise to the following question:

1. Are there useful ways of stating conditions on the set of finite-dimensional joint distribution functions $\{F_{\lambda_1,\ldots,\lambda_n}(x_{\lambda_1},\ldots,x_{\lambda_n})\}$ for the set $\{X_\lambda\}$ so that these conditions are necessary and/or sufficient for the truth of $[\mathrm{span}\{X_\lambda\}]^\perp = \{0\}$?

4. Quantum mechanical probability. Associated with each observable $\mathscr{R}$ in quantum theory is a resolution of the identity $E(t)$, $-\infty < t < \infty$, and with each state f (a unit vector in Hilbert space $\mathscr{H}$) there is associated a distribution function F_f defined by: $F_f(t) = (E(t)f, f)$, [10]. The number $F_f(t)$ is interpreted as the probability that in the state f the observable $\mathscr{R}$ assumes a value in $(-\infty, t]$. As a monotone increasing function F_f is differentiable almost everywhere. If $S_f = \{t_0 : F_f'(t_0) \text{ exists}\}$ then the Lebesgue measure $m(\mathbf{R} \setminus S_f) = 0$. On the other hand if $\operatorname{supp}(E(t)) = \mathbf{R} \setminus \{(a, b) : a < b, E(b) = E(a)\}$, then there obtains:

THEOREM 2. *Let K be a set of the second category in Σ_1, the surface of the unit spherre of $\mathscr{H}$, or in $\mathscr{H}$ itself. Then*

$$\operatorname{supp}(E(t)) \cap \left(\bigcap_{f \text{ in } K} S_f \right) = \varnothing.$$

Interpretation: For each f in Σ_1, for each positive e, and for each t_0 in S_f there is in Σ_1 a g such that $\|f - g\| < e$ and F_g is not differentiable at t_0. Thus F_f is and F_g is not differentiable at t_0 even though f and g are close.

Proof. For t_0 in $\operatorname{supp}(E(t))$ if $F_f'(t_0)$ exists for all f in K then

$$\lim_{h \to 0} \left((E(t_0 + h) - E(t_0))f, f \right)/h$$

exists in K. The Banach-Steinhaus theorem implies there is an $\mathbf{R}$ an M such that

$$\|E(t_0 + h) - E(t_0)\|/h \le M$$

for all h near 0. Since $E(t_0 + h) - E(t_0)$ is idempotent it follows that

$$\|E(t_0 + h) - E(t_0)\|/h \le hM^2$$

whence

$$\lim_{h \to 0} \|E(t_0 + h) - E(t_0)\|/h = 0.$$

However if t_0 is in $\operatorname{supp}(E(t))$ then either (a) t_0 is an isolated point of the set or (b) there is a sequence $\{t_n\}_{n=1}^\infty$ such that $t_n \uparrow t_0$ (or $t_n \downarrow t_0$) and $(E(t_{n+1}) - E(t_n)) \ne 0$. If (a) is true then there is an Σ_1 an f such that $E(t_0)f = f$ and if $t < t_0$, then $E(t)f = 0$. Hence if h is small and positive

$$\left((E(t_0 - h) - E(t_0))f, f \right)/h = -1/h$$

and so

$$\|E(t_0 - h) - E(t_0)\|/h \nrightarrow 0 \quad \text{as } h \to 0.$$

If (b) is true, for each n in $\mathbf{N}$ there is in Σ_1 an f_n such that $(E(t_{n+1}) - E(t_n))f_n = f_n$ and $\{f_n\}_{n=1}^{\infty}$ is orthonormal. If

$$b_n = ((t_n - t_{n-1})/(t_1 - t_0))^{1/2},$$

n in $\mathbf{N}$, then $\sum_{n=1}^{\infty} b_n^2 = 1$ and $\sum_{n=1}^{\infty} b_n f_n = f$ is in Σ_1. On the other hand

$$\sum_{n=m}^{\infty} ((E(t_n) - E(t_{n+1}))f, f) = -\frac{(t_m - t_0)}{(t_1 - t_0)}.$$

The left member above is also $((E(t_m) - E(t_0))f, f)$ and so if $F_f'(t_0)$ exists it must be $-(t_1 - t_0)^{-1}$ which is not 0, and a contradiction emerges. The result follows.

THEOREM 3. *If $\mathcal{H}$ is separable there is an $\mathcal{H}$ a sequence $\{f_n\}_{n=1}^{\infty}$ dense in $\mathcal{H}$ (or in Σ_1) and such that $m(\mathbf{R} \setminus \bigcap_{n=1}^{\infty} S_{f_n}) = 0$.*

Proof. If $\{f_n\}_{n=1}^{\infty}$ is dense in $\mathcal{H}$ (or in Σ_1) then $m(\mathbf{R} \setminus S_{f_n}) = 0$ and so $m(\mathbf{R} \setminus \bigcap_{n=1}^{\infty} S_{f_n}) = 0$.

REMARKS. If F is any (probability) distribution function there is a Hilbert space $\mathcal{H}$, a resolution of the identity $E(t)$, and in Σ_1 an f such that for all t, $F(t) = F_f(t)$. Indeed if $\mathcal{H} = L^2(R, P)$, P being the measure induced on $\mathbf{R}$ by F, let $E(t)k$ be $\chi_{(-\infty, t]} \cdot k$ for k in $\mathcal{H}$. Then the function f identically 1 is in $\mathcal{H}$ and $F = F_f$.

More generally, if $F_{a_1 a_2 \cdots a_n}$ is a family of multidimensional (probability) distribution functions satisfying the classical Kolmogorov consistency conditions (all a_i are members of a fixed parameter set A) and if $S = \mathbf{R}^A$, endowed *via* Kolmogorov's construction with the relevant measure P, then for each a in A and each f in $L^2(S, P)$ let $E_a(t)f$ be defined by: if $\{r_b\}$ is in S then

$$E_a(t)f(\{r_b\}) = f(\{r_b\}) \quad \text{if } r_a \le t = 0, \quad \text{otherwise.}$$

Then if $f \equiv 1$ on S

$$F_{a_1 a_2 \cdots a_n}(t_1, t_2, \ldots, t_n) = \left(\prod_{i=1}^{n} E_{a_i}(t_i)f, f \right).$$

Thus every stochastic process may be realized as a quantum mechanical process. By varying f over Σ_1 in $L^2(S, P)$ one achieves a family of (related) stochastic processes.

If X_1 and X_2 are random variables with distribution functions F_1 and F_2, then when X_1 and X_2 are independent, the distribution function for

$X_1 + X_2$ is $F_1 * F_2$, i.e.,

$$P(X_1 + X_2 \leq x) = F_1 * F_2(x) = \int_{\mathbf{R}} F_1(x - y) \, dF_2(y).$$

If X_1 and X_2 are not independent the formula above may fail to hold. On the other hand, if X_1, X_2 is realized in the manner given above, then, e.g.,

$$F_1(x) = (E_1(x)f, f) \qquad (f \equiv 1)$$
$$F_2(x) = (E_2(x)f, f).$$

Now, however, even if X_1 and X_2 are not independent,

$$P(X_1 + X_2 \leq x) = \int_{\mathbf{R}} ((E_1(x - y) \, dE_2(y))f, f),$$

i.e., a convolution-like formula may be used. Furthermore if $U_a(s) = \int_{\mathbf{R}} e^{ist} \, dE_a(t)$ then the joint characteristic function for $F_{a_1 a_2 \cdots a_n}$ is $(\prod_{i=1}^{} U_{a_i}(s_i)f, f)$, i.e., a product-like formula even if there is no independence indicated.

REFERENCES

[1]   G. Darmois, *Analyse générale des liaisons stochastiques*, Rev. Inst. Intern. Stat., **21** (1953), 2–8.

[2]   B. R. Gelbaum, *Independence of events and of random variables*, Zeitschrift für Wahrscheinlichkeitstheorie, **36** (1976), 333–342.

[3]   A. M. Kagan, Yu. V. Linnik and C. R. Rao, *Characterization Problems in Mathematical Statistics*, John Wiley & Sons, New York, 1973.

[4]   R. Hemasinha, Ph.D. dissertation, SUNY/Buffalo, 1983.

[5]   Yu. V. Linnik, *Decomposition of Probability Distributions*, Oliver and Boyd, Ltd. Edinburgh, 1964.

[6]   E. Nelson, *Probability Theory and Euclidean Field Theory*, in *Constructive quantum field theory*, Springer-Verlag, 1973.

[7]   I. Segal, *Tensor algebras over Hilbert spaces*, *I*, Trans. Amer. Math. Soc., **81** (1956), 106–134.

[8]   V. P. Skitovich, *On a property of the normal distribution*, DAN SSSR, **89** (1953), 217–217.

[9]   _____, *Linear forms in independent random variables and the normal distribution law*, Izvestia AN SSSR, Ser. Matem.

[10] J. von Neumann, *Grundlagen der Quantenmechanik*, Dover Publications, New York, 1943.

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