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RELATIONS BETWEEN THE MAXIMUM MODULUS AND MAXIMUM TERM OF ENTIRE FUNCTIONS

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In memory of Ernst Straus

Relations between the maximum modulus $M(R)$ and the maximum term $\mu(R)$ of an entire function are investigated. There are no upper bounds for $M(R)$ in terms of functions of R and $\mu(R)$ which are valid for all R. There are such bounds as functions of R, ε, $\mu(R)$ and $\mu(R+\varepsilon)$ for all $\varepsilon > 0$.

1. Introduction. For an entire function $F(z) = \Sigma a_n z^n$, we define the *maximum modulus*

$$M(R) = \max_{|z|=R} |F(z)|,$$

the *maximum term*

$$\mu(R) = \max_n |a_n| R^n$$

and the *central index* $N(R)$, which is the largest integer N so that

$$\mu(R) = |a_N| R^N.$$

If we set $L = \log R$ and plot $\log \mu(R)$ as a function of L, then the graph of a monomial $f(z) = a_n z^n$ is a straight line of slope n which passes through the point $(0, \log|a_n|)$. Hence the μ-graph of an entire function is convex polygonal line with edges that have increasing nonnegative integral slope. This implies that the L-coordinates of the vertices of a *μ-graph* have no limit point other than $+\infty$. In particular,

$$N(R) = \frac{d \log \mu(R+)}{dL}.$$

We introduce one more quantity, $\nu(R)$, the number of indices n for which $\mu(R) = |a_n| R^n$. Clearly $\nu(R) = 1$ except when R corresponds to a vertex of the *μ-graph*, where

$$2 \le \nu(R) \le 1 + \frac{d \log \mu(R+)}{dL} - \frac{d \log \mu(R-)}{dL} = 1 + N(R) - N(R-).$$

The Wiman-Valiron Theory (see e.g. [**1**], [**2**]) concentrates on "normal" values of R where the behavior of $\mu(R)$ and $M(R)$ are closely related. In this note we are interested in relations which hold for all R, or at least for all sufficiently large R.

In §2 we characterize the graphs which can arise as *μ-graphs* of an entire function. We also show that for any given function $\phi(R, \mu(R))$ it is possible to have arbitrarily large R with

$$\nu(R) > \phi(R, \mu(R)).$$

From this fact it follows immediately that there is no upper bound for $M(R)$ by a function of R and $\mu(R)$. On the other hand, in §3 we use the convexity of $\log\mu$ as a function of L to give an upper bound for $M(R)$ as a function of R, ε and $\mu(R+\varepsilon)$.

2. The μ-graphs and M-graphs of entire functions. As mentioned above, the *μ-graph* of an entire function is a convex polygonal line whose edges have (increasing) integral slopes. The converse is also true.

2.1. THEOREM. *Every convex polygonal line in the $(L, \log\mu)$-plane whose edges have nonnegative integral slopes has the property that every Taylor series $\Sigma a_n z^n$ with $\max_n|a_n|R^n = \mu(R)$ is the Taylor series of an entire function.*

Proof. Let the L-coordinates of the vertices be $L_1 < L_2 < L_3 < \cdots$ and the slopes to the right of L_i be N_i. Let $\lambda_i = \log\mu(R_i)$, where $\log R_i = L_i$. If $L_k \le L < L_{k+1}$, then $N = N_k$ and

$$(2.2)\quad \log|a_N| + NL = \log\mu(R) = \lambda_1 + N_1(L_2 - L_1) + \cdots + N_{k-1}(L_k - L_{k-1}) + N_k(L - L_k).$$

Hence

$$(2.3)\quad \frac{\log|a_N|}{N} = \frac{\lambda_1}{N} - \frac{1}{N}\big[L_k(N_k - N_{k-1}) + L_{k-1}(N_{k-1} - N_{k-2}) + \cdots + L_2(N_2 - N_1) + L_1N_1\big].$$

To show that $(1/N)\log|a_N| \to -\infty$ we pick the largest l so that $2N_l \le N$. Then for sufficiently large N, (2.3) yields

$$\frac{1}{N}\log|a_N| < \frac{\lambda_1}{N} - \frac{1}{N}L_l(N_k - N_l) \le \frac{\lambda_1}{N} - \frac{1}{2}L_l \to -\infty.$$

Since $l \to \infty$ as $N \to \infty$.

For those indices n for which $n \ne N(R)$ we have $N_{k-1} < n < N_k$ and

$$\log|a_n| + nL_k \le \log|a_{N_{k-1}}| + N_{k-1}L_k.$$

Hence

$$\frac{1}{n}\log|a_n| \le \frac{N_{k-1}}{n}\left(\frac{\log|a_{N_{k-1}}|}{N_{k-1}}\right) - L_k\left(1 - \frac{N_{k-1}}{n}\right) \tag{2.4}$$

$$\le \max\left(\frac{\log|a_{N_{k-1}}|}{N_{k-1}}, -L_k\right).$$

Thus $(1/n)\log|a_n| \to -\infty$ and $\Sigma a_n z^n$ is an entire function.

It is clear that two Taylor series $\Sigma a_n z^n$ and $\Sigma b_n z^n$ have the same *μ-graph* if and only if

(i) $|a_N| = |b_N|$ for all N which are slopes of edges of the graph.

(ii) $|a_n| \le s_n$, $|b_n| \le s_n$ where $\log \mu = nL + \log s_n$ is a line of support but not an edge of the μ-graph.

Thus the set of entire functions with the same *μ-graph* is infinite dimensional.

We now turn briefly to the M-graph which we get by plotting $\log M(R)$ as a function of L. By the Hadamard Three-Circle Theorem we know that this is a convex curve and by Cauchy's inequality we know that $\mu(R) \le M(R)$ with equality only when $F(z)$ is monomial. Thus the M-graph lies strictly above the *μ-graph* unless they are both a single straight line.

By Parseval's inequality we have

$$\sum |a_n|^2 R^{2n} \le M(R)^2$$

so that

$$\mu(R)\sqrt{\nu(R)} \le M(R). \tag{2.5}$$

In asking which entire functions have the same M-graph we note that for any real α, β we have

$$M(R, F) = M(R, e^{i\alpha}F) = M\big(R, F(e^{i\beta}z)\big) = M(R, \bar{F}), \tag{2.6}$$

where $\bar{F}$ is given by the Taylor series whose coefficients are the complex conjugates of those of F.

2.7. DEFINITION. Two entire functions $F(z)$ and $G(z)$ are *equivalent* if they are obtained from each other by a combination of the operations in (2.6).

This brings us to some conjectures which one of us has raised some time ago.

2.8. *Conjectures.* (i) If two entire functions have equal M-graphs then they are equivalent.

(ii) If two entire functions have both equal M-graphs and equal μ-*graphs* then they are equivalent.

(iii) If two entire functions have Taylor coefficients of equal absolute values and equal M-graphs then they are equivalent.

(iv) If F has a Taylor series with nonnegative real coefficients and $M(R, G) = M(R, F)$ then G is equivalent to F.

It is surprising that even Conjecture (iv) does not seem to be immediately obvious. However, following Valiron [2], we have the following.

2.9. THEOREM. *For every μ-graph there exists a unique equivalence class of entire functions with maximal $M(R)$. This class contains a function $G(z)$ with nonnegative real Taylor coefficients, hence this maximal $M(R)$ satisfies $M(R) = G(R)$ which is a totally monotonic analytic function of R.*

Proof. Define $G(z) = \Sigma g_n z^n$ where

$$\log \mu = \log g_n + nL$$

is a line of support of the μ-*graph*, provided the μ-*graph* has a line of support with slope n, and $g_n = 0$ otherwise. Thus $g_n = 0$ only for those indices which are less than the slope of the initial edge of the μ-*graph* and —in case the μ-*graph* is a finite polygon—those n which exceed the slope of the final edge.

If $F(z) = \Sigma a_n z^n$ and $\mu(R, F) = \mu(R, G)$ then clearly $|a_n| \leq g_n$ for all $n \geq 0$. Hence

$$M(R, F) \leq \sum |a_n| R^n \leq \sum g_n R^n = M(R, G). \tag{2.10}$$

Equality in (2.10) implies $|a_n| = g_n$ for all n and the existence of a β so that

$$\arg a_n e^{in\beta} = \alpha, \quad \text{a constant for all } n.$$

Thus $e^{-i\alpha}F(e^{i\beta}z) = G(z)$.

An examination of trinomials, say $F_\alpha(z) = e^{i\alpha} + 2z + z^2$, shows that there is no function of minimal $M(R)$ associated with a general μ-*graph*, because the values of α for which $M(R, F_\alpha)$ is minimal vary with R.

We close this section with one final observation and question. It is obvious that $\liminf_{R\to\infty} M(R)/\mu(R) \geq 1$ for all entire functions and that equality holds for all polynomials and for many transcendental functions with highly lacunary power series.

On the other hand inequality (2.5) shows that

$$\limsup_{R\to\infty} M(R)/\mu(R) \geq \sqrt{2}$$

for all transcendental entire functions and that

$$\limsup_{R\to\infty} M(R)/\mu(R) = 2$$

for transcendental entire functions with highly lacunary power series.

2.11. *Problem*. What is

$$\gamma = \inf \limsup_{R\to\infty} M(R)/\mu(R)$$

where the inf is taken over all transcendental entire functions?

We have seen that $\sqrt{2} \leq \gamma \leq 2$ and the upper limit appears to be the likely value of γ.

Finally, we observe that the maximal growth fucntion $G(z)$ which belongs to a *μ-graph* with infinitely many edges satisfies

$$M(R) = G(R) > \mu(R)(N(R+) - N(R-) + 1), \tag{2.12}$$

where $N(R)$ is the slope of the *μ-graph*.

For any function $\phi(R, \mu(R))$ and any sequence $R_1 < R_2 < R_3 < \cdots$ with $R_n \to \infty$ we can find a *μ-graph* so that

$$N(R_n +) - N(R_n -) > \phi(R_n, \mu(R_n)).$$

Thus inequality (2.12) yields the following.

2.13. THEOREM. *There is no bound for $M(R)$ which is a fixed function of R and $\mu(R)$.*

3. Upper bounds for $M(R)$ in terms of $\mu(R)$ and $\mu(R+\varepsilon)$. In contrast to Theorem 2.13 we have the following.

3.1. THEOREM. *For every $R > \varepsilon > 0$ we have*

$$M(R) < \left(\frac{4R+\varepsilon}{\varepsilon}\right)\left(1 + \log\frac{\mu(R+\varepsilon)}{\mu(R)}\right)\mu(R). \tag{3.2}$$

Proof. We set $\log(R+\varepsilon) = L + \delta_1 + \delta_2$ so that

$$\delta_1 + \delta_2 = \log\left(1 + \frac{\varepsilon}{R}\right). \tag{3.3}$$

It suffices to prove (3.2) for the maximal function $G(z)$ associated with $\mu(R)$. Now set $N_1 = N(Re^{\delta_1})$. Then

$$\sum_{n=0}^{N_1 - 1} g_n R^n \leq N_1 \mu(R) \tag{3.4}$$

and

$$N_1 \le \frac{1}{\delta_2}(\log \mu(Re^{\delta_1+\delta_2}) - \log \mu(Re^{\delta_1}))$$

$$\le \frac{1}{\delta_2}(\log \mu(R+\varepsilon) - \log \mu(R)) = \frac{1}{\delta_2} \log \frac{\mu(R+\varepsilon)}{\mu(R)}.$$

So (3.4) yields

$$\sum_{n=0}^{N_1-1} g_n R^n \le \frac{1}{\delta_2} \log \frac{\mu(R+\varepsilon)}{\mu(R)} \mu(R). \tag{3.5}$$

Now for $n \ge N_1$ we have, by the convexity of the *μ-graph*.

$$g_n R^n \le \mu(R) e^{-(n-N_1)\delta_1}.$$

Thus

$$\sum_{n=N_1}^{\infty} g_n R^n \le \mu(R)/(1-e^{-\delta_1}). \tag{3.6}$$

It remains to choose

$$\delta_1 = \log\left(1+\frac{\varepsilon}{R}\right) \Big/ \left(1+\log \frac{\mu(R+\varepsilon)}{\mu(R)}\right) \tag{3.7}$$

$$\delta_2 = \log\left(1+\frac{\varepsilon}{R}\right) \log \frac{\mu(R+\varepsilon)}{\mu(R)} \Big/ \left(1+\log \frac{\mu(R+\varepsilon)}{\mu(R)}\right).$$

Then (3.5) and (3.6) yield

$$M(R) \le G(R) \le \left(\frac{4R+\varepsilon}{\varepsilon}\right)\left(1+\log \frac{\mu(R+\varepsilon)}{\mu(R)}\right)\mu(R) \tag{3.8}$$

as was to be proved.

Note that Theorem 3.1 is similar to the inequality

$$M(R) < \mu(R)\left(2N\left(R+\frac{R}{N(R)}\right)+1\right) \tag{3.9}$$

of Valiron [**2**]. However the quantity $R/N(R)$ need not be small and so (3.2) cannot be directly deduced from (3.9). However it is obvious that any bound for $M(R)$ in terms of ε, R, $\mu(R)$, $\mu(R+\varepsilon)$ can also be expressed in terms of ε, R, $\mu(R)$ and $N(R+\varepsilon)$.

REFERENCES

[1] W. K. Hayman, *The local growth of power series: a survey of the Wiman-Valiron method*, Canad. Math. Bull., **17** (1974), 317–358.

[2] G. Valiron, *Fonctions entières d'orde fini et fonctions méromorphes*, L'Ensiegnement Math. Ser. II, **IV** (1958), 1–150.

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