AN EXTENSION OF E. G. STRAUS' PERFECT LATIN 3-CUBE OF ORDER 7

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In memory of Ernst Straus

In a letter dated January, 1976 E. G. Straus sent this author a detailed construction of a $7 \times 7 \times 7$ perfect magic cube written to base 7 with digits 000 to 666. In this construction he superimposed 3 Latin cubes of order 7 (a Latin 3-cube of order 7) to get what may be the lowest possible order of a perfect cube. Before Straus' construction the smallest known perfect cube was of order 8. The Straus cube is perfect in the following way: the sum (2331) of the elements in each minor diagonal and in each pan-diagonal is equal to the sum of the elements of a row in each of the 2 directions in each of the respective squares (layers) that make up this perfect Latin 3-cube of order 7. The sum (2331) of the elements of a row in each direction of the cube is equal to the sum of the elements in each of the 4 major diagonals and the sum on all the diagonals of the cube is the same (namely 2331). The construction of the cube is based on the 3 orthogonal cubes

$$A_{ijk}^{(2)} = x_i + 2x_j - 3x_k, \quad A_{ijk}^{(-2)} = x_i - 2x_j - 3x_k,$$

$$A_{ijk}^{(3)} = x_i + 3x_j + 2x_k$$

where $(x_1, \dots, x_7) = (0, 1, \dots, 6)$ and arithmetic is (mod 7).

In this paper we superimpose 6 orthogonal Latin cubes of order 7 where each ordered triple (000, 001, ..., 666) occurs in every one of the 6 possible positions to form 20 separate Straus cubes.

A *Latin square of order n* is an $n \times n$ square in which each of the numbers $0, 1, \dots, n-1$ occurs exactly once in each row and exactly once in each column. For example

$$\begin{array}{ccc} 0 & 1 & \\ 1 & 0 & \end{array} \quad \begin{array}{ccc} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{array} \quad \begin{array}{cccc} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{array}$$

are Latin squares of order 2, 3, 4, respectively. Two Latin squares of order n are *orthogonal*, if when one is superimposed on the other, every ordered pair $00, 01, \dots, n-1, n-1$ occurs. Thus

$$\begin{array}{ccc} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{array} \quad \text{and} \quad \begin{array}{ccc} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{array} \quad \text{superimpose to} \quad \begin{array}{ccc} 0 & 0 & 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 0 & 0 & 1 \\ 2 & 1 & 0 & 2 & 1 & 0 \end{array}$$

and therefore are orthogonal squares of order 3. A set of Latin squares of order n is orthogonal if every two of them are orthogonal. As an example the 4×4 square of triples

0 0 0	1 1 1	2 2 2	3 3 3
1 2 3	0 3 2	3 0 1	2 1 0
2 3 1	3 2 0	0 1 3	1 0 2
3 1 2	2 0 3	1 3 0	0 2 1

represents three mutually orthogonal squares of order 4 since each of the 16 pairs 00, 01, ..., 33 occurs in each of the three possible positions among the 16 triples.

We can generalize all these concepts to $n \times n \times n$ cubes and cubes of higher dimensions. A Latin cube of order n is an $n \times n \times n$ cube (n rows, n columns and n files) in which the numbers $0, 1, \dots, n - 1$ are entered so that each number occurs exactly once in each row, column and file. If we list the cube in terms of the n squares of order n which form its different levels we can list the cubes

0 1	1 0		0 1 2	1 2 0	2 0 1
1 0	0 1	and	1 2 0	2 0 1	0 1 2
			2 0 1	0 1 2	1 2 0

as Latin cubes of order 2 and 3, respectively.

Orthogonality of Latin cubes is the following relation among three Latin cubes: three Latin cubes of order n are orthogonal if, when superimposed, each ordered triple $000, 001, \dots, n - 1 n - 1 n - 1$ will occur. For example the pair of 3×3 Latin squares

0 0	1 1	2 2
1 2	2 0	0 1
2 1	0 2	1 0

leads to the four $3 \times 3 \times 3$ cubes

A:	0 1 2	1 2 0	2 0 1
	1 2 0	2 0 1	0 1 2
	2 0 1	0 1 2	1 2 0
B:	0 1 2	1 2 0	2 0 1
	2 0 1	0 1 2	1 2 0
	1 2 0	2 0 1	0 1 2
C:	0 2 1	1 0 2	2 1 0
	2 1 0	0 2 1	1 0 2
	1 0 2	2 1 0	0 2 1
D:	0 2 1	1 0 2	2 1 0
	1 0 2	2 1 0	0 2 1
	2 1 0	0 2 1	1 0 2

Superimposed these lead to a cube of quadruples in three levels with I over II over III,

ABCD:

0000	1122	2211	1111	2200	0022	2222	0011	1100
1221	2010	0102	2002	0121	1210	0110	1202	2021
2112	0201	1020	0220	1012	2101	1001	2120	0212
I			II			III		

where each ordered triple occurs in every one of the four possible positions in the quadruples.

Note. We define a cube of triples (say ABC) where each ordered triple occurs in some order in the 27 cells of the three levels (I over II over III) of the cube ABC as a *Latin 3-cube of order 3*.

In this paper we superimpose the construction for 6 orthogonal Latin cubes of order 7 where each ordered triple (000, 001, ..., 666) occurs in every one of the 6 possible positions to form 20 separate Straus cubes is as follows.

363330	525524	050011	212205	444462	606656	131143	I.
416016	641203	103460	335654	560141	022335	254522	
532465	064652	226146	451333	613520	145014	300201	
655144	110331	342525	504012	036206	261463	423650	
001523	233010	465204	620461	152655	314142	546336	
124202	356466	511653	043140	205334	430521	662015	
240651	402145	634332	166526	321013	553200	015464	
006065	231252	463446	625633	150120	312314	544501	II.
122444	354631	516125	041312	203506	435063	660250	
245123	400310	632504	164061	326255	551442	013636	
361502	523066	055253	210440	442634	604121	136315	
414251	646445	101632	333126	565313	020500	252064	
530630	062124	224311	456505	611062	143256	305443	
653316	115503	340060	502254	034441	266635	421122	
412423	644610	106104	331361	563555	025042	250236	III.
535102	060366	222553	454040	616234	141421	303615	
651551	113045	345232	500426	032613	264100	426364	
004230	236424	461611	623105	155362	310556	542043	
120616	352103	514360	046554	201041	433235	665422	
243365	405552	630046	162233	324420	556614	011101	
366044	521231	053425	215612	440106	602363	134550	
125151	350345	512532	044026	206213	431400	663664	IV.
241530	403024	635211	160405	322662	554156	016343	
364216	526403	051660	213154	445341	600535	132022	
410665	642152	104346	336533	561020	023214	255401	
533344	065531	220025	452212	614406	146663	301150	
656023	111210	343404	505661	030155	262342	424536	
002402	234666	466153	621340	153534	315021	540215	

531516	063003	225260	450454	612641	144135	306322	V.
654265	116452	341646	503133	035320	260514	422001	
000644	232131	464325	626512	151006	313263	545450	
123323	355510	510004	042261	204455	436642	661136	
246002	401266	633453	165640	320134	552321	014515	
362451	524645	056132	211326	443513	605000	130264	
415130	640324	102511	334005	566262	021456	253643	
244244	406431	631625	163112	325306	550563	012050	VI.
360623	522110	054304	216561	441055	603242	135436	
413302	645566	100053	332240	564434	026621	251115	
536051	061245	223432	455626	610113	142300	304564	
652430	114624	346111	501305	033562	265056	420243	
005116	230303	462560	624054	156241	311435	543622	
121565	353052	515246	040433	202620	434114	666301	
650602	112166	344353	506540	031034	263221	425415	VII.
003351	235545	460032	622226	154413	316600	541164	
126030	351224	513411	045605	200162	432356	664543	
242416	404603	636160	161354	323541	555035	010222	
365165	520352	052546	214033	446220	601414	133601	
411544	643031	105225	330412	562606	024163	256350	
534223	066410	221604	453161	615355	140542	302036	

In the above construction each ordered triple (a Straus cube) occurs in every one of the six possible positions in the septuples.

Note. The Straus cube is mentioned in *The Scientific American*, February 1976, p. 127.

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