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CANCELLATION OF LOW-RANK VECTOR BUNDLES

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The purpose of this paper is to show that even though vector bundles cannot in general be cancelled from direct sums (Whitney sums), in certain low-rank situations vector bundles can be cancelled at the expense of complexifying or quaternionifying the remaining terms. To be specific, let λ, ξ_1, ξ_2 be vector bundles over a paracompact space X, such that $\lambda \oplus \xi_1 \cong \lambda \oplus \xi_2$. First assume that these are real vector bundles. If λ is a line bundle of finite type, then the complexifications of ξ_1 and ξ_2 are isomorphic, and hence $2\xi_1 \cong 2\xi_2$ (where $2\xi_i$ denotes the direct sum of two copies of ξ_i), while if λ is a direct sum of two line bundles of finite type, then the quaternionifications of ξ_1 and ξ_2 are isomorphic, and hence $4\xi_1 \cong 4\xi_2$. Now assume that these are complex vector bundles. If λ is the complexification of a real line bundle of finite type (in particular, λ could be a trivial complex vector bundle of rank 1), then the quaternionifications of ξ_1 and ξ_2 are isomorphic, and hence $\xi_1 \oplus \bar{\xi}_1 \cong \xi_2 \oplus \bar{\xi}_2$ (where $\bar{\xi}_i$ denotes the conjugate vector bundle to ξ_i). These results are independent of the dimension of the space X, and also independent of the dimensions of the fibres of ξ_1 and ξ_2. The same results also hold for smooth vector bundles over a smooth manifold.

I. Background on vector bundles. Let $\mathbf{F}$ be one of $\mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$. We work with $\mathbf{F}$-vector bundles over a topological space X in the same generality as Swan [9], so that we do not require the dimensions of the fibres of a vector bundle ξ to be constant. (However, the local triviality condition on ξ forces the fibre-dimension of ξ to be locally constant, and hence the fibre-dimension will be constant if X is connected.) A vector bundle ξ over X is said to be of *finite type* provided that there exists a finite open covering $U_1, \ldots, U_n$ of X such that the restriction of ξ to each U_i is trivial. (In particular, this places a bound on the fibre-dimension of ξ.) Of course if X is compact, then all vector bundles over X are of finite type.

The basic mechanism for algebraic investigations of $\mathbf{F}$-vector bundles over X is the relationship between such vector bundles and modules over the ring $C(X, \mathbf{F})$ of continuous $\mathbf{F}$-valued functions on X. This relationship is effected through the section functor Γ, which assigns to each $\mathbf{F}$-vector bundle ξ the $C(X, \mathbf{F})$-module $\Gamma(\xi)$ of continuous sections of ξ, and which assigns to each $\mathbf{F}$-vector bundle map $f\colon \xi \to \eta$ the induced $C(X, \mathbf{F})$-module homomorphism $\Gamma(f)\colon \Gamma(\xi) \to \Gamma(\eta)$. (In the case where $\mathbf{F} = \mathbf{H}$, a choice of right versus left vector spaces and modules is required for consistency. As we prefer to write homomorphisms on the left, let us stipulate that all our

H-vector spaces are right vector spaces, and that all our $C(X, \mathbf{H})$-modules are right modules.) An obvious property of the section functor Γ is that it preserves direct sums, that is, $\Gamma(\xi \oplus \eta) \cong \Gamma(\xi) \oplus \Gamma(\eta)$. The additional properties of Γ which we shall need may be outlined as follows.

THEOREM 1.1. *Let* $\mathbf{F}$ *be one of* $\mathbf{R}$, $\mathbf{C}$, *or* $\mathbf{H}$, *let* X *be a paracompact topological space, and let* ξ, η *be* $\mathbf{F}$*-vector bundles over* X.

(a) *Given any* $C(X, \mathbf{F})$*-module homomorphism* $f\colon \Gamma(\xi) \to \Gamma(\eta)$, *there exists a unique* $\mathbf{F}$*-vector bundle map* $g\colon \xi \to \eta$ *such that* $\Gamma(g) = f$.

(b) $\xi \cong \eta$ *if and only if* $\Gamma(\xi) \cong \Gamma(\eta)$.

(c) *If* ξ *is of finite type, then* $\Gamma(\xi)$ *is a finitely generated projective* $C(X, \mathbf{F})$*-module.*

(d) *If* A *is a finitely generated projective* $C(X, \mathbf{F})$*-module, then there exists an* $\mathbf{F}$*-vector bundle* ζ *over* X *of finite type such that* $\Gamma(\zeta) \cong A$.

Proof. (a), (b) These properties hold more generally for vector bundles over a normal topological space [**9**, Theorem 1, Corollary 4].

(c) We need to know that there exists an $\mathbf{F}$-vector bundle ζ over X such that $\xi \oplus \zeta$ is trivial. In case the fibre-dimension of ξ is constant, this is proved in [**7**, Chapter 3, Proposition 5.8]. In general, since the fibre-dimension of ξ is locally constant, and also bounded (because ξ is of finite type), X may be expressed as a disjoint union of clopen sets $U_1, \ldots, U_n$ such that the restriction of ξ to each U_i has constant fibre-dimension. Applying the theorem above to each of these restrictions then provides us with the desired vector bundle ζ. Now $\Gamma(\xi) \oplus \Gamma(\zeta)$ is a free $C(X, \mathbf{F})$-module of finite rank, and hence $\Gamma(\xi)$ is a finitely generated projective $C(X, \mathbf{F})$-module.

(d) The proof of [**9**, Theorem 2] may be used to show that there exists an $\mathbf{F}$-vector bundle ζ over X such that $\Gamma(\zeta) \cong A$ and ζ is a direct summand of a trivial bundle. In particular, it follows that the fibre-dimension of ζ is bounded. Applying [**7**, Chapter 3, Proposition 5.8] to clopen subsets of X as above, we conclude that ζ must be of finite type. □

In particular, it follows from Theorem 1.1 that over a paracompact space X, the section functor Γ provides a category equivalence between the category of $\mathbf{F}$-vector bundles over X of finite type and the category of finitely generated projective $C(X, \mathbf{F})$-modules.

We denote the complexification of a real vector bundle ξ by $\xi^{\mathbf{C}}$, and we denote the quaternionification of ξ by $\xi^{\mathbf{H}}$. Note that as real vector bundles, $\xi^{\mathbf{C}} \cong 2\xi$ (the direct sum of two copies of ξ), and $\xi^{\mathbf{H}} \cong 4\xi$. We denote the quaternionification of a complex vector bundle η by $\eta^{\mathbf{H}}$. Note that as complex vector bundles, $\eta^{\mathbf{H}} \cong \eta \oplus \bar{\eta}$, where $\bar{\eta}$ denotes the conjugate vector bundle of η. Complexification and quaternionification commute with the section functor in the following straightforward manner, the details of which we leave to the reader. If, for instance, ξ is a real vector bundle over a space X, then the $C(X, \mathbf{R})$-module $\Gamma(\xi)$ may be extended to a $C(X, \mathbf{C})$-module by passing to the tensor product $\Gamma(\xi) \otimes_{C(X,\mathbf{R})} C(X, \mathbf{C})$. This tensor product is naturally isomorphic to $\Gamma(\xi) \otimes_{\mathbf{R}} \mathbf{C}$, because $\{1, i\}$ is simultaneously a basis for $\mathbf{C}$ as an $\mathbf{R}$-vector space and for $C(X, \mathbf{C})$ as a free $C(X, \mathbf{R})$-module. Similarly, we may identify $\Gamma(\xi) \otimes_{C(X,\mathbf{R})} C(X, \mathbf{H})$ with $\Gamma(\xi) \otimes_{\mathbf{R}} \mathbf{H}$, and given a complex vector bundle η over X, we may identify $\Gamma(\eta) \otimes_{C(X,\mathbf{C})} C(X, \mathbf{H})$ with $\Gamma(\eta) \otimes_{\mathbf{C}} \mathbf{H}$.

PROPOSITION 1.2. *Let X be a topological space.*

(a) *If ξ is a real vector bundle over X, then $\Gamma(\xi^{\mathbf{C}}) \cong \Gamma(\xi) \otimes_{\mathbf{R}} \mathbf{C}$ as $C(X, \mathbf{C})$-modules, and $\Gamma(\xi^{\mathbf{H}}) \cong \Gamma(\xi) \otimes_{\mathbf{R}} \mathbf{H}$ as $C(X, \mathbf{H})$-modules.*

(b) *If η is a complex vector bundle over X, then $\Gamma(\eta^{\mathbf{H}}) \cong \Gamma(\eta) \otimes_{\mathbf{C}} \mathbf{H}$ as $C(X, \mathbf{H})$-modules.* □

PROPOSITION 1.3. *Let $\mathbf{F}$ be either $\mathbf{R}$ or $\mathbf{C}$, and let X be a paracompact topological space. Let λ be an $\mathbf{F}$-vector bundle over X, and let φ be the natural ring homomorphism of $C(X, \mathbf{F})$ into the endomorphism ring of the $C(X, \mathbf{F})$-module $\Gamma(\lambda)$. If λ is a line bundle of finite type, then φ is an isomorphism.*

Proof. Set $R = C(X, \mathbf{F})$ and $E = \mathrm{End}_R(\Gamma(\lambda))$. Since λ is of finite type, Theorem 1.1 shows that $\Gamma(\lambda)$ is a finitely generated projective R-module.

For $x \in X$, let $M(x)$ denote the maximal ideal of R consisting of all functions in R that vanish at x. By [**9**, Corollary 3], $\Gamma(\lambda)/M(x)\Gamma(\lambda)$ is isomorphic to the fibre of λ at x, which is a line. Consequently, the localized module $\Gamma(\lambda)_{M(x)}$ is a rank 1 free module. As the intersection of the maximal ideals of the form $M(x)$ is zero, the set of these maximal ideals is dense (in the Zariski topology) in $\mathrm{Spec}(R)$. In addition, the map which assigns to each $P \in \mathrm{Spec}(R)$ the rank of the free R_P-module $\Gamma(\lambda)_P$ is locally constant [**2**, Chapter II, §5.2, Theorem 1]. As a result, $\Gamma(\lambda)_M$ is a rank 1 free R_M-module for every maximal ideal M of R.

It now follows that for each maximal ideal M of R, the localization of φ at M is an isomorphism of R_M onto E_M (which can be identified with the endomorphism ring of $\Gamma(\lambda)_M$ because $\Gamma(\lambda)$ is a finitely generated projective module). Therefore φ is an isomorphism. □

The analogs of Theorem 1.1 and Propositions 1.2 and 1.3 hold for smooth **F**-vector bundles over a smooth manifold X, in which case $C(X, \mathbf{F})$ is replaced by the ring $C^\infty(X, \mathbf{F})$ of smooth **F**-valued functions on X. For proofs of the relevant properties in the case of smooth vector bundles of constant fibre-dimension, see [**6**, §§2.14, 2.23, 2.24].

II. Background on cancellation. The standard cancellation results for vector bundles over a topological space X require that one of the vector bundles involved have fibre-dimension everywhere greater than the dimension of X. For example, let X be an n-dimensional CW-complex, and let θ, ξ_1, ξ_2 be real vector bundles over X such that θ is trivial and $\theta \oplus \xi_1 \cong \theta \oplus \xi_2$. If the fibres of ξ_1 have dimension at least $n + 1$, then $\xi_1 \cong \xi_2$ [7, Chapter 8, Theorem 1.5]. Even if the fibres of ξ_1 have small dimension, those of $(n + 1)\xi_1$ have large enough dimension, and hence $(n + 1)\xi_1 \cong (n + 1)\xi_2$.

We are interested in cancellation results of the form

$$\eta \oplus \xi_1 \cong \eta \oplus \xi_2 \Rightarrow r\xi_1 \cong r\xi_2$$

in which the positive integer r does not depend on the dimension of the base space. One such result is immediate from the following theorem of T. Y. Lam. To keep our module notation in line with our vector bundle notation, we write sM to denote the direct sum of s copies of a module M.

THEOREM 2.1. [*Lam*] *Let R be a commutative ring, and let A, B_1, B_2 be R-modules such that $A \oplus B_1 \cong A \oplus B_2$. Assume that A is free of rank n and that B_2 is free of rank k, for some positive integers n, k. Then $rB_1 \cong rB_2$ for all integers $r \geq n + (n/k)$.*

Proof. [**8**, Theorem 2]. □

COROLLARY 2.2. *Let X be a paracompact topological space, and let θ, ξ_1, ξ_2 be real or complex vector bundles over X such that $\theta \oplus \xi_1 \cong \theta \oplus \xi_2$. Assume that θ is trivial of rank n and that ξ_2 is trivial of rank k, for some positive integers n, k. Then $r\xi_1 \cong r\xi_2$ for all integers $r \geq n + (n/k)$.*

Proof. Theorems 1.1 and 2.1. □

In particular, Corollary 2.2 implies that if θ, ξ_1, ξ_2 are real or complex vector bundles over a paracompact space satisfying $\theta \oplus \xi_1 \cong \theta \oplus \xi_2$, and if θ, ξ_2 are trivial with θ having rank n, then $r\xi_1 \cong r\xi_2$ for all integers $r \geq 2n$. While this conclusion is open if θ and ξ_2 are allowed to be non-trivial, we can at least guarantee that $r\xi_1 \cong r\xi_2$ for some positive integers r, as follows.

THEOREM 2.3. *Let A, B_1, B_2 be finitely generated projective modules over a commutative ring R. If $A \oplus B_1 \cong A \oplus B_2$, then there exists a positive integer s such that $rB_1 \cong rB_2$ for all integers $r \geq s$.*

Proof. The modules A, B_1, B_2 and the inverse isomorphisms between $A \oplus B_1$ and $A \oplus B_2$ may be defined in terms of a finite set of matrices over R. Let S be the subring of R generated by the entries of these matrices. Then there exist finitely generated projective S-modules C, D_1, D_2 such that $C \otimes_S R \cong A$ and each $D_i \otimes_S R \cong B_i$, while also $C \oplus D_1 \cong C \oplus D_2$. Thus there is no loss of generality in assuming that R is a finitely generated ring. Now R is a homomorphic image of a polynomial ring over $\mathbf{Z}$ in finitely many indeterminates, and hence R is a noetherian ring with finite Krull dimension d.

Because of [**2**, Chapter II, §5.2, Theorem 1], there is a ring decomposition $R = R' \times R''$ such that $R'B_1 = 0$, while $(R''B_1)_P \neq 0$ for all prime ideals P of R''. Hence, we need only consider the cases in which either $B_1 = 0$ or else B_1 is nonzero at all localizations. If $B_1 = 0$, then $A \cong A \oplus B_2$, and by localization we find that $B_2 = 0$. Therefore there is no loss of generality in assuming that $(B_1)_P \neq 0$ for all prime ideals P of R.

Now given an integer $r \geq d + 1$, for each maximal ideal M of R the free R_M-module $r(B_1)_M$ has rank at least $d + 1$. Applying the Bass Cancellation Theorem [**1**, Chapter IV, Corollary 3.5] to the relation

$$rA \oplus rB_1 \cong rA \oplus rB_2,$$

we conclude that $rB_1 \cong rB_2$. □

COROLLARY 2.4. *Let X be a paracompact topological space, and let η, ξ_1, ξ_2 be real or complex vector bundles over X, of finite type. If $\eta \oplus \xi_1 \cong \eta \oplus \xi_2$, then there exists a positive integer s such that $r\xi_1 \cong r\xi_2$ for all integers $r \geq s$.*

Proof. Theorems 1.1 and 2.3. □

III. Stable range one for ring homomorphisms. A result of Evans [**3**, Theorem 2] states that any module whose endomorphism ring has 1 in its stable range may be cancelled from direct sums. (A ring E has 1 in its stable range provided that whenever $fg + h = 1$ in E, there exists $k \in E$ such that $f + hk$ is right invertible in E. It can be shown that this condition is left-right symmetric.) In fact, if A is such a module, then given any module M having internal direct sum decompositions $M = A_1 \oplus B_1 = A_2 \oplus B_2$ with $A_1 \cong A_2 \cong A$, there exists a submodule C of M such that $M = C \oplus B_1 = C \oplus B_2$, and so the cancellation occurs because $B_1 \cong M/C \cong B_2$. This so-called substitution property (the substitution of C for the two copies of A in such direct sum decompositions) is actually equivalent to the endomorphism ring of A having 1 in its stable range [**11**, Theorem 2.1].

We obtain our cancellation results from these methods by simply changing module categories in mid-stream. For instance, to prove that a module A has the property that whenever B_1, B_2 are modules satisfying $A \oplus B_1 \cong A \oplus B_2$, then $nB_1 \cong nB_2$ (for a pre-assigned positive integer n), it suffices to show that the diagonal map from the endomorphism ring of A to the $n \times n$ matrix ring over the endomorphism ring of A satisfies a suitable version of stable range 1. (For a detailed development of these ideas in the case when n is allowed to vary, see [**4**, **5**].)

If a ring E has 1 in its stable range, then it follows that all right or left invertible elements in E are actually invertible (e.g., [**11**, Theorem 2.1]). Consequently, the condition that E have 1 in its stable range may be restated in the form: $fg + h = 1$ in E implies that there exists $k \in E$ such that $f + hk$ is invertible. It is this latter version of the condition that we need to apply to ring homomorphisms.

DEFINITION. Let φ: $E \to F$ be a ring homomorphism. (We are assuming that rings have units, and that ring homomorphisms are unital.) Let us say that 1 *is in the stable range of* φ provided that for any f, g, $h \in E$ satisfying $fg + h = 1$, there exists $k \in F$ such that $\varphi(f) + \varphi(h)k$ is invertible in F. (Strictly speaking, this condition should be called something like "1 is in the strong stable range of φ", to distinguish it from the possibly weaker condition in which it is only required that $\varphi(f) + \varphi(h)k$ be right invertible in F.) To prove that this condition is left-right symmetric, we use the following result of Vasershtein, adapted from the proof of [**10**, Theorem 2]. We would like to thank R. K. Dennis for bringing this argument to our attention.

LEMMA 3.1. [*Vasershtein*] *Let a, b, c be elements of a ring F, such that $ab + c = 1$. If there exists $x \in F$ such that $a + cx$ is invertible, then there exists $y \in F$ such that $b + yc$ is invertible.*

Proof. Set $u = a + cx$, and set

$$v = b + (1 - bx)u^{-1}c \quad \text{and} \quad w = a + x(1 - ba).$$

The lemma is established by showing that v is invertible with inverse w.

First, observe that

$$(1) \qquad va = ba + (1 - bx)u^{-1}ca,$$

$$(2) \qquad vx = bx + (1 - bx)u^{-1}(u - a) = 1 - (1 - bx)u^{-1}a,$$

$$(3) \qquad \begin{aligned} vx(1 - ba) &= (1 - ba) - (1 - bx)u^{-1}(1 - ab)a \\ &= 1 - ba - (1 - bx)u^{-1}ca. \end{aligned}$$

Adding equations (1) and (3) yields $vw = 1$. Next, observe that

$$(4) \qquad wb = ab + xb(1 - ab) = ab + xbc,$$

$$(5) \qquad \begin{aligned} w(1 - bx) &= a + x(1 - ba) - abx - xbcx \\ &= a + (1 - ab)x - xb(a + cx) \\ &= a + cx - xbu = (1 - xb)u, \end{aligned}$$

$$(6) \qquad w(1 - bx)u^{-1}c = (1 - xb)c.$$

Adding equations (4) and (6) yields $wv = ab + c = 1$. □

PROPOSITION 3.2. *Let φ: $E \to F$ be a ring homomorphism. Then φ has 1 in its stable range if and only if for any f, g, $h \in E$ satisfying $gf + h = 1$, there exists $k \in F$ such that $\varphi(f) + k\varphi(h)$ is invertible in F.*

Proof. First assume that φ has 1 in its stable range. Given f, g, $h \in E$ with $gf + h = 1$, there exists $k' \in F$ such that $\varphi(g) + \varphi(h)k'$ is invertible in F. Since $\varphi(g)\varphi(f) + \varphi(h) = 1$, Lemma 3.1 shows that there exists $k \in F$ such that $\varphi(f) + k\varphi(h)$ is invertible in F. The converse is proved symmetrically. □

In a similar manner, Lemma 3.1 may be used to show that the "power-substitution" condition studied in [4, 5] is left-right symmetric.

We use $\mathfrak{M}$od-R to denote the category of all right modules over a ring R.

THEOREM 3.3. *Let R and S be rings, and let T: $\mathfrak{M}$od-$R \to \mathfrak{M}$od-S be an additive functor. Let A be a right R-module, and let*

$$\varphi\colon \mathrm{End}_R(A) \to \mathrm{End}_S(T(A))$$

be the induced ring homomorphism, $e \mapsto T(e)$. *Then* φ *has* 1 *in its stable range if and only if the following condition holds*:

(*) *Given any direct sum decomposition* $M = A_1 \oplus B_1 = A_2 \oplus B_2$ *of right R-modules with* $A_1 \cong A_2 \cong A$, *there exists an S-submodule C of* $T(M)$ *such that*

$$T(M) = C \oplus T(B_1) = C \oplus T(B_2).$$

[In the statement of (*), we have identified $T(B_i)$ with the image of T of the inclusion map $B_i \to M$. This is legitimate because T preserves split monomorphisms.]

Proof. First assume that 1 is in the stable range of φ, and consider a right R-module direct sum decomposition

$$M = A_1 \oplus B_1 = A_2 \oplus B_2$$

such that $A_1 \cong A_2 \cong A$. For $i = 1, 2$, let r_i: $A \to M$ be the composition of an isomorphism $A \to A_i$ with the inclusion map $A_i \to M$, and let s_i: $B_i \to M$ be the inclusion map. There exist projection maps p_i: $M \to A$ and q_i: $M \to B_i$ such that

$$p_i r_i = 1_A; \quad q_i s_i = 1_{B_i}; \quad p_i s_i = 0; \quad q_i r_i = 0$$

for each i, and also $r_i p_i + s_i q_i = 1_M$ for each i. (See Diagram I.)

$$\begin{array}{ccccc} A & & & & A \\ r_1 \downarrow \uparrow p_1 & & & & r_2 \downarrow \uparrow p_2 \\ A_1 \oplus B_1 & = & M & = & A_2 \oplus B_2 \\ s_1 \uparrow \downarrow q_1 & & & & s_2 \uparrow \downarrow q_2 \\ B_1 & & & & B_2 \end{array}$$

DIAGRAM I

Note that since

$$T(p_i)T(s_i) = 0 \quad \text{and} \quad T(r_i)T(p_i) + T(s_i)T(q_i) = 1_{T(M)},$$

we must have $\ker(T(p_i)) = \operatorname{im}(T(s_i)) = T(B_i)$.

Observe that

$$1_A = p_2 r_2 = p_2(r_1 p_1 + s_1 q_1)r_2 = (p_2 r_1)(p_1 r_2) + (p_2 s_1 q_1 r_2),$$

and that the maps $p_2 r_1$, $p_1 r_2$, $p_2 s_1 q_1 r_2$ all lie in $\operatorname{End}_R(A)$. As φ has 1 in its stable range, there exists a map k in $\operatorname{End}_S(T(A))$ such that the map

$$\varphi(p_2 r_1) + \varphi(p_2 s_1 q_1 r_2)k = T(p_2 r_1) + T(p_2 s_1 q_1 r_2)k$$

is an automorphism of $T(A)$. Set

$$f = T(r_1) + T(s_1 q_1 r_2)k\colon T(A) \to T(M),$$

so that $T(p_2)f$ is an automorphism of $T(A)$. Thus if $C = f(T(A))$, then

$$T(M) = C \oplus \ker(T(p_2)) = C \oplus T(B_2).$$

In addition, $T(p_1)f = T(p_1 r_1) = 1_{T(A)}$, and hence

$$T(M) = C \oplus \ker(T(p_1)) = C \oplus T(B_1),$$

so that (*) is proved.

Conversely, assume that (*) holds, and consider maps f, g, $h \in \mathrm{End}_R(A)$ satisfying $fg + h = 1$. Set $A_1 = B_1 = A$ and $M = A_1 \oplus B_1$. Let

$$p_1\colon M \to A_1; \quad q_1\colon M \to B_1; \quad r_1\colon A_1 \to M; \quad s_1\colon B_1 \to M$$

be the projection and injection maps for this direct sum. Define maps

$$p_2 = fp_1 + hq_1\colon M \to A, \qquad r_2 = r_1 g + s_1\colon A \to M,$$

and note that $p_2 r_2 = fg + h = 1_A$. Thus if $A_2 = r_2(A)$ and $B_2 = \ker(p_2)$, then $M = A_2 \oplus B_2$ with $A_2 \cong A$.

By (*), there exists an S-submodule C of $T(M)$ such that

$$T(M) = C \oplus T(B_1) = C \oplus T(B_2).$$

As C and $T(A_1)$ are each complements for $T(B_1)$ in $T(M)$, we must have

$$C \cong T(M)/T(B_1) \cong T(A_1) = T(A).$$

Consequently, there exists a monomorphism $t\colon T(A) \to T(M)$ such that $t(T(A)) = C$. Since each of the maps $T(p_i)\colon T(M) \to T(A)$ is a (split) epimorphism, and since

$$T(M) = C \oplus T(B_i) = t(T(A)) \oplus \ker(T(p_i)),$$

we see that $T(p_i)t$ is an automorphism of $T(A)$. Now

$$T(p_2)t = T(f)T(p_1)t + T(h)T(q_1)t,$$

and hence

$$[T(p_2)t][T(p_1)t]^{-1} = T(f) + T(h)[T(q_1)t][T(p_1)t]^{-1}.$$

Therefore the map $k = [T(q_1)t][T(p_1)t]^{-1}$ is an element of $\mathrm{End}_S(T(A))$ such that $\varphi(f) + \varphi(h)k$ is invertible in $\mathrm{End}_S(T(A))$, which proves that φ has 1 in its stable range. □

COROLLARY 3.4. *Let R and S be rings, and let T: $\mathfrak{M}$od-$R \to \mathfrak{M}$od-S be an additive functor. Let A be a right R-module and assume that the induced ring homomorphism*

$$\mathrm{End}_R(A) \to \mathrm{End}_S(T(A))$$

has 1 *in its stable range. If B_1 and B_2 are any right R-modules such that $A \oplus B_1 \cong A \oplus B_2$, then $T(B_1) \cong T(B_2)$.*

Proof. We may assume that there exists a right R-module M with direct sum decompositions

$$M = A_1 \oplus B_1 = A_2 \oplus B_2$$

such that $A_1 \cong A_2 \cong A$. By Theorem 3.3, there is an S-submodule C of $T(M)$ such that

$$T(M) = C \oplus T(B_1) = C \oplus T(B_2).$$

Therefore $T(B_1) \cong T(M)/C \cong T(B_2)$. □

IV. Vector bundle cancellation results. We apply Corollary 3.4 to certain modules of sections of vector bundles to obtain our cancellation results for vector bundles. We remind the reader of the notation $n\xi$ for the direct sum of n copies of a vector bundle ξ.

THEOREM 4.1. *Let X be a paracompact topological space, let λ, ξ_1, ξ_2 be real vector bundles over X, and assume that λ is a line bundle of finite type. If $\lambda \oplus \xi_1 \cong \lambda \oplus \xi_2$, then:*

(a) $\xi_1^{\mathbf{C}} \cong \xi_2^{\mathbf{C}}$ *(as complex vector bundles).*

(b) $2\xi_1 \cong 2\xi_2$ *(as real vector bundles).*

Proof. As (b) follows immediately from (a), we need only prove (a).

Set $R = C(X, \mathbf{R})$ and $S = C(X, \mathbf{C})$, and let T: $\mathfrak{M}$od-$R \to \mathfrak{M}$od-S be the complexification functor $(-) \otimes_{\mathbf{R}} \mathbf{C}$. Set $A = \Gamma(\lambda)$ and $B_i = \Gamma(\xi_i)$ (for $i = 1, 2$). Then $A \oplus B_1 \cong A \oplus B_2$, and in view of Theorem 1.1 and Proposition 1.2, it suffices to show that $T(B_1) \cong T(B_2)$.

Let φ: $\mathrm{End}_R(A) \to \mathrm{End}_S(T(A))$ be the ring homomorphism induced by T. We claim that φ has 1 in its stable range. Since λ is a real line bundle of finite type, Proposition 1.3 shows that the natural ring homomorphism of R into $\mathrm{End}_R(A)$ is an isomorphism. Similarly, as $\lambda^{\mathbf{C}}$ is a complex line bundle of finite type, the natural ring homomorphism of S

into $\mathrm{End}_S(T(A))$ is an isomorphism. Consequently, we obtain a commutative diagram as follows:

$$\begin{array}{ccc} R & \overset{\subseteq}{\rightarrow} & S \\ \cong\downarrow & & \downarrow\cong \\ \mathrm{End}_R(A) & \overset{\varphi}{\rightarrow} & \mathrm{End}_S(T(A)) \end{array}$$

DIAGRAM II

Hence, to prove that φ has 1 in its stable range, it suffices to prove the same for the inclusion map $R \to S$.

Thus, consider any f, g, h in R satisfying $fg + h = 1$. Then f and h are real-valued functions which do not simultaneously vanish at any point of X. Consequently, the complex-valued function $f + ih$ never vanishes, and so $f + ih$ is an invertible element of S. This shows that the inclusion map $R \to S$ has 1 in its stable range and, hence, φ does also.

As $A \oplus B_1 \cong A \oplus B_2$, we can now conclude from Corollary 3.4 that $T(B_1) \cong T(B_2)$. □

The second conclusion of Theorem 4.1 was noted (for the case that X is compact) in [**5**, Theorem 14], as a consequence of the observation that the diagonal map from $C(X, \mathbf{R})$ to the 2×2 matrix ring over $C(X, \mathbf{R})$ has 1 in its stable range.

In proving a complex analog of Theorem 4.1, there is an obstacle, due to the fact that the complex field is not contained in the center of the quaternions. Namely, given an arbitrary complex line bundle λ over X of finite type, the endomorphism ring of $\Gamma(\lambda)$ can be identified with $C(X, \mathbf{C})$, but there is no natural way of identifying the endomorphism ring of $\Gamma(\lambda^{\mathbf{H}})$ with $C(X, \mathbf{H})$; in fact, it is not even clear whether the endomorphism ring of $\Gamma(\lambda^{\mathbf{H}})$ is necessarily isomorphic to $C(X, \mathbf{H})$. This difficulty can be circumvented in case λ is a trivial vector bundle of rank 1, or, more generally, if λ is the complexification of a real line bundle.

THEOREM 4.2. *Let X be a paracompact topological space, let λ, ξ_1, ξ_2 be complex vector bundles over X, and assume that λ is the complexification of a real line bundle of finite type.* (*In particular, λ could be trivial of rank* 1.) *If $\lambda \oplus \xi_1 \cong \lambda \oplus \xi_2$, then*:

(a) $\xi_1^{\mathbf{H}} \cong \xi_2^{\mathbf{H}}$ (*as quaternionic vector bundles*).

(b) $\xi_1 \oplus \bar{\xi}_1 \cong \xi_2 \oplus \bar{\xi}_2$ (*as complex vector bundles*).

Proof. Set $R = C(X, \mathbf{C})$ and $S = C(X, \mathbf{H})$, and let $A = \Gamma(\lambda)$. As in Theorem 4.1, the desired conclusions follow from Corollary 3.4 provided that we show that the induced ring homomorphism from $\operatorname{End}_R(A)$ to $\operatorname{End}_S(A \otimes_{\mathbf{C}} \mathbf{H})$ has 1 in its stable range.

By assumption, there exists a real line bundle λ_0 over X of finite type such that $\lambda_0^{\mathbf{C}} = \lambda$. Set $Q = C(X, \mathbf{R})$ and $B = \Gamma(\lambda_0)$. By Proposition 1.2, we may identify A with $B \otimes_{\mathbf{R}} \mathbf{C}$, and then $A \otimes_{\mathbf{C}} \mathbf{H}$ becomes identified with $B \otimes_{\mathbf{R}} \mathbf{H}$. Remember that $B \otimes_{\mathbf{R}} \mathbf{C}$ and $B \otimes_{\mathbf{R}} \mathbf{H}$ are to be identified with $B \otimes_Q R$ and $B \otimes_Q S$, and that $B \otimes_Q S$ is a *right* S-module. Thus the problem now is to show that the induced ring homomorphism φ from $\operatorname{End}_R(B \otimes_Q R)$ to $\operatorname{End}_S(B \otimes_Q S)$ has 1 in its stable range.

According to Proposition 1.3, the natural ring homomorphism from R to $\operatorname{End}_R(B \otimes_Q R)$ is an isomorphism. Under this isomorphism, an element $x \in R$ corresponds to the endomorphism sending any $b \otimes r$ to $b \otimes xr$. Since the ring Q is the center of the ring S, we may define a ring homomorphism

$$\psi\colon S \to \operatorname{End}_S(B \otimes_Q S)$$

so that $\psi(y)(b \otimes s) = b \otimes ys$ for all $b \in B$ and all $y, s \in S$. We now have a commutative diagram as follows:

$$\begin{array}{ccc} R & \xrightarrow{\subseteq} & S \\ \cong\downarrow & & \downarrow\psi \\ \operatorname{End}_R(B \otimes_Q R) & \xrightarrow{\varphi} & \operatorname{End}_S(B \otimes_Q S) \end{array}$$

DIAGRAM III

To show that φ has 1 in its stable range, it suffices to show that the inclusion map $R \to S$ does (since the composition of this map with ψ will then have 1 in its stable range).

However, this is as easy as before. Given f, g, h in R satisfying $fg + h = 1$, then f and h are complex-valued functions which do not simultaneously vanish at any point of X. Consequently, the quaternion-valued function $f + jh$ never vanishes, and so $f + jh$ is invertible in S. Therefore the inclusion map $R \to S$ does indeed have 1 in its stable range. □

COROLLARY 4.3. *Let X be a paracompact topological space, let λ, ξ_1, ξ_2 be real vector bundles over X, and assume that λ is a direct sum of two line bundles of finite type. If $\lambda \oplus \xi_1 \cong \lambda \oplus \xi_2$, then*:

(a) $\xi_1^{\mathbf{H}} \cong \xi_2^{\mathbf{H}}$ (*as quaternionic vector bundles*).

(b) $\xi_1^{\mathbf{C}} \oplus \overline{\xi_1^{\mathbf{C}}} \cong \xi_2^{\mathbf{C}} \oplus \overline{\xi_2^{\mathbf{C}}}$ (*as complex vector bundles*).
(c) $4\xi_1 \cong 4\xi_2$ (*as real vector bundles*).

Proof. By assumption, $\lambda = \lambda_1 \oplus \lambda_2$ for some line bundles λ_1, λ_2 of finite type. Then

$$\lambda_1 \oplus (\lambda_2 \oplus \xi_1) \cong \lambda_1 \oplus (\lambda_2 \oplus \xi_2).$$

Applying Theorem 4.1, we obtain $\lambda_2^{\mathbf{C}} \oplus \xi_1^{\mathbf{C}} \cong \lambda_2^{\mathbf{C}} \oplus \xi_2^{\mathbf{C}}$. Then Theorem 4.2 applies, yielding $(\xi_1^{\mathbf{C}})^{\mathbf{H}} \cong (\xi_2^{\mathbf{C}})^{\mathbf{H}}$, or $\xi_1^{\mathbf{H}} \cong \xi_2^{\mathbf{H}}$. The remaining conclusions follow immediately. □

The last conclusion of Corollary 4.3 may be derived directly from Corollary 3.4, using the fact that the inclusion map from the 2×2 matrix ring over $C(X, \mathbf{R})$ into the 2×2 matrix ring over $C(X, \mathbf{H})$ has 1 in its stable range. (The proof of this fact requires a certain amount of matrix manipulation.)

Theorems 4.1 and 4.2, and Corollary 4.3, also hold for smooth vector bundles over a smooth manifold X, since the inclusion maps from $C^\infty(X, \mathbf{R})$ to $C^\infty(X, \mathbf{C})$ and from $C^\infty(X, \mathbf{C})$ to $C^\infty(X, \mathbf{H})$ each have 1 in their stable range.

It might seem plausible to try proving one further result of the type we have derived by using the Cayley numbers. However, as our method applies to endomorphism rings, which are necessarily associative, there is no way for the Cayley numbers to fit into our method.

We would like to raise the question of cancellation results of this type for higher-rank vector bundles, at least in the real case. Here is one possible suggestion.

Problem. Let X be a paracompact topological space, let λ, ξ_1, ξ_2 be real vector bundles over X, and assume that λ is trivial of rank n. If $\lambda \oplus \xi_1 \cong \lambda \oplus \xi_2$, does it follow that $2n\xi_1 \cong 2n\xi_2$? [The answer is positive in case $n = 1$ or 2, by Theorem 4.1 and Corollary 4.3, and the answer is also positive in case ξ_2 is trivial, by Lam's result (Corollary 2.2).]

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