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ON THE STRUCTURE OF SIMPLE-SEMIABELIAN LIE-ALGEBRAS

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Simple Lie-algebras, all whose proper subalgebras are abelian, and their algebras of derivations are studied. In many cases the algebra of outer derivations of such a Lie-algebra turns out to be abelian.

0. Introduction. In this paper the structure of simple Lie-algebras having only abelian subalgebras, in the following referred to as simple-semiabelian, will be investigated. It has been shown in [3] that this class of simple Lie-algebras depends on the properties of the underlying base field: there are, for instance, no simple-semiabelian Lie-algebras over algebraically closed fields. Questions concerning the field theoretical aspects are not studied here; we will approach the problem from a purely Lie-algebraic point of view.

In order to apply the results of Kaplansky ([6], [7]) some introductory remarks on base field extensions are necessary. Although according to the nature of the topic, many structural aspects of simple-semiabelian Lie-algebras vanish after base field extension, some features can be retrieved. This applies in particular to the index one case studied in §4 which makes it possible to illustrate the scarcity of examples of low dimension. At present only three-dimensional representatives of this class are known (cf. [3]) and it is an interesting open problem to construct such objects of higher dimension.

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1. Remarks on base field extensions. In the following, let k be a perfect field and let K be an algebraic closure of k. The Galois group of $K: k$ will be denoted by $\mathrm{Gal}(K: k)$. Throughout this paper we will consider a finite dimensional Lie-algebra G, together with the Lie-algebra $G' := G \otimes_k K$ obtained by base field extension.

LEMMA 1.1. *Let $H \subset G$ be a Cartan subalgebra. Then the following statements hold*:

(1) *$H' := H \otimes_k K$ is a Cartan subalgebra of G'.*

(2) *Let* $G' = H' \oplus \bigoplus_{\alpha \in R} G'_\alpha$ *be the Cartan decomposition of* G' *relative* H'. *For* γ *in* $\mathrm{Gal}(K : k)$ *and* α *in* R *define*

$$\gamma \cdot \alpha := \gamma \circ \alpha \circ \mathrm{id}_G \otimes \gamma^{-1}.$$

Then $(\mathrm{id}_G \otimes \gamma)(G'_\alpha) = G'_{\gamma \cdot \alpha}$.

(3) $\mathrm{Gal}(K : k)$ *acts on* R *via*

$$\sigma : \begin{cases} \mathrm{Gal}(K : k) \times R \to R \\ (\gamma, \alpha) \mapsto \gamma \cdot \alpha \end{cases}$$

The orbit of $\alpha \in R$ *under* $\mathrm{Gal}(K : k)$ *will be denoted by* $[\alpha]$.

DEFINITION. G is called ad-semisimple if ad_x is semisimple $\forall\, x \in G$.

According to (1.2) of [3] every ad-semisimple solvable Lie-algebra is abelian. Every subalgebra and every homorphic image of an ad-semisimple Lie-algebra is ad-semisimple.

Let $H \subset G$ be a Cartan subalgebra. Then there exists an H-module $V \subset G$ such that $G = H \oplus V$ (Theorem 4, p. 39 of [4]). This decomposition will be referred to as the Fitting decomposition of G relative to H. We obviously have $V \otimes_k K = \bigoplus_{\alpha \in R} G'_\alpha$.

PROPOSITION 1.2. *Let G be ad-semisimple. Then $H \subset G$ is abelian and V is a completely reducible H-module. Moreover*

$$G'_\alpha = \{x \in G';\, [h, x] = \alpha(h) \cdot x\, \forall\, h \in H'\} \quad \forall\, \alpha \in R.$$

Proof. H is nilpotent, ad-semisimple and, by virtue of (1.2) of [3], abelian. Consequently ad_h is diagonable for every $h \in H'$. Since $\alpha(h)$ is the only eigenvalue of $\mathrm{ad}_h|_{G'_\alpha}$, we obtain $\mathrm{ad}_h|_{G'_\alpha} = \alpha(h) \cdot \mathrm{id}_{G'_\alpha}$. The H'-module $V \otimes_k K$ is obviously completely reducible, therefore the H-module V has the same property.

PROPOSITION 1.3. *Let G be ad-semisimple and consider the Fitting decomposition $G = H \oplus V$ relative to a Cartan subalgebra H, as well as the induced Cartan decomposition $G' = H' \oplus \bigoplus_{\alpha \in R} G'_\alpha$*

(1) *Let $W \subset V$ be an irreducible H-submodule. Then there is $\alpha \in R$ such that $(W \otimes_k K) \cap G'_\alpha \neq 0$ and $W \otimes_k K = \bigoplus_{\sigma \in [\alpha]} (W \otimes_k K) \cap G'_\sigma$.*

(2) *Let $\mathrm{Irr}_H(V)$ denote the set of irreducible H-submodules of V. Then there is a mapping Q: $\mathrm{Irr}_H(V) \to R/\mathrm{Gal}(K : k)$, such that $Q(W) = [\alpha]$ if $W \otimes_k K = \bigoplus_{\sigma \in [\alpha]} (W \otimes_k K) \cap G'_\alpha$.*

(3) *Suppose* $\dim_K G'_\alpha = 1 \ \forall \alpha \in R$. *Then* Q *is bijective and if* $V = \bigoplus_{i=1}^n V_i$, *where each* V_i *is irreducible, then* $\mathrm{Irr}_H(V) = \{V_1, \ldots, V_n\}$.

Proof. (1). By assumption, H' is abelian and every ad_h is diagonable. Consequently, there is a common eigenvector x in $W \otimes_k K$. This yields the existence of a root $\alpha \in R$ such that G'_α meets $W \otimes_k K$. Consider

$$U' := \sum_{\gamma \in \mathrm{Gal}(K:\,k)} \mathrm{id}_G \otimes \gamma\big((W \otimes_k K) \cap G'_\alpha\big).$$

By virtue of (1.2) U' is an H'-module which is obviously contained in $W \otimes_k K$. Since U' is invariant under the action of the Galois group there exists, by general theory, a subspace $U \subset W$ such that $U \otimes_k K = U'$. Now U is an H-module and by virtue of the irreducibility of W we obtain $U = W$. It is easy to see that $U' = \bigoplus_{\sigma \in [\alpha]} (W \otimes_k K) \cap G'_\sigma$.

(2) We need only prove Q is well-defined. This is obviously a consequence of the uniqueness of the direct sum decomposition.

(3). Suppose $\dim_K G'_\alpha = 1 \ \forall \alpha \in R$. By (1) we obtain

$$W \otimes_k K = \bigoplus_{\sigma \in Q(W)} G'_\sigma \quad \forall W \in \mathrm{Irr}_H(V).$$

Let U, W be irreducible submodules of V such that $Q(U) = Q(W)$. Then $U \otimes_k K = W \otimes_k K$. Since U, W are irreducible they are either equal or have trivial intersection. By the equation above the latter case cannot occur. To verify the surjectivity let $\alpha \in R$ be a root and consider $U' := \bigoplus_{\sigma \in [\alpha]} G'_\sigma$. Clearly, U' is an H'-module which is invariant under the action of the Galois group (1.1). There is an H-module $U \subset V$ such that $U \otimes_k K = U'$. Let $W \subset U$ be an irreducible submodule. Then there is $\sigma_0 \in [\alpha]$ such that

$$W \otimes_k K = \bigoplus_{\sigma \in [\sigma_0]} G'_\sigma = \bigoplus_{\sigma \in [\alpha]} G'_\sigma = U \otimes_k K.$$

Consequently $\dim_k W = \dim_k U$ and we obtain $U = W$. By definition of U the equation $Q(U) = [\alpha]$ holds.

Now consider a direct sum decomposition $V = \bigoplus_{i=1}^n V_i$. This yields $R = \bigcup_{i=1}^n Q(V_i)$, therefore we have $\mathrm{Irr}_H(V) = \{V_1, \ldots, V_n\}$.

REMARK. One can show that the condition of (3) holds if V is an irreducible H-module. According to (1.3) this also implies the Galois group acts transitively on R. Suppose, conversely, that a transitive Galois group action is given and all the root spaces are one-dimensional; then $\mathrm{Irr}_H(V)$ contains only one element, and by complete reducibility of V we see that V has to be irreducible.

We proceed by proving a lemma which is analogous to the classical result concerning the Killing form and will be applied in the study of the structure of simple-semiabelian Lie-algebras of index 1. Following Dixmier [1] we define for every linear form $f \in G^*$ the associated alternating bilinear form

$$B_f: \begin{cases} G \times G \to k \\ (x, y) \mapsto f([x, y]). \end{cases}$$

Note that $\operatorname{rad}(B_f) = \{x \in G;\ B_f(x, y) = 0\ \forall y \in G\}$ is a subalgebra of G.

LEMMA 1.4. *Let $H \subset G$ be a Cartan subalgebra and suppose there is $f \in G^*$ such that $H = \operatorname{rad}(B_f)$. Consider the extended linear form f': $G' \to K$. Then the following statements hold*:

(1) $H' = \operatorname{rad}(B_{f'})$.

(2) $V' := \bigoplus_{\alpha \in R} G'_\alpha \subset \ker f'$.

(3) $B_{f'}|_{V' \times V'}$ *is non-singular.*

(4) $B_{f'}(G'_\alpha, G'_\beta) = 0\ \forall \beta \neq -\alpha \in R$.

(5) $G'_\alpha \cap G'^{\perp}_{-\alpha} = 0\ \forall \alpha \in R$.

(6) $\alpha \in R \Rightarrow -\alpha \in R$.

Proof. (1). We have $H' = H \otimes_k K = \operatorname{rad}(B_f) \otimes_k K = \operatorname{rad}(B_{f'})$.

(2) Let $\alpha \in R$ be a root. Then there is $h \in H'$ such that $\alpha(h) = 1$ and we consequently have $\ker \operatorname{ad}_h|_{G'_\alpha} = 0$. For $w \in G'_\alpha$ there is $v \in G'_\alpha$ such that $w = [h, v]$. This yields $f'(w) = f'([h, v]) = B_{f'}(h, v) = 0$.

(3) This follows directly from the definition of V' and (1).

(4) If $\beta \neq -\alpha$ then $\beta + \alpha \neq 0$ and $[G'_\alpha, G'_\beta] \subset V'$. The result now follows from (2).

(5) is a direct consequence of (3) and (4).

(6) According to (5) $B_{f'}|_{G'_\alpha \times G'_{-\alpha}}$ is non-singular. Hence there is an isomorphism $G'_\alpha \simeq G'^*_{-\alpha}$ proving that $G'_{-\alpha} \neq 0$.

2. The algebra of derivations of a simple-semiabelian Lie-algebra. We adopt the notation and the assumptions of the preceding section.

DEFINITION. G is called simple-semiabelian if G is simple and every proper subalgebra is abelian.

Note that the maximal subalgebras of G are the Cartan subalgebras. Moreover, for every maximal subalgebra H of G, H is equal to the centralizer $\mathrm{Cen}_G(x)$ for every non-zero element x of H. It has been shown in [**2**] that the maximal subalgebras of G are of the form $H = \mathrm{rad}(B_f)$, $f \in G^*$. Every derivation of a simple-semiabelian Lie-algebra is semisimple, by (4.1) of [**3**]. We use this to prove:

LEMMA 2.1. *Let G be simple-semiabelian and let $D \in \mathrm{Der}_k(G)$ be a derivation. Then every eigenvalue of D which lies in k is zero.*

Proof. Let $\alpha \in k$ be an eigenvalue of D. Then there is $x_0 \in G \backslash \{0\}$ such that $D(x_0) = \alpha \cdot x_0$; this implies that $H := kD + k \cdot \mathrm{ad}_{x_0}$ is a solvable subalgebra of $\mathrm{Der}_k(G)$. Since $\mathrm{Der}_k(G)$ is (by (4.1) of [**3**] and (1.3.22) of [**1**]) ad-semisimple it follows that H is abelian. This yields $0 = [D, \mathrm{ad}_{x_0}] = \alpha \cdot \mathrm{ad}_{x_0}$. Thus $\alpha = 0$.

LEMMA 2.2. *Let G be simple-semiabelian. Then the following statements hold*: (1) *If $H \subset G$ is a proper subalgebra and $D \in \mathrm{Der}_k(G)$ is a drivation such that $D(H) \subset H$, then $H \subset \ker D$.*

(2) Let $D \in \mathrm{Der}_k(G)$ *and* $x_0 \in \ker D \backslash \{0\}$, *then the centralizer* $\mathrm{Cen}_G(x_0)$ *of* x_0 *lies in* $\ker D$.

Proof. (1). Consider $H_1 := \mathrm{ad}(H) + kD \subset \mathrm{Der}_k(G)$. Then H_1 is a solvable subalgebra of $\mathrm{Der}_k(G)$. Consequently, H_1 is abelian and, in particular, H is contained in $\ker D$.

(2). Let $x_0 \in \ker D \backslash \{0\}$ and let $y \in \mathrm{Cen}_G(x_0)$. Then $[D(y), x_0] = D([y, x_0]) - [y, D(x_0)] = 0$. Consequently, we obtain the desired result by applying (1).

THEOREM 2.3. *Let G be simple-semiabelian of characteristic $p > 0$. Consider a maximal subalgebra $H \subset G$, as well as $\mathfrak{A}_H := \{D \in \mathrm{Der}_k(G);$ $D(H) = 0\}$. Then the following statements hold*:

(1) *$\mathfrak{A}_H$ is a self-normalizing p-subalgebra of* $\mathrm{Der}_k(G)$.

(2) *If $\mathfrak{H} \subset \mathfrak{A}_H$ is a Cartan subalgebra of $\mathfrak{A}_H$, then $\mathfrak{H}$ is an abelian Cartan subalgebra of* $\mathrm{Der}_k(G)$.

(3) *If $G = H \oplus V$ is the Fitting decomposition of G relative to H then* $\mathrm{Der}_k(G) = \mathfrak{A}_H \oplus \mathrm{ad}(V)$.

(4) $\mathrm{Der}_k(G)/\mathrm{ad}(G) \simeq \mathfrak{A}_H/\mathrm{ad}(H)$.

(5) $[\mathrm{Der}_k(G), \mathrm{Der}_k(G)] = \mathrm{ad}(G)$ *if and only if $\mathfrak{A}_H$ is abelian.*

Proof. (1) Let D be an element of $\mathrm{Nor}_{\mathrm{Der}_k(G)}(\mathfrak{A}_H)$, the normalizer of $\mathfrak{A}_H$ in $\mathrm{Der}_k(G)$. Then we have $[D, \mathfrak{A}_H] \subset \mathfrak{A}_H$, which, in particular, yields

$\mathrm{ad}_{D(h)} \in \mathfrak{A}_H$ for every $h \in H$. Consequently H is a D-stable subspace and D is an element of $\mathfrak{A}_H$ by virtue of (2.2). A self-normalizing subalgebra of a restricted Lie-algebra is necessarily a p-subalgebra.

(2) Every Cartan subalgebra of an ad-semisimple Lie-algebra is abelian. Let D be an element of $\mathrm{Nor}_{\mathrm{Der}_k(G)}(\mathfrak{H})$. Then $\mathfrak{H} + kD$ is solvable and therefore abelian. As a Cartan subalgebra of $\mathfrak{A}_H$, $\mathfrak{H}$ obviously contains the center $\mathfrak{Z}(\mathfrak{A}_H)$. This yields particularly $\mathrm{ad}(H) \subset \mathfrak{H}$ and we therefore have $[\mathrm{ad}(H), D] \subset [\mathfrak{H}, D] = 0$, which, in turn, means that D lies in $\mathfrak{A}_H$. We finally apply (1) in order to see that D is actually an element of $\mathfrak{H}$.

(3) Let $G = H \oplus V$ be the Fitting decomposition of G relative H. Consider in addition the Fitting decomposition of $\mathrm{Der}_k(G)$ relative to $\mathfrak{H}$ and write $\mathrm{Der}_k(G) = \mathfrak{H} \oplus W$. According to (1.2), W is a completely reducible $\mathfrak{H}$-module, so we may write $W = (W \cap \mathfrak{A}_H) \oplus U$. We claim that U lies in $\mathrm{ad}(V)$. Let D be an element of U. If $D(h) = 0$, for an element $h \in H\backslash\{0\}$, then $H = \mathrm{Cen}_G(h) \subset \ker D$ (2.2). This implies $D \in \mathfrak{A}_H$ and hence $D = 0$. Consequently, for a non-zero element h_0 of H, the map

$$T\colon \begin{cases} U \to U \\ D \mapsto [\mathrm{ad}_{h_0}, D] \end{cases}$$

is injective and hence surjective. For every element $D \in U$ there exists an element $D_1 \in U$ such that $D = [D_1, \mathrm{ad}_{h_0}] = \mathrm{ad}_{D_1(h_0)}$.

This proves the inclusion $U \subset \mathrm{ad}(G)$ and we obtain

$$\mathrm{Der}_k(G) = \mathfrak{H} \oplus (W \cap \mathfrak{A}_H) \oplus U = \mathfrak{A}_H + \mathrm{ad}(G) = \mathfrak{A}_H \oplus \mathrm{ad}(V)$$

since $\mathrm{ad}(H) \subset \mathfrak{A}_H$ and $\mathfrak{A}_H \cap \mathrm{ad}(V) = 0$.

(4)

$$\begin{aligned} \mathrm{Der}_k(G)/\mathrm{ad}(G) &= \mathfrak{A}_H + \mathrm{ad}(G)/\mathrm{ad}(G) \simeq \mathfrak{A}_H/\mathfrak{A}_H \cap \mathrm{ad}(G) \\ &= \mathfrak{A}_H/\mathrm{ad}(H). \end{aligned}$$

(5) If $\mathfrak{A}_H$ is abelian, so is $\mathrm{Der}_k(G)/\mathrm{ad}(G)$ by (4). Together with the simplicity of $\mathrm{ad}(G) \simeq G$ this yields the asserted equality. Suppose, conversely, that $\mathrm{Der}_k(G)/\mathrm{ad}(G)$ is abelian, then $\mathfrak{A}_H$ is solvable and hence abelian.

REMARK. In the situation above, let v be an element of V and $x_0 \in H\backslash\{0\}$. Then $\ker(\mathrm{ad}_{x_0}|_V) = 0$. Consequently there exists $v_1 \in V$ such that $v = [x_0, v_1]$. Write $D(v_1) = h_1 + v_2$, $h_1 \in H$, $v_2 \in V$. Then

$$D(v) = D([x_0, v_1]) = [x_0, D(v_1)] = [x_0, v_2] \in V.$$

Therefore we obtain the structure of a restricted $\mathfrak{A}_H$-module on V by defining $D \cdot v = D(v)$.

Now let v be a non-zero element of V. The definition of $\mathfrak{A}_H$ gives rise to the injectivity of the mapping

$$S_v : \begin{cases} \mathfrak{A}_H \to V \\ D \to D(v). \end{cases}$$

Since 1 is not an eigenvalue of any $D \in \mathfrak{A}_H$ (2.1), we have $v \notin \operatorname{im} S_v$. Thus $\dim_k \mathfrak{A}_H < \dim_k V$. Combining this with (3) of the preceding theorem we obtain $\dim_k \operatorname{Der}_k(G) < 2 \dim_k G/H$.

In some cases, notably when G possesses an invariant non-singular bilinear form, $\mathfrak{A}_H$ can be shown to be abelian (cf. §3). At the moment, we investigate the case of a "minimal" simple-semiabelian Lie-algebra.

PROPOSITION 2.4. *Let G be simple-semiabelian of minimal dimension. Then* $[\operatorname{Der}_k(G), \operatorname{Der}_k(G)] = \operatorname{ad}(G)$.

Proof. Let $H \subset G$ be a maximal subalgebra and suppose $\mathfrak{A}_H$ is not abelian. Let B be a minimal non-abelian subalgebra of $\mathfrak{A}_H$ and let $J \lhd B$ be a maximal ideal of B. The subquotient B/J is not abelian since otherwise B would be solvable and hence abelian. Consequently B/J is simple. According to the choice of B and J every proper subalgebra of B/J is abelian. This contradicts the minimality of $\dim_k G$, since $\dim_k B/J \leq \dim_k \mathfrak{A}_H < \dim_k G$. This shows that $\mathfrak{A}_H$ is abelian and the assertion now follows from (2.3).

PROPOSITION 2.5. *Let G be simple-semiabelian and let $G = H \oplus V$ be the Fitting decomposition relative to a maximal subalgebra H. Suppose V is $\mathfrak{A}_H$-irreducible and consider $\mathfrak{A}_V := \{D|_V;\ D \in \mathfrak{A}_H\}$, as well as $A := \operatorname{alg}_k(\operatorname{ad}_h|_V;\ h \in H)$ and $B := \operatorname{alg}_k(\mathfrak{A}_V)$, the associative k-algebras generated by $\{\operatorname{ad}_h|_V;\ h \in H\}$ and $\mathfrak{A}_V$, respectively. Then the following statements hold*:

(1) *$A \subset Z(B)$ is a field ($Z(B)$ denotes the center)*

(2) *V is H-irreducible if and only if A is equal to B*

Proof. (1) By the definition of $\mathfrak{A}_H$, we have $D \circ \operatorname{ad}_h = \operatorname{ad}_h \circ D$ $\forall D \in \mathfrak{A}_H$, $\forall h \in H$. $\mathfrak{A}_V$ is therefore contained in the centralizer $\operatorname{Cen}_B(\operatorname{ad}_h|_V)$ $\forall h \in H$. Consequently, $B \subset \operatorname{Cen}_B(\operatorname{ad}_h|_V)$ $\forall h \in H$, proving that $\mathrm{ad}_h|_V$ lies centrally in B. This gives rise to $A \subset Z(B)$. The Lie-algebras $\mathfrak{A}_H$ and $\mathfrak{A}_V$ are canonically isomorphic and V is, by virtue of our assumption, an irreducible $\mathfrak{A}_V$-module. Hence B is a primitive, finite-dimensional k-algebra and, by general theory, therefore, simple. This in turn implies $Z(B)$ is a field and so is A as a finite-dimensional integral k-algebra.

(2) If A is equal to B, then V is obviously H-irreducible. Suppose, conversely, that V is H-irreducible. Then V is A-irreducible and we infer from (1) that $\dim_A V = 1$. Since A lies centrally in B, B is a subalgebra of $\mathrm{End}_A(V)$. We therefore obtain $1 = \dim_A \mathrm{End}_A(V) \geq \dim_A B$. Thus $A = B$.

Consider the extended Lie-algebra G' and, for a subalgebra $H \subset G$, the associated subalgebra H'. We define $\mathfrak{A}_{H'} := \{D \in \mathrm{Der}_K(G');\ D(H') = 0\}$.

LEMMA 2.6. *Let G be simple-semiabelian and $H \subset G$ maximal subalgebra. Consider the Cartan decomposition $G' = H' \oplus \bigoplus_{\alpha \in R} G'_\alpha$. Then the following statements hold*:

(1) *There is a Lie-algebra isomorphism t: $\mathfrak{A}_H \otimes_k K \to \mathfrak{A}_{H'}$ such that $t(D \otimes \alpha) = D \otimes \alpha\,\mathrm{id}_K$.*

(2) $D(G'_\alpha) \subset G'_\alpha\ \forall\, \alpha \in R, \forall\, D \in \mathfrak{A}_{H'}$.

(3) $D|_{G'_\alpha} = 0 \Rightarrow D|_{G'_{-\alpha}} = 0$.

Proof. (1) By general theory there is an isomorphism of associative algebras t: $\mathrm{End}_k(G) \otimes_k K \to \mathrm{End}_K(G \otimes_k K)$, such that $t(f \otimes \alpha) = f \otimes \alpha\,\mathrm{id}_K$. It is easy to check that $t(\mathrm{Der}_k(G) \otimes_k K) = \mathrm{Der}_K(G')$ and $t(\mathfrak{A}_H \otimes_k K) = \mathfrak{A}_{H'}$.

(2) Let D be an element of $\mathfrak{A}_H$ and $x \in G'_\alpha$. Then

$$\begin{aligned}[h, D \otimes \mathrm{id}_K(x)] &= D \otimes \mathrm{id}_K([h, x]) - [D \otimes \mathrm{id}_K(h), x] \\ &= \alpha(h) D \otimes \mathrm{id}_K(x)\end{aligned}$$

by (1.2) $\forall\, h \in H'$. This proves $D \otimes \mathrm{id}_k(G'_\alpha) \subset G'_\alpha\ \forall\, D \in \mathfrak{A}_H$. Applying (1) we obtain the desired result.

(3) According to (4.3) of [3] there is $f \in G^*$ such that $H = \mathrm{rad}(B_f)$. Let $f' \in G^*$ be the extended linear form and suppose $D|_{G'_\alpha} = 0$. Let x be an element of $G'_{-\alpha}$; then

$$0 = D([x, y]) = [x, D(y)] + [D(x), y] = [D(x), y]$$

for every $y \in G'_\alpha$. This yields, in particular, $D(x) \in G'_{-\alpha} \cap G'^{\perp}_\alpha$. Applying (1.4) we find that $D(x) = 0$.

PROPOSITION 2.7. *Let $G = H \oplus V$ be the Fitting decomposition of a simple-semiabelian Lie-algebra relative to a maximal subalgebra H. Then the following statements hold*:

(1) *If V is not $\mathfrak{A}_H$-irreducible, then* $\dim_k \mathfrak{A}_H \leq \frac{1}{2} \dim_k G/H$.

(2) *If $\mathfrak{A}_H$ is abelian, then* $\dim_k \mathfrak{A}_H \leq \frac{1}{2} \dim_k G/H$.

(3) *If* $\dim_K G'_\alpha = 1\ \forall \alpha \in R$ *then $\mathfrak{A}_H$ is abelian.*

Proof. (1) By assumption there exists an $\mathfrak{A}_H$-irreducible submodule $W \subset V$. Let $H = \operatorname{rad}(B_f)$ for an appropriate $f \in G^*$. It is easy to verify that $V \cap W^\perp$ is an $\mathfrak{A}_H$-submodule of V. Consequently we have $W \cap W^\perp = 0$ or $W \subset V \cap W^\perp$. In the second case W is a totally isotropic subspace of V and therefore its dimension is bounded by $\frac{1}{2} \dim_k V$. If $W \cap W^\perp = 0$ then $\dim_k W \leq \frac{1}{2} \dim_k V$ or $\dim_k W^\perp \leq \frac{1}{2} \dim_k V$. In either case there exists an $\mathfrak{A}_H$-submodule $U \subset V$ such that $\dim_k U \leq \frac{1}{2} \dim_k V$. For $u \in U \backslash \{0\}$ consider the injective linear map:

$$S_u: \begin{cases} \mathfrak{A}_H \to U \\ D \mapsto D(u). \end{cases}$$

We obtain $\dim_k \mathfrak{A}_H \leq \dim_k U \leq \frac{1}{2} \dim_k V$.

(2) By virtue of (1) we only have to consider the case where V is $\mathfrak{A}_H$-irreducible. It is a result of [2] that this yields $\dim_K G'_\alpha = 1\ \forall \alpha \in R$ and $\operatorname{char}(k) \neq 2$. Let $R = \{\alpha_1, \ldots, \alpha_n, -\alpha_1, \ldots, \alpha_n\}$ and write $G'_{\alpha_i} = Kx_i$, $1 \leq i \leq n$. According to (2.6) the linear map is injective and therefore we obtain:

$$g: \begin{cases} \mathfrak{A}_{H'} \to G' \\ D \mapsto \sum_{i=1}^{n} D(x_i) \end{cases}$$

observing (2.6)

$$\dim_k \mathfrak{A}_H = \dim_K \mathfrak{A}_{H'} \leq n = \tfrac{1}{2} \dim_k G/H.$$

(3) This is an immediate consequence of (2.6).

We finally use the results established above in order to estimate the dimension of maximal subalgebras.

PROPOSITION 2.8. *Let G be simple semiabelian and let $H \subset G$ be a maximal subalgebra. Then* $\dim_k H \leq \frac{1}{3} \dim_k G$.

REMARK. It can be shown (cf. [2]) that $\dim_k H < \frac{1}{3} \dim_k G$ for finite k.

Proof. Let $G = H \oplus V$ be the Fitting decomposition relative to H. V is completely reducible and decomposes into a direct sum of irreducible submodules $V = V_1 \oplus \cdots \oplus V_n$. Since for $v \in V_i$ the map $S_v\colon H \to V_i$, where $S_v(h) = [h, v]$ is injective, we obtain $\dim_k H \leq \dim_k V_i$, which yields the desired result in case $n \geq 2$. If V is irreducible then by applying (2.5) and (2.7) consecutively we obtain $2 \cdot \dim_k H \leq \dim_k V$. This yields $\dim_k H \leq \frac{1}{3} \dim_k G$.

3. Simple-semiabelian Lie-algebras having a non-singular invariant bilinear form. In this section we assume k to be perfect of positive characteristic $p > 3$. All the results stated in the sequel hold in the non-modular case as well, however they are even stronger and well known so that we dispense with stating them explicitly.

Let K be an algebraic closure of k and let G be a finite-dimensional Lie-algebra over k. We assume G to carry a non-singular invariant symmetric bilinear form $f\colon G \times G \to k$. On G', consider the extended form $f'\colon G' \times G' \to K$, which is non-singular.

THEOREM 3.1. *Let G be simple-semiabelian. Then any two maximal subalgebras are of the same dimension.*

Proof. Let $H \subset G$ be a maximal subalgebra. We claim that G' is classical with respect to H' (cf. [**8**] p. 28). We have to show (a) $\mathfrak{z}(G') = 0$, (b) $[G', G'] = G'$, (c) $[h, x] = \alpha(h) \cdot x \quad \forall\, h \in H', \quad \forall\, x \in G'_\alpha$, (d) $\dim_k[G'_\alpha, G'_{-\alpha}] = 1 \;\forall\, \alpha \in R$, (e) $\forall\, \alpha, \beta \in R \;\exists\, i \in \mathrm{GF}(p)\colon \alpha + i\beta \notin R$.

(a) and (b) are direct consequences of the simplicity of G. Property (c) follows from (1.2). By applying Theorems 90 and 89 of [**6**] we obtain $\dim_K G'_\alpha = 1 \quad \forall\, \alpha \in R$ which in turn yields $\dim_K[G'_\alpha, G'_{-\alpha}] \leq 1$. Since $[G'_\alpha, G'_{-\alpha}] \neq 0$, by (1.4), (d) holds. Finally (e) is a consequence of Theorems 90 and 92 of [**6**].

Now let H_1 and H_2 be two maximal subalgebras of G. According to the above H'_i is a classical Cartan subalgebra of the classical algebra G'. By virtue of Theorem III.4.1 of [**8**] we have $\dim_K H'_1 = \dim_K H'_2$ which yields the asserted result.

THEOREM 3.2. *Let G be simple-semiabelian. Let H be a maximal subalgebra of G and write $G' = H' \oplus \bigoplus_{\alpha \in R} G'_\alpha$. Then the following statements hold*: (1) $\dim_K G'_\alpha = 1 \;\forall\, \alpha \in R$.

(2) $[\mathrm{Der}_k(G), \mathrm{Der}_k(G)] = \mathrm{ad}(G)$.

(3) $\dim_k \mathrm{Der}_k(G) \leq \frac{3}{2} \dim_k G/H$.

Proof. (1) According to (1.2), G' is a V-algebra in the sense of [**6**] p. 75. Since G' is centerless we may apply Theorems 90 and 89 consecutively in order to obtain the desired result.

(2) By virtue of (2.7) $\mathfrak{A}_H$ is abelian and the assertion is a consequence of (2.3).

(3) Since $\mathfrak{A}_H$ is abelian, (2.7) applies and, combining this with (2.3), we obtain

$$\dim_k \operatorname{Der}_k(G) = \dim_k \mathfrak{A}_H + \dim_k G/H \leq \tfrac{3}{2} \dim_k G/H.$$

4. Simple-semiabelian Lie-algebras of index one. Except for the existence of a non-singular invariant form, we adopt the assumptions of the preceding section. The number $\operatorname{ind}(G) := \min_{f \in G^*} \dim_k \operatorname{rad}(B_f)$ will be called the index of G.

THEOREM 4.1. *Let G be simple-semiabelian of index 1 and let $H \subset G$ be a one-dimensional maximal subalgebra. Write $G' = H' \oplus \bigoplus_{\alpha \in R} G'_\alpha$. Then the following statements hold*:

(1) $\dim_K G'_\alpha = 1 \; \forall \, \alpha \in R$.

(2) $\dim_k G = 3$ *or there is* $n \in \mathbf{N}$ *such that* $\dim_k G = p^n$.

(3) *If* $\dim_k G \neq 3$ *then* $R \cup \{0\} = \Sigma_{\sigma \in [\alpha]} \mathrm{GF}(p) \cdot \sigma \; \forall \, \alpha \in R$.

(4) $[\operatorname{Der}_k(G), \operatorname{Der}_k(G)] = \operatorname{ad}(G)$.

Proof. (1) Let f be a linear form of G such that $H = \operatorname{rad}(B_f)$ and let f' denote the linear form of G' defined by f. Then $H' = \operatorname{rad}(B_{f'})$. Let $\alpha \in R$ be a root and suppose $x \in G'_\alpha$ has the property $[x, G'_{-\alpha}] = 0$. Then $x \in G'_\alpha \cap G'^{\perp}_{-\alpha} = 0$ (cf. (1.4)). The assertion now follows from [7] Theorem 4.

(2) Suppose $\dim_k G \neq 3$. By virtue of [7] Theorem 4, $R \cup \{0\}$ is an abelian group and, since $p \cdot \alpha = 0 \; \forall \, \alpha \in R$, it also has the structure of a $\mathrm{GF}(p)$-vector space. Let n denote the dimension of $R \cup \{0\}$ over $\mathrm{GF}(p)$. Then $R \cup \{0\}$ has p^n elements and by (1) we obtain $\dim_k G = \dim_K G' = |R \cup \{0\}| = p^n$.

(3) Let $\alpha \in R$ be a root and consider $\Delta := \Sigma_{\sigma \in [\alpha]} \mathrm{GF}(p)\sigma$. Obviously, Δ is a subspace of $R \cup \{0\}$. The equality

$$\gamma \cdot \left(\sum_{\sigma \in [\alpha]} \tau_\sigma \sigma \right) = \sum_{\sigma \in [\alpha]} \tau_\sigma (\gamma \cdot \sigma), \qquad \tau_\sigma \in \mathrm{GF}(p)$$

proves the invariance of Δ under the action of the Galois group. Consequently $\mathfrak{G}' := H' \oplus \bigoplus_{\lambda \in \Delta \setminus \{0\}} G'_\lambda$ is a subalgebra of G' having the property $(\mathrm{id}_G \otimes \gamma)(\mathfrak{G}') \subset \mathfrak{G}' \; \forall \, \gamma \in \operatorname{Gal}(K : k)$. This gives rise to a subalgebra

$\mathfrak{G} \subset G$ such that $\mathfrak{G} \otimes_k K = \mathfrak{G}'$. Since $\mathfrak{G}'$ is not abelian we must have $\mathfrak{G} = G$ and $\mathfrak{G}' = G'$. Hence $\Delta = R \cup \{0\}$.

(4) This is a direct consequence of (1), (2.7) and (2.3).

REMARK. Using Theorem 88 of [**6**] one can show that every simple-semiabelian Lie-algebra of index 1 possessing a non-singular invariant symmetric form is three dimensional.

Let W be an irreducible H-submodule of V. Since $\dim_K G'_\alpha = 1$ $\forall \alpha \in R$ we have $W \otimes_k K = \sum_{\gamma \in \mathrm{Gal}(K:k)} G'_{\gamma \cdot \alpha}$ for an appropriate $\alpha \in R$ (cf. (1.3)). By applying (2.6) it is now clear that $D(W) \subset W\, \forall D \in \mathfrak{A}_H$.

The following proposition illustrates the scarcity of simple-semiabelian Lie-algebras of low dimensions. Let $W(1)$ denote the Witt-algebra.

PROPOSITION 4.2. *Let G be simple-semiabelian. Then*:

(1) $\dim_k G \neq 4$.

(2) $\dim_k G = 5 \Rightarrow \mathrm{char}(k) = 5$ *and* $G' \simeq W(1)$.

(3) $\dim_k G = 6 \Rightarrow G' \simeq \mathrm{sl}(2) \oplus \mathrm{sl}(2)$.

(4) $\dim_k G = 7 \Rightarrow \mathrm{char}(k) = 7$ *and* $G' \simeq W(1)$.

Proof. (1) Suppose $\dim_k G = 4$ and let $H \subset G$ be a maximal subalgebra. Since $\dim_k G/H$ is even ((4.3) [**3**]) we necessarily have $\dim_k H = 2$, contradicting (2.8).

(2) By virtue of (2.8) every maximal subalgebra of G is one dimensional. The equation $5 = \dim_k G = p^n$ yields $n = 1$ and $p = 5$. Consequently $R \cup \{0\} = \mathrm{GF}(p)\alpha$ $\forall \alpha \in R$ and the assertion follows from Theorem 2 of [**7**].

(3) If $H \subset G$ is a maximal subalgebra, its dimension is not greater than 2. As in (1) we find that H has dimension 2. It is a result of [**2**] that G' then decomposes into a direct sum of copies of sl(2).

(4) Analogous to (2).

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