CONTINUITY AND CONVEXITY OF PROJECTIONS AND BARYCENTRIC COORDINATES IN CONVEX POLYHEDRA

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If $s_0, \cdots, s_n$ are linearly independent points of real n-dimensional Euclidean space R^n then each point x of their convex hull S has a (unique) representation $x = \sum_{i=0}^{n} \lambda_i(x) s_i$ with $\lambda_i(x) \geqq 0$ $(i = 0, \cdots, n)$ and $\sum_{i=0}^{n} \lambda_i(x) = 1$, and the barycentric coordinates $\lambda_0, \cdots, \lambda_n$ are continuous convex functions on S (cf. [3, p. 288]). We shall show in this paper that given any finite set $s_0, \cdots, s_m$ of points of R^n we can assign barycentric coordinates $\lambda_0, \cdots, \lambda_m$ to their convex hull S in such a way that each coordinate is continuous on S and that one prescribed coordinate (λ_0 say) is convex on S (Theorem 2); the author does not know whether it is always possible to make all the coordinates convex simultaneously (cf. Example 3). In proving Theorem 2 we shall use certain "projections" which we now define; these projections are in general distinct from those of [1, p. 614] and [2, p. 12]. Given two distinct points s_0 and s of R^n, let $s_0 s$ be the open half-line consisting of all points $s_0 + \lambda(s - s_0)$ with $\lambda > 0$; given a point s_0 of R^n and a closed subset S of R^n such that $s_0 \notin S$, let $C(s_0, S)$ be the "cone" formed by the union of all open half-lines $s_0 s$ with s in S; and given a point x in such a cone $C(s_0, S)$, let $\pi(x)$ be the (unique) point of $s_0 x \cap S$ which is closest to s_0. Then we shall call the function π the "projection of $C(s_0, S)$ on S." Our proof of Theorem 2 depends on the fact that if S is a convex polyhedron then π is continuous (Theorem 1). This result may appear to be obvious, but it is not immediately obvious how a formal proof should be given; moreover, as we shall show (Examples 1 and 2), the conclusion need not remain true for polyhedra S which are not convex or for convex sets S which are not polyhedra. The author is indebted to the referee for improvements to Lemma 3, Example 1, and Example 2, and for the remark at the end of § 1.

1. Projections. For any subset A of R^n we shall denote by $H(A)$ the convex hull of A and by $L(A)$ the affine subspace of R^n spanned by A (cf. [2, pp. 21, 15]). If $A = \{s_1, \cdots, s_p\}$ we shall write $H(A) = H(s_1, \cdots, s_p)$ and $L(A) = L(s_1, \cdots, s_p)$. Given two points x and y of R^n we shall denote by (x, y) the inner product of x and y and by $|x - y|$ the Euclidean distance $\sqrt{(x - y, x - y)}$ between x and y.

LEMMA 1. *Let s_0 be a point of R^n, let S be a closed convex subset of R^n such that $s_0 \notin S$, and let π be the projection of $C(s_0, S)$ on S. Suppose that points $x, s_1, \cdots, s_p$ of S and real numbers $\lambda_1, \cdots, \lambda_p$ are*

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such that $x = \sum_{i=1}^{p} \lambda_i s_i$, $\lambda_i > 0$ $(i = 1, \cdots, p)$, $\sum_{i=1}^{p} \lambda_i = 1$, *and* $\pi(x) = x$. *Then*

(i) $\pi(y) = y$ *for all* y *in* $H(s_1, \cdots, s_p)$; *and*

(ii) $s_0 \notin L(s_1, \cdots, s_p)$.

Proof. (i) Given y in $H(s_1, \cdots, s_p)$ we can find nonnegative real numbers $\mu_1, \cdots, \mu_p$ such that $y = \sum_{i=1}^{p} \mu_i s_i$ and $\sum_{i=1}^{p} \mu_i = 1$. Since each $\lambda_i > 0$, there exists α with $0 < \alpha < 1$ such that $\lambda_i - \alpha\mu_i > 0$ for each $i = 1, \cdots, p$. Let

$$z = \frac{x}{1-\alpha} - \frac{\alpha y}{1-\alpha} = \sum_{i=1}^{p} \left(\frac{\lambda_i - \alpha\mu_i}{1-\alpha} \right) s_i ;$$

then $z \in H(s_1, \cdots, s_p) \subseteqq S$ and $x = \alpha y + (1-\alpha)z$. We now use an indirect argument. Suppose that $\pi(y) \neq y$; then for some β with $0 < \beta < 1$ we have $\pi(y) = (1-\beta)s_0 + \beta y$ and

$$\text{(1)} \qquad \frac{\alpha(1-\beta)s_0 + \beta x}{\alpha(1-\beta) + \beta} = \frac{\alpha\pi(y) + \beta(1-\alpha)z}{\alpha + \beta(1-\alpha)} = x'$$

say. It follows from (1) that $x' \in s_0x \cap S$ and that $|s_0 - x'| < |s_0 - x|$, contradicting the hypothesis that $\pi(x) = x$. This completes the proof of (i).

(ii) Suppose that $s_0 \in L(s_1, \cdots, s_p)$. Then we can find real numbers $\nu_1, \cdots, \nu_p$ such that $s_0 = \sum_{i=1}^{p} \nu_i s_i$ and $\sum_{i=1}^{p} \nu_i = 1$. Since each $\lambda_i > 0$, there exists γ with $0 < \gamma < 1$ such that $\lambda_i - \gamma(\lambda_i - \nu_i) > 0$ for each $i = 1, \cdots, p$. But then if

$$w = \gamma s_0 + (1-\gamma)x = \sum_{i=1}^{p} [\lambda_i - \gamma(\lambda_i - \nu_i)]s_i$$

we have $w \in s_0x \cap S$ and $|s_0 - w| < |s_0 - x|$, contradicting the hypothesis that $\pi(x) = x$. This completes the proof of (ii).

Let s_0, S, and π be as in Lemma 1. Then we shall call a subset A of S "π-admissible" if $\pi(x) = x$ for all x in $H(A)$.

LEMMA 2. *Let* s_0, S, *and* π *be as in Lemma* 1, *let* A *be a finite* π*-admissible subset of* S, *and let* π' *be the projection of* $C(s_0, H(A))$ *on* $H(A)$. *Then*

(i) $\pi(x) = \pi'(x)$ *for all* x *in* $C(s_0, H(A))$; *and*

(ii) π' *is a continuous mapping of* $C(s_0, H(A))$ *into* $H(A)$.

Proof. (i) Let x be any point of $C(s_0, H(A))$. Then $\pi(\pi'(x)) = \pi'(x)$ since A is π-admissible, hence $\pi'(x)$ is the point of $s_0\pi'(x) \cap S = s_0x \cap S$ which is closest to s_0, and hence $\pi(x) = \pi'(x)$.

(ii) Let $A = \{s_1, \cdots, s_p\}$ and let $x_0 = \sum_{i=1}^{p} (1/p)s_i$; then $\pi(x_0) = x_0$

since A is π-admissible, and hence $s_0 \notin L(A)$ by Lemma 1. It follows that if s_* is the point of $L(A)$ which is closest to s_0, and x is any point of $C(s_0, H(A))$, then

$$\pi'(x) = \frac{x - \lambda(x)s_0}{1 - \lambda(x)}, \quad \text{where} \quad \lambda(x) = \frac{(x - s_*, s_0 - s_*)}{|s_0 - s_*|^2}.$$

Hence π' is continuous.

LEMMA 3. *Let s_0 be a point of R^n and let T be a closed bounded subset of R^n such that $s_0 \notin T$. Then $\{s_0\} \cup C(s_0, T)$ is a closed subset of R^n.*

With the help of the Bolzano-Weierstrass theorem it is not difficult to prove Lemma 3.

THEOREM 1. *Let $s_0, s_1, \cdots, s_m$ be points of R^n such that $s_0 \notin H(s_1, \cdots, s_m) = S$ say, and let π be the projection of $C(s_0, S)$ on S. Then π is a continuous mapping of $C(s_0, S)$ into S.*

Proof. Let $A_1, \cdots, A_q$ be the subsets of $\{s_1, \cdots, s_m\}$ which are π-admissible subsets of S. Then each x in $C(s_0, S)$ belongs to at least one $C(s_0, H(A_j))$ $(1 \leqq j \leqq q)$; indeed, given x in $C(s_0, S)$, there exist positive integers $x(1), \cdots, x(p)$ and positive real numbers $\lambda_1, \cdots, \lambda_p$ such that $\pi(x) = \sum_{i=1}^{p} \lambda_i s_{x(i)}$ and $\sum_{i=1}^{p} \lambda_i = 1$, and then $A = \{s_{x(1)}, \cdots, s_{x(p)}\}$ is π-admissible by Lemma 1 (i), and $x \in C(s_0, H(A))$. For each $j = 1, \cdots, q$ let π_j be the projection of $C(s_0, H(A_j))$ on $H(A_j)$.

To prove the theorem it will be enough to show that, if $x, x_1, x_2, \cdots$ in $C(s_0, S)$ are such that $x = \lim_k x_k$, then it follows that $\pi(x) = \lim_k \pi(x_k)$. Let J be the set of all j $(1 \leqq j \leqq q)$ such that $x_k \in C(s_0, H(A_j))$ for infinitely many values of k, and for each j in J let $j(1) < j(2) < \cdots$ be the values of k such that $x_k \in C(s_0, H(A_j))$. Now, for each j in J, $x \in C(s_0, H(A_j))$ by Lemma 3, and hence, by Lemma 2, $\pi(x) = \pi_j(x) = \lim_l \pi_j(x_{j(l)}) = \lim_l \pi(x_{j(l)})$. Since all but a finite number of the positive integers are of the form $j(l)$ for some j in J and some $l = 1, 2, \cdots$, it follows that $\pi(x) = \lim_k \pi(x_k)$, as we wished to prove.

The following example shows that if S is a non-convex polyhedron in R^2, and $s_0 \notin S$, then the projection of $C(s_0, S)$ on S need not be continuous.

EXAMPLE 1. Let $s_0 = (0, 2)$, $s_1 = (0, 1)$, $s_2 = (0, 0)$, and $s_3 = (1, 0)$; and let $S = H(s_1, s_2) \cup H(s_2, s_3)$. Then the projection of $C(s_0, S)$ on S is not continuous at s_1.

The following example shows that if S is a closed convex set in R^3, and $s_0 \notin S$, then the projection of $C(s_0, S)$ on S need not be continuous.

EXAMPLE 2. Let $s_0 = (0, 0, 2)$, let $s_1 = (0, 0, 1)$, let K be the circle consisting of all points (ξ, η, ζ) in R^3 such that $(\xi - 1)^2 + \eta^2 = 1$ and $\zeta = 0$, let $S = H(\{s_1\} \cup K)$, and let π be the projection of $C(s_0, S)$ on S. Then if we set $x_k = (1 - \cos k^{-1}, \sin k^{-1}, 0)$ $(k = 1, 2, \cdots)$ we have $x_k \in C(s_0, S)$ and $\pi(x_k) = x_k$ $(k = 1, 2, \cdots)$. When $k \to \infty$, $x_k \to (0, 0, 0) = s_2$ say, and $\pi(x_k) \to s_2$; since $\pi(s_2) = s_1$, this shows that π is not continuous at s_2.

REMARK. Theorem 1 is valid for each closed convex set $S \subseteqq R^2$, and for each strictly convex closed set $S \subseteqq R^n$.

2. Barycentric coordinates.

Let s_0 be a point of R^n, let S be a closed convex subset of R^n such that $s_0 \notin S$, and let $D(s_0, S)$ be the union of all segments $H(s_0, s)$ joining s_0 to points s of S; then $D(s_0, S)$ is a convex set. Define a real-valued function λ_0 on $D(s_0, S)$ as follows: let $\lambda_0(s_0) = 1$, let $\lambda_0(x) = 0$ if $x \in S$, and if $x \neq s_0$ and $x \notin S$ let $\lambda_0(x)$ be defined by the equation $x = \lambda_0(x)s_0 + [1 - \lambda_0(x)]\pi(x)$, where π is the projection of $C(s_0, S)$ on S; then each x in $D(s_0, S)$ has a representation of the form

$$x = \lambda_0(x)s_0 + [1 - \lambda_0(x)]s \, , \tag{2}$$

with s in S. We shall call λ_0 the "barycentric function of $D(s_0, S)$."

LEMMA 4. *Let s_0 be a point of R^n, let S be a closed convex subset of R^n such that $s_0 \notin S$, and let λ_0 be the barycentric function of $D(s_0, S)$. Then $0 \leqq \lambda_0(x) \leqq 1$ for all x in $D(s_0, S)$ and λ_0 is a convex function on $D(s_0, S)$. If S is a polyhedron then λ_0 is continuous on $D(s_0, S)$.*

Proof. It is clear that $\lambda_0(x) \leqq 1$ for all x in $D(s_0, S)$; the proof that $\lambda_0(x) \geqq 0$ for all x in $D(s_0, S)$ depends on the convexity of S, and will be left to the reader. To prove that λ_0 is convex on $D(s_0, S)$ we show that if $x, x' \in D(s_0, S)$ and $0 < \alpha < 1$ then

$$\lambda_0(\alpha x + (1 - \alpha)x') \leqq \alpha\lambda_0(x) + (1 - \alpha)\lambda_0(x') \, . \tag{3}$$

Let $x^* = \alpha x + (1 - \alpha)x'$ and let $\beta = \alpha\lambda_0(x) + (1 - \alpha)\lambda_0(x')$; we may assume that $\beta < 1$ since otherwise (3) is trivial. Then if $\gamma = \alpha[1 - \lambda_0(x)](1 - \beta)^{-1}$, and s, s' in S are such that

$$x = \lambda_0(x)s_0 + [1 - \lambda_0(x)]s \, , \quad x' = \lambda_0(x')s_0 + [1 - \lambda_0(x')]s'$$

(cf. (2)), we have

$$\gamma s + (1 - \gamma)s' = -\beta(1 - \beta)^{-1}s_0 + (1 - \beta)^{-1}x^* \, , \tag{4}$$

and $\gamma s + (1 - \gamma)s' \in S$ since S is convex. It follows from (4) that $x^* \neq s_0$.

If $x^* \notin S$ and π is the projection of $C(s_0, S)$ on S then

$$\pi(x^*) = -\lambda_0(x^*)[1 - \lambda_0(x^*)]^{-1}s_0 + [1 - \lambda_0(x^*)]^{-1}x^* ,$$

and hence by (4) and the definition of π, $\lambda_0(x^*) \leqq \beta$, as asserted by (3). If $x^* \in S$ then (3) is trivial. This completes the proof that λ_0 is convex on $D(s_0, S)$.

We next show that λ_0 is continuous at s_0. Given ε with $0 < \varepsilon < 1$, let $\delta = M\varepsilon$, where $M > 0$ is the shortest distance from s_0 to S. Then if $x \in D(s_0, S)$ and $0 < |x - s_0| < \delta$ we have $x \neq s_0$, $x \notin S$, and

$$M \leqq |\pi(x) - s_0| = [1 - \lambda_0(x)]^{-1} |x - s_0| < [1 - \lambda_0(x)]^{-1}M\varepsilon ,$$

and hence $0 < 1 - \lambda_0(x) < \varepsilon$. This proves that λ_0 is continuous at s_0. It remains to prove that λ_0 is continuous on $D(s_0, S) - \{s_0\}$ if S is a polyhedron. For each x in $C(s_0, S)$ define $\mu_0(x)$ by the equation $x = \mu_0(x)s_0 + [1 - \mu_0(x)]\pi(x)$; then

$$(5) \qquad \mu_0(x) = 1 - |x - s_0|/|\pi(x) - s_0| .$$

It follows that $\mu_0(x) \leqq 0$ if $x \in S$, and that $\mu_0(x) = \lambda_0(x) > 0$ if $x \in D(s_0, S)$, $x \neq s_0$, and $x \notin S$; thus

$$(6) \qquad \lambda_0(x) = \max[\mu_0(x), 0] \quad (x \in D(s_0, S),\ x \neq s_0) .$$

If S is a polyhedron then μ_0 is continuous on $C(s_0, S)$ by Theorem 1 and (5), and hence λ_0 is continuous on $D(s_0, S) - \{s_0\}$ by (6). This completes the proof of the lemma.

THEOREM 2. *Let $s_0, \cdots, s_m$ be points of R^n, and let $S = H(s_0, \cdots, s_m)$. Then there exist nonnegative real-valued continuous functions $\lambda_0, \cdots, \lambda_m$ on S, with λ_0 a convex function, such that, for each x in S,*

$$x = \sum_{i=0}^{m} \lambda_i(x)s_i , \quad \text{and} \quad \sum_{i=0}^{m} \lambda_i(x) = 1 .$$

Proof. We use induction on m. The case $m = 0$ is trivial. We assume the theorem to have been proved for $m = M - 1$ and deduce it for $m = M$. Let $T = H(s_1, \cdots, s_M)$. If $s_0 \in T$ we may set $\lambda_0(x) = 0$ for all x in S, and deduce the existence of $\lambda_1, \cdots, \lambda_M$ directly from the induction hypothesis; we therefore assume that $s_0 \notin T$. By the induction hypothesis there exist nonnegative real-valued continuous functions $\mu_1, \cdots, \mu_M$ on T such that, for each y in T, $y = \sum_{i=1}^{M} \mu_i(y)s_i$, and $\sum_{i=1}^{M} \mu_i(y) = 1$. Let λ_0 be the barycentric function of $D(s_0, T)$. Then each x in $S = D(s_0, T)$ has a representation of the form $x = \lambda_0(x)s_0 + [1 - \lambda_0(x)]s_x$ with s_x in T (cf. (2)), and if we now set $\lambda_i(x) = \mu_i(s_x)[1 - \lambda_0(x)]$ $(x \in S;\ i = 1, \cdots, M)$ then it follows that the λ_i $(i = 1, \cdots, M)$ are well-defined functions on S, and, by Lemma 4, that the functions $\lambda_0, \cdots, \lambda_M$ satisfy all the conditions in the statement of the theorem.

To show that the functions λ_i defined in the proof of Theorem 2 need not all be convex we can let s_0, s_1, s_2, and s_3 be the points $(0, 2)$, $(1, 0)$, $(-1, 0)$, and $(0, 1)$ respectively of R^2 and let $S = H(s_0, s_1, s_2, s_3)$; however in this example we obtain convex barycentric coordinates if we interchange the roles of s_0 and s_3. In the following example some of the barycentric coordinates determined as in the proof of Theorem 2 fail to be convex no matter how s_0 is chosen.

EXAMPLE 3. Define $t_0, \cdots, t_4$ in R^3 as follows: $t_0 = (0, 0, 1)$, $t_1 = (0, 1, 0)$, $t_2 = (0, -1, 0)$, $t_3 = (1, 0, -1)$, and $t_4 = (-1, 0, -1)$; let $S = H(t_0, \cdots, t_4)$; and let barycentric coordinates be defined for S as in the proof of Theorem 2, with

(i) t_0,

(ii) t_1 or t_2, and

(iii) t_3 or t_4 playing the role of s_0. Then if we write $\theta_\pm$ for $\max[\pm\theta, 0]$ (θ real) we obtain

(i) $(\xi, 0, 0) = |\xi| t_0 + (\frac{1}{2} - |\xi|)(t_1 + t_2) + \xi_+ t_3 + \xi_- t_4 \qquad (|\xi| \leqq \frac{1}{2})$,

(ii) $(0, \eta, 0) = \frac{1}{2}(1 - |\eta|)t_0 + \eta_+ t_1 + \eta_- t_2 + \frac{1}{4}(1 - |\eta|)(t_3 + t_4) \qquad (|\eta| \leqq 1)$, and

(iii) $(0, 0, \zeta) = \zeta_+ t_0 + \frac{1}{2}(1 - |\zeta|)(t_1 + t_2) + \frac{1}{2}\zeta_-(t_3 + t_4) \qquad (|\zeta| \leqq 1)$,

respectively, and hence in no case are the barycentric coordinates all convex.

The argument in the proof of Theorem 2 amounts to determining barycentric coordinates $\lambda_0, \cdots, \lambda_m$ for $H(s_0, \cdots, s_m)$ by first choosing λ_0 as small as possible, then choosing λ_1 as small as possible with this choice of λ_0, etc. We remark in conclusion that if we first choose λ_0 as large as possible, then choose λ_1 as large as possible with this choice of λ_0, etc., we do not in general obtain convex barycentric coordinates; this may be seen by considering the case of a square in R^2.

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