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EXTREME POINTS IN THE HAHN-BANACH-KANTOROVIČ SETTING

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This paper presents an existence and characterization theorem for the extreme points of the convex set of all extensions of a linear operator from a real vector space into an order complete real vector lattice which are dominated by a sublinear operator. This result is applied to positive extensions, contractions, and dominated invariant extensions.

The paper falls into four sections.

Section 1 is reserved for preliminaries.

In §2 we consider the convex set of all extensions of a linear operator defined on a vector subspace of a real vector space X with values in an order complete real vector lattice Y which are dominated by a sublinear operator P from X into Y. We present a characterization of the extreme points of this set being useful for applications. This part is related to papers of Kutateladze [7], [8] and Portenier [15].

In §3 we give two applications of the preceding result. The first one yields another proof of an existence and characterization theorem due to Lipecki [10], [11] concerning extreme positive extensions of a linear operator which is defined on a subspace of an ordered vector space. The second one yields a new characterization theorem for extreme contractions from a separable Banach space into the space of real valued continuous functions on a compact extremally disconnected space.

In §4 the results of §2 are extended to P-dominated extensions which are positive on a given cone in X, and we apply them to P-dominated extensions which are invariant with respect to a set of mappings from X into X. Furthermore, we obtain a refinement of a dominated extension theorem for positive linear operators due to Luxemburg and Zaanen.

1. Preliminaries. We adhere to the notation of Schaefer's monograph [16]. Throughout X stands for a real vector space, M for its vector subspace and Y for an order complete real vector lattice. $P\colon X \to Y$ denotes a sublinear mapping, i.e. P is positively homogeneous and subadditive. The space of all linear operators from M into Y is denoted by $L(M, Y)$. Given a vector subspace N of X with $M \subset N$ and $T \in L(M, Y)$, we put

$$E_N(P, T) = \{S \in L(N, Y)\colon S \le P \mid N \text{ and } S \mid M = T\}.$$

The notation $E_X(P, T)$ is abbreviated to $E(P, T)$ and $E(P)$ stands for $E_X(P, T)$, provided $M = \{0\}$, i.e.

$$E(P) = \{S \in L(X, Y) \colon S \leq P\}.$$

Finally, ex $E_N(P, T)$ denotes the set of all extreme points of the convex set $E_N(P, T)$.

PROPOSITION 1.1. (*Hahn-Banach-Kantorovič Theorem*, [**6**, 2.5.7, 2.5.8]). *If $T \in L(M, Y)$ and $T \leq P \mid M$, then $E(P, T) \neq \varnothing$. In particular, given $x \in X$ and $y \in [-P(-x), P(x)]$, then there exists an operator $S \in E(P)$ such that $Sx = y$.*

Bonnice, Silverman [**3**] and To [**21**] (cf. also Ioffe [**5**]) have proved that a preordered vector space is order complete, if it has the Hahn-Banach extension property according to Proposition 1.1. Thus the order completeness of Y is indispensable in our investigation of extreme extensions.

2. Existence and characterization of extreme extensions. With $P \colon X \to Y$ sublinear and $S \in L(X, Y)$ (and the linear subspace M) we associate the map $P^S \colon X \to Y \cup \{-\infty\}$ defined by

$$P^S(x) = \inf\{(P - S)(u + z + x) + (P - S)(u - z - x) \colon z \in M, u \in X\}.$$

LEMMA 2.1. *The following conditions are equivalent.*
(i) $S \leq P$,
(ii) $0 \leq P^S \leq 2(P - S)$,
(iii) $P^S \mid M = 0$,
(iv) $P^S(0) > -\infty$,
(v) $P^S \colon X \to Y$ *is sublinear.*

The simple proof is left to the reader. (The fact that matters subsequently is that (i) implies the other statements.) Moreover, if $S \leq P$ and $S \mid M = P \mid M$, then the definition of P^S reduces to

$$P^S(x) = \inf\{(P - S)(u + x) + (P - S)(u - x) \colon u \in X\}$$

for all $x \in X$.

If $T \in L(M, Y)$ and $T \leq P \mid M$, then the existence of extreme points of $E(P, T)$ is known, see the sophisticated result of Vincent-Smith [**22**, Addendum to Theorem 1] and recent results of Kutateladze [**7**], [**8**] and Lipecki [**11**]. We shall give a proof using the characterization of extreme extensions stated in part (b) of the following theorem. This characterization turns out to be useful for applications. For $Y = \mathbf{R}$ and

$M = \{0\}$ it follows from Proposition 2.2 in Portenier [15] and for $M = \{0\}$ some other characterizations may be found in [7], [8].

THEOREM 2.2. *Let* $T \in L(M, Y)$.
(a) *If (and only if)* $T \leq P \mid M$, *then* $\operatorname{ex} E(P, T) \neq \varnothing$.
(b) *Suppose* $S \in E(P, T)$. *Then* $S \in \operatorname{ex} E(P, T)$ *if and only if* $P^S(x) = 0$ *for each* $x \in X$.

Proof. Both parts (a) and (b) are proved simultaneously. In Step 1 and Step 2 we show the existence of $S \in E(P, T)$ such that $P^S = 0$ by means of the Kuratowski-Zorn lemma; Step 3 proves the "if" part and Step 4 proves the "only if" part of (b) which completes the proof.

Step 1. By $\mathbf{M}$ we denote the class of all pairs (N, R), where N is a vector subspace of X with $M \subset N$ and R is in $L(N, Y)$ such that $R \mid M = T$, $R \leq P \mid N$ and $P^R = 0$ ($P^R(x) := \inf\{(P - R)(u + z + x) + (P - R)(u - z - x)\colon z \in M,\ u \in N\}$, $x \in N$). Let $\ll$ be an ordering in $\mathbf{M}$ defined by $(N_1, R_1) \ll (N_2, R_2)$ if and only if $N_1 \subset N_2$ and $R_2 \mid N_1 = R_1$. Given $(N, R) \in \mathbf{M}$ and $x_0 \in X$, we shall show that $(N_0, R_0) \in \mathbf{M}$ and $(N, R) \ll (N_0, R_0)$, where $N_0 = \operatorname{lin}(N \cup \{x_0\})$ and $R_0\colon N_0 \to Y$ is defined by

$$R_0(v + tx_0) = Rv + ty_0$$

$(v \in N, t \in \mathbf{R})$ with

$$y_0 = \inf\{P(v + x_0) - Rv\colon v \in N\}.$$

As easily seen we have $R_0 \in L(N_0, Y)$ and $R_0 \mid N = R$ and $R_0 \leq P \mid N_0$. It remains to show that $P^{R_0} = 0$. In view of $0 \leq P^{R_0} \mid N \leq P^R = 0$ we get

$$\begin{aligned} 0 \leq P^{R_0}(v + tx_0) &\leq P^{R_0}(v) + P^{R_0}(tx_0) = |t|\, P^{R_0}(x_0) \\ &= |t|\, P^{R_0}(w + x_0) \leq 2|t|\,(P - R_0)(w + x_0) \end{aligned}$$

for all $v, w \in N$ and $t \in \mathbf{R}$. This yields $P^{R_0} = 0$, since

$$\inf\{(P - R_0)(v + x_0)\colon v \in N\} = 0$$

by the definition of y_0.

Step 2. Let $\mathbf{M}_0 \subset \mathbf{M}$ be a chain and put $N_0 = \bigcup\{N\colon (N, R) \in \mathbf{M}_0\}$ and $R_0 \mid N = R$ for all $(N, R) \in \mathbf{M}_0$. Then (N_0, R_0) in $\mathbf{M}$ is an upper bound for $\mathbf{M}_0$, since $0 \leq P^{R_0}(x) \leq P^R(x)$ for all $x \in N$ and $(N, R) \in \mathbf{M}_0$. Thus, by the Kuratowski-Zorn lemma, $\mathbf{M}$ has a maximal element (N, R) and we obtain $N = X$ by Step 1.

Step 3. Let $S \in E(P, T)$ with $P^S = 0$ and $S_0 \in L(X, Y)$ such that $S \pm S_0 \in E(P, T)$. Then $S_0 \mid M = 0$ and $\pm S_0 \leq P - S$. Hence, for each $u, x \in X$ and $z \in M$ we have

$$\begin{aligned} 2S_0 x &= S_0(u + z + x) - S_0(u - z - x) \\ &\leq (P - S)(u + z + x) + (P - S)(u - z - x) \end{aligned}$$

which implies $2S_0 \leq P^S = 0$. Therefore, $S_0 = 0$ whence $S \in \operatorname{ex} E(P, T)$.

Step 4. Given $S \in \operatorname{ex} E(P, T)$ and $x_0 \in X$, there exists $S_0 \in L(X, Y)$ such that $S_0(x_0) = P^S(x_0)$ and $S_0 \leq P^S$ by Proposition 1.1. Thus,

$$\pm S_0 x = S_0(\pm x) \leq P^S(\pm x) = P^S(x) \leq 2(P - S)(x)$$

for all $x \in X$ which implies $S \pm \frac{1}{2}S_0 \leq P$. Moreover, $\pm S_0 z \leq P^S(z) = 0$ for all $z \in M$ whence $S \pm \frac{1}{2}S_0 \in E(P, T)$. Therefore, $P^S(x_0) = 0$.

The proof of Theorem 2.2 suggests to associate with $P\colon X \to Y$ sublinear and $T \in L(M, Y)$ the map $P_T\colon X \to Y \cup \{-\infty\}$ defined by

$$P_T(x) = \inf\{P(x + z) - Tz\colon z \in M\}.$$

LEMMA 2.3. *Suppose $T \in L(M, Y)$.*
(a) *The following conditions are equivalent.*
(i) $T \leq P \mid M$,
(ii) $T = P_T \mid M$,
(iii) $P_T(0) > -\infty$,
(iv) $P_T\colon X \to Y$ *is sublinear.*
(b) *Suppose $S \in L(X, Y)$. Then $S \in E(P, T)$ if and only if $S \leq P_T$.*

The simple proof is left to the reader. It is worth mentioning that P^S and $(P_T)^S$ coincide for each $S \in E(P, T)$. Especially, this implies

$$P^S(x) = \inf\{(P_T - S)(u + x) + (P_T - S)(u - x)\colon u \in X\}$$

for all $x \in X$. Moreover, if $T \in L(M, Y)$ and $T \leq P \mid M$, then $E(P, T) = E(P_T)$ holds by Lemma 2.3 and according to Proposition 1.1 we obtain

$$\{Sx\colon S \in E(P, T)\} = [-P_T(-x), P_T(x)]$$

for all $x \in X$. Following up these ideas, we obtain

COROLLARY 2.4. *Suppose $T \in L(M, Y)$ with $T \leq P \mid M$ and $x \in X$. Then*

$$\operatorname{ex}[-P_T(-x), P_T(x)] \subset \{Sx\colon S \in \operatorname{ex} E(P, T)\} \subset [-P_T(-x), P_T(x)].$$

Proof. We only have to show the first inclusion. Given $y \in \operatorname{ex}[-P_T(-x), P_T(x)]$, we define

$$H = \{S \in E(P,T): Sx = y\}$$

and

$$Q(u) = \sup\{Su: S \in H\}, \qquad u \in X.$$

Then $H \neq \varnothing$, $Q: X \to Y$ is sublinear, and $E(Q) = H$. By Theorem 2.2, $\operatorname{ex} H \neq \varnothing$. Moreover, H is an extreme subset of $E(P,T)$ by virtue of $y \in \operatorname{ex}[-P_T(-x), P_T(x)]$. Thus $\operatorname{ex} H \subset \operatorname{ex} E(P,T)$ which implies the assertion.

REMARK 2.5. Both inclusions are proper in general. They provide precise bounds for $\{Sx: S \in \operatorname{ex} E(P,T)\}$ as will be shown by the following examples. For this let $(\Omega, \mathcal{A}, \mu)$ be a probability space. We put $X = L_1(\mu)$, $Y = \mathbf{R}$, $P(u) = \int |u|\, d\mu$ and $x = 1_\Omega$. Then

$$E(P) = \{f \in L_\infty(\mu): |f| \leq 1_\Omega\}$$

and

$$\operatorname{ex} E(P) = \{f \in L_\infty(\mu): |f| = 1_\Omega\},$$

and we have $P(x) = 1$ and $-P(-x) = -1$.

If μ is the one-point measure δ_ω in $\omega \in \Omega$, then $\{\int f\, d\mu: f \in \operatorname{ex} E(P)\} = \{-1, 1\} \underset{\neq}{\subset} [-1, 1]$ which shows that the first inclusion turns into equality and the second inclusion is proper.

If μ is non-atomic, then $\{\int f\, d\mu: f \in \operatorname{ex} E(P)\} = [-1, 1]$. Indeed, given $|y| \leq 1$, let $y_0 = (1/2)(|y| + 1) \in [0, 1]$ and $A \in \mathcal{A}$ with $\mu(A) = y_0$. Then $f = 1_A - 1_{A^c}$ is in $\operatorname{ex} E(P)$ with $\int f\, d\mu = |y|$. Thus the first inclusion is proper and the second inclusion turns into equality.

REMARK 2.6. Obviously

$$\{-P_T(-x), P_T(x)\} \subset \operatorname{ex}[-P_T(-x), P_T(x)]$$

and this inclusion is proper in general. Indeed, let $(\Omega, \mathcal{A}, \mu)$ be a probability space, where μ is not $\{0,1\}$-valued, let $X = \mathbf{R}$, $Y = L_\infty(\mu)$, $P(u) = |u| \cdot 1_\Omega$ for all $u \in X$ and $x = 1$. Then $\{-P(-x), P(x)\} = \{-1_\Omega, 1_\Omega\} \underset{\neq}{\subset} \operatorname{ex}[-1_\Omega, 1_\Omega] = \{f \in L_\infty(\mu): |f| = 1_\Omega\}$.

Suppose now that M, N are linear subspaces of X with $M \subset N \subset X$ and $R \in E_N(P,T)$. Theorem 2.7 and the succeeding counterexamples clear up the connections between the sets $\operatorname{ex} E_N(P,T)$, $\operatorname{ex} E(P,T)$ and ex $E(P, R)$. In particular, it follows that $S \in \text{ex } E(P, T)$ implies $S \in \text{ex } E(P, S \mid N)$, whereas $S \mid N \notin \text{ex } E_N(P, T)$ in general. Conversely, $S \in \text{ex } E(P, S \mid N)$ and $S \mid N \in \text{ex } E_N(P, T)$ imply $S \in \text{ex } E(P, T)$.

THEOREM 2.7. *Let M, N be subspaces of X with $M \subset N \subset X$. Suppose $T \in L(M, Y)$ and $R \in E_N(P, T)$. Then*

(a) ex $E(P, R) \supset E(P, R) \cap \text{ex } E(P, T)$.

(b) *Suppose $R \in \text{ex } E_N(P, T)$. Then*

$$\text{ex } E(P, R) = E(P, R) \cap \text{ex } E(P, T).$$

Proof. (a) is obvious and as regards (b), $R \in \text{ex } E_N(P, T)$ implies that $E(P, R)$ is an extreme subset of $E(P, T)$ which proves the assertion.

REMARK 2.8. The inclusion in (a) is proper in general. Indeed, let $N = X = Y = \mathbf{R}$, $M = \{0\}$, $P(x) = |x|$ and $Rx = 0$ for all $x \in \mathbf{R}$. Then $\{R\} = E(P, R) = \text{ex } E(P, R)$ and ex $E(P) = \{\text{id}_{\mathbf{R}}, -\text{id}_{\mathbf{R}}\}$.

The converse in (b) does not hold as the following example shows (cf. also Singer [**19**, p. 106]). Let $X = \mathbf{R}^2$, $Y = \mathbf{R}$, $M = \{(0,0)\}$, $N = \mathbf{R} \times \{0\}$, $Rx = 0$ for all $x \in N$ and let $P((x_1, x_2)) = \max\{|x_1|, |x_2|\}$ and $\text{pr}_i((x_1, x_2)) = x_i$ for all $(x_1, x_2) \in \mathbf{R}^2$. Then ex $E(P, R) = \{\text{pr}_2, -\text{pr}_2\} \subset \text{ex } E(P) = \{\text{pr}_1, -\text{pr}_1, \text{pr}_2, -\text{pr}_2\}$, whereas $R \notin \text{ex } E_N(P) = \{\text{pr}_1 \mid N, -\text{pr}_1 \mid N\}$.

In particular, this example yields an operator $S \in \text{ex } E(P)$ such that $S \mid N \notin \text{ex } E_N(P)$; put $S = \text{pr}_2$.

3. Applications. By Theorem 2.2 we obtain immediately a result due to Lipecki concerning extreme positive extensions of an operator defined on a subspace of an ordered vector space with values in an order complete vector lattice.

A1. (Lipecki [**10**, Theorem 1], [**11**, Theorem 2 and Remark 2]). Let X be an ordered vector space with positive cone C, M a majorizing vector subspace of X and $T: M \to Y$ a positive linear operator. Then

(a) $\text{ex}\{S \in L(X, Y): S \mid M = T \text{ and } S \mid C \geq 0\} \neq \varnothing$.

(b) Suppose C is generating. Then $S \in \{S' \in L(X, Y): S' \mid M = T, S' \mid C \geq 0\}$ is an extreme point of this set if and only if

$$\inf\{Su: \pm(x + z) \leq u \in X, z \in M\} = 0$$

for each $x \in X$.

Proof. Obviously, $P: X \to Y$ defined by

$$P(x) = \inf\{Tz: x \leq z \in M\}$$

is sublinear and we have

$$E(P,T) = \{S \in L(X,Y) \colon S \mid M = T, S \mid C \geq 0\}.$$

Moreover, it is readily verified that

$$P^S(x) = 2 \cdot \inf\{Su \colon \pm(x+z) \leq u \in X, z \in M\}$$

holds for all $x \in X$ and $S \in E(P,T)$. Thus the assertion follows from Theorem 2.2.

We now apply Theorem 2.2 to prove a new result concerning extreme contractions into spaces of continuous functions. This subject was started by Blumenthal et al. [**2**].

Let X be a normed vector space and $Y = C(H)$, where H is an extremally disconnected compact space. Recall that a compact space is extremally disconnected (i.e. open subsets have an open closure) if and only if the space of continuous real valued functions is order complete (cf. [**16**, II. 7.7]). X' denotes the continuous dual of X, $U(X')$ is the unit ball of X' and $U(X, C(H))$ is the unit ball of $L(X, C(H))$. An operator $S \in U(X, C(H))$ is called almost nice, if $S' \colon H \to U(X')$ defined by $S'(h) = \delta_h \circ S$ maps a dense subset of H into $\operatorname{ex} U(X')$.

It is immediate from Theorem 2.2 (with $M = \{0\}$ and $P(x) = \|x\|$) that $x' \in U(X')$ is an extreme point of this set if and only if

$$\inf\{\|u + x\| + \|u - x\| + 2x'(u) \colon u \in X\} = 0$$

for each $x \in X$. (Incidentally, this characterization may be used to prove the well known result that $\operatorname{ex} U(C'(K)) = \{\alpha\delta_k \colon |\alpha| = 1, k \in K\}$, where K is a compact space [**4**, V.8.6]). Furthermore, let us point out that an almost nice operator S in $U(X, C(H))$ is an extreme point of this set, since S' is weak* continuous [**4**, VI.7.1]. The converse does not hold in general (cf. Remark 3.1). In addition, we note (cf. Oates [**14**, Theorem 1.2]) that $U(X, C(H))$ is the closed convex hull of its extreme points with respect to the strong (or equivalently weak) operator topology on the space of all continuous linear operators from X into $C(H)$. More general Krein-Milman type theorems may be found in Morris and Phelps [**13**] and (without proofs) in Levashov [**9**].

A2. Let X be a separable normed space and let H be an extremally disconnected compact space. Then $S \in U(X, C(H))$ is an extreme point of this set if and only if S is almost nice.

Proof. We only have to show the "only if" part. Suppose $S \in \operatorname{ex} U(X, C(H))$. By Theorem 2.2 (with $M = \{0\}$ and $P(x) = \|x\| 1_H$), we have

$$\inf\{\|u + x\| 1_H + \|u - x\| 1_H - 2Su \colon u \in X\} = 0$$

for each $x \in X$. Let $\varphi \colon X \times H \to \mathbf{R}$ be defined by

$$\varphi(x, h) = \inf\{\|u + x\| + \|u - x\| - 2\delta_h(Su) \colon u \in X\}.$$

The set $\{\varphi(x, \cdot) > 0\} \subset H$ is meager for each $x \in X$, since the lattice infimum and the point infimum in $C(H)$ differ on a meager subset (cf. Stone [**20**]). If A is a countable dense subset of X, then

$$K = \bigcap_{x \in A} \{\varphi(x, \cdot) = 0\}$$

is a dense subset of H by Baire's category theorem. Since $\varphi(\cdot, h)$ is continuous for all $h \in H$, we obtain $\varphi(\cdot, h) = 0$ for all $h \in K$. Thus by Theorem 2.2 T is almost nice.

Without proof we note the following slight generalization of A2.

A3. Let X be a separable normed space, M a vector subspace of X and $T \in U(M, C(H))$, where H is an extremally disconnected compact space. Then

$$S \in \{R \in U(X, C(H)) \colon R \mid M = T\}$$

is an extreme point of this set if and only if

$$S'(h) \in \operatorname{ex}\{x' \in U(X') \colon x' \mid M = T'(h)\}$$

for all h in some dense subset of H.

REMARK 3.1. A2 and A3 fail for non-separable normed spaces X. Indeed, let H be a compact extremally disconnected space such that the set H_0 of isolated points of H is not dense (e.g. if $(\Omega, \mathcal{Q}, \mu)$ is a positive σ-finite non-atomic measure space and $\mathfrak{N}$ the ideal of μ-null sets, then the Stone representation space of $\mathcal{Q}/\mathfrak{N}$ is extremally disconnected and has no isolated points [**18**, p. 28 and p. 86]). By virtue of a result due to Blumenthal et al. [**2**, Theorem 2] there is a (non-separable) Banach space X and an operator $S \in \operatorname{ex} U(X, C(H))$ such that $\{h \in H \colon S'(h) \in \operatorname{ex} U(X')\} = H_0$. Hence S is not almost nice.

Nevertheless, the separability of X can be removed if $X = C(K)$ for some compact space K. More generally, the separability assumption in A2 can be replaced by the assumption that $\operatorname{ex} U(X')$ is weak* closed. This result is due to Sharir [**17**] and it may also be proved by an application of Theorem 2.2 and Theorem 2.7.

4. Generalizations. In this section the preceding results are generalized to P-dominated extensions, which are positive on a (pointed convex) cone $C \subset X$. With $P\colon X \to Y$ sublinear and a cone $C \subset X$ we associate the map $P_C\colon X \to Y \cup \{-\infty\}$ defined by

$$P_C(x) = \inf\{P(x+u)\colon u \in C\}$$

and, given $T \in L(M, Y)$, we put

$$E(P, T, C) = \{S \in E(P, T)\colon S \mid C \geq 0\}.$$

LEMMA 4.1. *Let C be a cone in X.*

(a) *The following conditions are equivalent.*

(i) $P \mid C \geq 0$,

(ii) $P_C(0) > -\infty$,

(iii) $P_C\colon X \to Y$ *is sublinear.*

(b) *Suppose $S \in L(X, Y)$. Then $S \leq P$ and $S \mid C \geq 0$ if and only if $S \leq P_C$.*

The simple proof will be omitted (for $Y = \mathbf{R}$ compare Anger and Lembcke [**1**, Lemma 1.9, Lemma 3.2]). In view of the preceding lemma the following corollary is merely a restatement of Theorem 2.2.

COROLLARY 4.2. *Let $T \in L(M, Y)$ and C be a cone in X.*

(a) *If (and only if) $T \leq P_C \mid M$, then* ex $E(P, T, C) \neq \varnothing$.

(b) *An operator $S \in E(P, T, C)$ is an extreme point of this set if and only if $(P_C)^S(x) = 0$ for each $x \in X$.*

REMARK 4.3. Lemma 2.3 yields equivalent assertions for the statement $T \leq P_C \mid M$. Furthermore, $T \leq P_C \mid M$ yields

$$E(P, T, C) = E(P_C, T) = E(P_T, C) = E(P_{TC}),$$

where P_{TC} stands for both (coinciding) operators $(P_C)_T$ and $(P_T)_C$. Finally, we note that $T \leq P_C \mid M$ readily implies $T \leq P \mid M$, $T \mid M \cap C \geq 0$, $P \mid C \geq 0$, but the converse does not hold in general. Indeed, let $X = \mathbf{R}^2$, $M = \mathbf{R} \times \{0\}$, $C = \{0\} \times \mathbf{R}$, $Y = \mathbf{R}$, $P((x_1, x_2)) = |x_1 + x_2|$ for all $(x_1, x_2) \in X$ and $T(x_1, 0) = x_1$ for all $(x_1, 0) \in M$ and note that $P_C \mid M = 0$.

An application of Corollary 4.2 yields a refinement of a result due to Luxemburg and Zaanen.

A4. (Luxemburg and Zaanen; cf. [**6**, 2.6.3]). Suppose that X is a vector lattice with positive cone C, M is a vector sublattice, and P is lattice-increasing, i.e. $|x_1| \leq |x_2|$ implies $P(x_1) \leq P(x_2)$, $x_i \in X$. Let $T \in L(M, Y)$.

(a) If $T \leq P \mid M$ and $T \mid M \cap C \geq 0$, then ex $E(P, T, C) \neq \varnothing$.

(b) An operator $S \in E(P, T, C)$ is extreme in this set if and only if

$$\inf\{P((u+z+x)^+) + P((u-z-x)^+) - 2Su\colon z \in M, u \in X\} = 0$$

for each $x \in X$.

Proof. First note that $P_C(x) = P(x^+)$ holds for all $x \in X$. Indeed, $x \leq x^+$ implies $P_C(x) \leq P(x^+)$; conversely, $x \leq u \in X$ implies $x^+ \leq u^+$, and since P is lattice-increasing we obtain $P(x^+) \leq P(u^+) \leq P(u)$ whence $P(x^+) \leq P_C(x)$. Moreover, we have $Tz \leq Tz^+ \leq P(z^+)$ for all $z \in M$. Hence the assertions follow from Corollary 4.2.

In the remaining part of this section we deal with invariant P-dominated extensions. Let $\mathcal{G}$ be a set of mappings from X into X. A linear operator $S\colon X \to Y$ is called invariant if $SV = S$ for all $V \in \mathcal{G}$. The vector space of all invariant linear operators from X into Y is denoted by $L(X, Y)_{\mathcal{G}}$ and, given $T \in L(M, Y)$, we put

$$E(P, T)_{\mathcal{G}} = E(P, T) \cap L(X, Y)_{\mathcal{G}}.$$

Furthermore, let G denote the linear hull of the set $\{Vx - x\colon V \in \mathcal{G}, x \in X\}$. Obviously, $S \in L(X, Y)$ is invariant if and only if $S \mid G = 0$, i.e. $G \subset \ker S$.

A5. Let $T \in L(M, Y)$ and $\mathcal{G}$ be a set of mappings from X into X.
(a) If (and only if) $T \leq P_G \mid M$, then $\operatorname{ex} E(P, T)_{\mathcal{G}} \neq \varnothing$.
(b) $S \in E(P, T)_{\mathcal{G}}$ is an extreme point of this set if and only if

$$\inf\{(P - S)(u + z + x) + (P - S)(u - z - x)\colon \\ z \in M + G, u \in X\} = 0$$

for each $x \in X$.

Proof. Obviously, $E(P, T)_{\mathcal{G}} = E(P, T, G)$, and it is easily seen that $(P_G)^S = (P^S)_G$ holds for all $S \in E(P, T, G)$. Hence the assertions follow from Corollary 4.2.

A6. Let T and $\mathcal{G}$ be as in A5. Assume that there exists an operator $R \in L(X, X)$ such that $RM \subset M$, $PR \leq P$ and $P_{\ker R} \leq P_G$.

(a) If $TR = T$ on M and $T \leq P \mid M$, then $\operatorname{ex} E(P, T)_{\mathcal{G}} \neq \varnothing$.

(b) If additionally R is a projection with $P_{\ker R} = P_G$, then an operator S in $E(P, T)_{\mathcal{G}}$ is an extreme point of this set if and only if $P^S(x) = 0$ for each x in the fixed space of R.

Proof. (a) For $z \in M$ and $u \in \ker R$ we get $Tz = TRz \leq P(Rz) = P(R(z + u)) \leq P(z + u)$ whence $T \leq P_{\ker R} \mid M$. The assertion follows from A5.

(b). Suppose $S \in E(P, T)_{\mathcal{G}}$. Then $S \leq P_G = P_{\ker R}$ implies $SR = S$ whence $(PR)^S = P^S R$. Furthermore, it is readily verified that $PR = P_{\ker R}$. Therefore $(P_G)^S = P^S R$. By virtue of Corollary 4.2 S is an extreme point of $E(P, T)_{\mathcal{G}}$ if and only if $P^S(Rx) = 0$ for each $x \in X$, i.e. $P^S(x) = 0$ for each x in the fixed space of R.

We conclude our considerations by applying this result to a topological setting. A similar result for positive invariant extensions was stated by the first-named author in [**12**].

A7. Suppose that X is a locally convex space, that $\mathcal{G}$ is a mean ergodic semigroup of continuous linear operators on X [**16**, III.7.1] and that the order complete vector lattice Y is a topological vector space with a closed normal positive cone (e.g. we may assume that Y is an order complete topological vector lattice). Let M be a closed subspace and let P be continuous such that $VM \subset M$ and $PV \leq P$ for all $V \in \mathcal{G}$. Suppose $T \in L(M, Y)$.

(a) If T is invariant and $T \leq P \mid M$, then ex $E(P, T)_{\mathcal{G}} \neq \varnothing$.

(b) An operator S in $E(P, T)_{\mathcal{G}}$ is an extreme point of this set if and only if $P^S(x) = 0$ for each x in the fixed space of $\mathcal{G}$.

Proof. Let R be the zero element of the closed convex hull of $\mathcal{G}$ in the space of all continuous linear operators on X equipped with the topology of pointwise convergence. R is a continuous linear projection onto the fixed space of $\mathcal{G}$ with kernel the closure of G [**16**, III.7.2]. In view of the properties of P and $\mathcal{G}$ we obtain $P_{\ker R} = P_G$, $PR \leq P$ and $RM \subset M$. Furthermore, each $S \in E(P)$ is continuous since P is continuous and the positive cone of Y is normal. Thus T is continuous by virtue of $E(P, T) \neq \varnothing$ and employing the invariance of T we obtain $TR = T$ on M. The assertions follow now from A6.

REMARK 4.4. If $\mathcal{G}$ is a set of mappings from X into X, then invariant versions of A2 and A3 are valid.

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