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CARLESON MEASURES FOR FUNCTIONS ORTHOGONAL TO INVARIANT SUBSPACES

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Let $D=\{z: |z|<1\}$ be the unit disk. Suppose φ is an inner function with singular support K and let $M^{\perp}=H^2\ominus\varphi H^2$ where H^2 is the usual class of functions holomorphic on D. If μ is a positive measure on $\bar{D}$, the closed disk, which assigns zero mass to K, then call μ a *Carleson measure for* $M^{\perp}$ if for a $c>0$,

$$\int |f|^2 d\mu \leqq c\|f\|_2^2$$

for all $f\in M^{\perp}$. (Here and elsewhere, $\|f\|_2$ denotes the H^2 norm of an H^2 function.) In this paper the Carleson measures for $M^{\perp}$ are characterized for all inner functions φ such that for some ε, $0<\varepsilon<1$, the set $\{z: |\varphi(z)|<\varepsilon\}$ is connected.

If μ is a positive measure on D, then recall that μ is a *Carleson measure* if there is a positive constant c such that

$$\mu(R(I)) \leqq c|I| ,$$

where I is an arc on the unit circle with center $e^{i\theta_0}$ and length $|I|$, and $R(I)$ is the "curvilinear rectangle" $\{re^{i\theta}: 1-|I|/2\pi \leqq r < 1$ and $|\theta-\theta_0| \leqq 1/2|I|\}$.

In [2], Carleson proved that there is a constant $c>0$ such that

$$\int |f(z)|^2 d\mu(z) \leqq c\|f\|_2^2$$

for all $f\in H^2$, if and only if μ is a Carleson measure.

Clearly, any Carleson measure is a Carleson measure for $M^{\perp}$. Functions in $M^{\perp}$, however, can be better behaved than typical H^2 functions. Thus one is lead to suspect that there are more Carleson measures for $M^{\perp}$ than just the Carleson measures alone. This in fact turns out to be the case.

For the sake of simplicity, we state an abridged version of our main result.

THEOREM. *Suppose φ is inner and $\{z: |\varphi(z)|<\varepsilon\}$ is connected for some ε, $0<\varepsilon<1$. Let μ be a measure which assigns zero mass off $T\backslash K$, where T is the unit circle. Then μ is a Carleson measure for $M^{\perp}$ if and only if, for some constant $c>0$,*

$$\int_T \frac{1-|\xi|^2}{|1-\bar{\xi}e^{i\theta}|^2} d_\mu \leqq \frac{c}{1-|\varphi(\xi)|^2}$$

for all $\xi \in D$.

Now, it is easy to see that a measure μ on the unit circle T has the property that

$$\int |f|^2 d\mu \leqq c\|f\|_2^2$$

for all $f \in H^2$ if and only if $d_\mu = b d\theta$ where b is a bounded function. In this case,

$$\int_T \frac{1-|\xi|^2}{|1-\bar{\xi}e^{i\theta}|^2} d\mu \leqq c$$

for all $\xi \in D$. Thus one can see how the situation changes when dealing with $M^\perp$ instead of H^2.

This paper is divided into four sections. The assumption that $\{z: |\varphi(z)| < \varepsilon\}$ is connected implies that φ is a covering map onto the annulus $\{w: \varepsilon < |w| < 1/\varepsilon\}$. This is proven in § 1. In § 2 the covering map hypothesis is used to characterize those Carleson measures restricted to certain subsets of $\{z: \varepsilon < |\varphi(z)| < 1\}$. A corollary of this characterization is that for $\varepsilon < \delta < 1$, arc length on $\{z: |\varphi(z)| = \delta\}$ is a Carleson measure. In § 3 we prove a theorem about $M^\perp$ functions which is the key to our main results. Essentially, we show that $M^\perp$ functions belong to a Hardy space of functions defined on a larger domain than the disk. Section 4 contains some examples and applications.

The measures we consider are always positive measures, even if we do not specifically say so. The constant "c" which appears in various theorems changes each time it is used in a different context. If F and E are sets, $F \backslash E$ denotes their set theoretic difference. The symbol $\bar{F}$ denotes the closure of F, and ∂F denotes the topological boundary of F.

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1. Recall that any inner function φ has the form

$$\varphi(z) = cB(z)\exp\left(-\int_T \frac{\eta+z}{\eta-z} d\sigma(\eta)\right),$$

where $|c| = 1$, B is a Blaschke product, and σ is a positive measure on T which is singular with respect to arc length measure. Let

K be the closure of the union of the zero set of φ and the support of σ; K is called the *singular support* of φ.

For any complex number $z, z \neq 0$, define $z^* = 1/\bar{z}$. Let 0^* be the point at ∞ on S^2, the Riemann sphere. Then z^* is the reflection of z through the unit circle. If $E \subseteqq S^2$, let $E^* = \{z^*: z \in E\}$ be the reflected set. The equation

$$\varphi(z)\overline{\varphi(z^*)} = 1$$

defines an extension of φ which is holomorphic on $S^2 \backslash K^*$. Thus, for $t > 0$, the sets

$$D_t = \{z: \varphi \text{ is holomorphic at } z \text{ and } |\varphi(z)| < t\}$$

form a collection of open sets such that

$$D_t \subseteqq D_s \text{ for } t \leqq s$$

and

$$\bigcup_{t>0} D_t = S^2 \backslash K^* .$$

If $0 < \varepsilon < 1$, let $A_{\varepsilon,1/\varepsilon}$ be the annulus

$$A_{\varepsilon,1/\varepsilon} = \{w: \varepsilon < |w| < 1/\varepsilon\} .$$

Define

$$R_\varepsilon = \{z: \varphi \text{ is holomorphic at } z \text{ and } \varepsilon < |\varphi(z)| < 1/\varepsilon\} .$$

Then $R_\varepsilon = \bigcup_{n=1}^\infty \Omega_n$, where the Ω_n are the distinct connected components of R_ε. (The union may be finite.) We will be interested in the situation that D_ε is connected for some ε, $0 < \varepsilon < 1$.

THEOREM 1.1. *Suppose for some ε, $0 < \varepsilon < 1$, D_ε is connected. If $T \cap K \neq \emptyset$ then $R_\varepsilon = \bigcup_{n=1}^\infty \Omega_n$ where:*

(i) *each Ω_n is a simply connected set which is symmetric with respect to T;*

(ii) *the map $\varphi: \Omega_n \to A_{\varepsilon,1/\varepsilon}$ is a covering map.*

Proof. Fix n. Let $z_0 \in \partial\Omega_n$: we may suppose $z_0 \notin T$. Consider the case where $|z_0| < 1$.

Let Γ_ε be the set $\{z: |z| < 1, |\varphi(z)| = \varepsilon\}$. Thus $z_0 \in \Gamma_\varepsilon$. Observe that φ' never vanishes on Γ_ε, since D_ε is connected. Let γ be the component of Γ_ε which contains z_0. Since D_ε is connected and $T \cap K \neq \emptyset$, $\bar{\gamma}$ is not contained in D. Thus γ is a simple arc whose closure intersects $T \cap K$. It is well known that either $\bar{\gamma} \cap T$ consists of one point or two points.

In the first case, $\bar{\gamma}$ is a Jordan curve, and D_ε must consist entirely of the region which $\bar{\gamma}$ bounds. Thus $R_\varepsilon = \Omega_n$, $\partial\Omega_n = \bar{\gamma} \cup \bar{\gamma}^*$, and (i) holds.

If $\bar{\gamma}$ contains two points, then γ divides D into two components, one of which must contain D_ε. The other component is entirely contained in Ω_n. It follows $\bar{\gamma} \cup \bar{\gamma}^*$ is a Jordan curve, and Ω_n is the simply connected region which $\bar{\gamma} \cup \bar{\gamma}^*$ bounds. Thus (i) is true.

If the original point z_0 lies outside of $\bar{D}$, then $|\varphi(z_0)| = 1/\varepsilon$. By considering z_0^*, for which $|\varphi(z_0^*)| = \varepsilon$, and repeating the arguments above, we complete the proof of property (i).

To prove (ii), let $\psi: D \to \Omega_n$ be a conformal map of the unit disk onto Ω_n. Since $\partial\Omega_n$ is a Jordan curve, ψ extends to a homeomorphism of T. By symmetry we may assume that

(a) $\psi(\{w: |w| = 1, \operatorname{Im} w > 0\}) = \partial\Omega_n \cap D$

(b) $\psi(\{w: |w| = 1, \operatorname{Im} w < 0\}) = \partial\Omega_n \cap D^*$.

Let $g(w) = \varphi(\psi(w))$. Then $\varepsilon < |g| < 1/\varepsilon$, and therefore g is an outer function. Furthermore,

$$|g(\xi)| = \varepsilon$$

for $\xi \in T \cap \{\operatorname{Im} \xi > 0\}$, and

$$|g(\xi)| = 1/\varepsilon$$

for $\xi \in T \cap \{\operatorname{Im} \xi < 0\}$. This proves that $g: D \to A_{\varepsilon,1/\varepsilon}$ is a universal cover. Since ψ is conformal, the theorem is proved.

As a corollary of the proof of Theorem 1.1, we make the following observation.

COROLLARY 1.1. *If D_ε is connected, then $D_{1/\varepsilon}$ is simply connected.*

Proof. We first show that $D_{1/\varepsilon}$ is connected. Clearly, $D \subseteqq D_{1/\varepsilon}$. Let $z \in D_{1/\varepsilon}$. Then $z \in R_\varepsilon$, and hence, $z \in \Omega_n$, for some n. By the proof of Theorem 1.1, $\Omega_n \cap D \neq \varnothing$. Thus $D_{1/\varepsilon}$ is connected.

To show that $D_{1/\varepsilon}$ is simply connected, it suffices to show that $S^2 \backslash D_{1/\varepsilon}$ is connected. But the map $z \to z^*$ defines a homeomorphism of $S^2 \backslash D_{1/\varepsilon}$ and $\bar{D}_\varepsilon$. Since D_ε is connected, so is $\bar{D}_\varepsilon$. This finishes the proof.

It may occur that $K \subseteqq D$. In this case, φ is a finite Blaschke product and we have the following result.

THEOREM 1.2. *Suppose φ is a Blaschke product with n zeros, counted according to multiplicity. If D_ε is connected then the map $\varphi: R_\varepsilon \to A_{\varepsilon,1/\varepsilon}$ is an n:1 covering map. Furthermore, $D_{1/\varepsilon}$ is simply connected.*

Proof. Since φ is a finite Blaschke product, we need only show

R_ε contains no point where φ' vanishes. But if $\varphi'(z) = 0$ for some z such that $\varepsilon \leqq |\varphi(z)| \leqq 1/\varepsilon$, it follows that D_ε has at least two components. This is a contradiction. The rest of the proof is elementary and is omitted.

We finish this section with an observation which will prove useful later.

COROLLARY 1.2. *If D_ε is connected and $\varepsilon < \delta < 1$, then D_δ is connected.*

Proof. By Theorem VIII. 31 in [8] any component of D_δ is simply connected and if ψ is a conformal mapping of the unit disk onto one such component, then $s = 1/\delta\, \varphi(\psi)$ is an inner function. Since $|s|$ takes values less than ε, D_ε intersects every component of D_δ. Thus D_δ is connected.

We immediately get the next result.

COROLLARY 1.3. *If D_ε is connected and $\varepsilon < \delta < 1$, then $D_{1/\delta}$ is simply connected.*

2. Suppose $s \in H^\infty$, and $\|s\|_\infty \leqq 1$. Let $0 < \varepsilon < 1$ and set $A_{\varepsilon,1} = \{w: \varepsilon < |w| < 1\}$. Suppose further that $s: s^{-1}(A_{\varepsilon,1}) \to A_{\varepsilon,1}$ is a covering map. The main result of this section is a characterization of Carlson measures which take all their mass on certain subsets of $s^{-1}(A_{\varepsilon,1})$.

Let $|z| = (1 + \varepsilon)/2$ and set $B(z, (1 - \varepsilon)/2)$ equal to the open disk centered at z with radius $(1 - \varepsilon)/2$. Since s is a covering map, we have

$$s^{-1}\left(B\left(z,\ \frac{1-\varepsilon}{2}\right)\right) = \cup\, C_{n,z}\ ,$$

where the $C_{n,z}$ are pairwise disjoint and $s: C_{n,z} \to B(z, (1 - \varepsilon)/2)$ is a homeomorphism. Let $\mathscr{C}$ be the collection of all such $C_{n,z}$, where z ranges over the circle of radius $(1 + \varepsilon)/2$.

We prove the following theorem.

THEOREM 2.1. *Let F be a compact subset of $A_{\varepsilon,1}$ and let μ be a measure on D which assigns zero mass off $s^{-1}(F)$. Then the following conditions are equivalent:*

(i) *μ is a Carleson measure.*

(ii) *There is a constant $c > 0$ such that $\int_{C_{n,z}} |s'(z)|\, d\mu(z) \leqq c$ for all $C_{n,z} \in \mathscr{C}$.*

Proof. Since F is compact, it is contained in the finite union of noneuclidean disks of the form

$$N(z_0, r) = \left\{\xi : \left|\frac{\xi - z_0}{1 - \bar{z}_0\xi}\right| < r\right\} .$$

We may also assume that for each $N(z_0, r)$, there is an t, $r < t < 1$, and a z, $|z| = (1 + \varepsilon)/2$, such that

$$N(z_0, r) \subseteqq N(z_0, t) \subseteqq B\left(z, \frac{1 - \varepsilon}{2}\right) .$$

Thus we may assume that μ assigns zero mass off the set $s^{-1}(N(z_0, r))$.

Write $s^{-1}(N(z_0, r)) = \cup G_n$, and $s^{-1}(N(z_0, t)) = \cup R_n$, where $G_n \subseteqq R_n$ and $s: R_n \rightarrow N(z_0, t)$ is a homeomorphism. Let a_n be the point in G_n for which $s(a_n) = z_0$. We make the following observation.

LEMMA 2.1. *The sequence* $\{a_n\}$ *is uniformly separated.*

Proof. Let $h(z) = (z_0 - s(z))/(1 - \bar{z}_0 s(z))$. Then $\{a_n\}$ is the zero set of h, and $|h| \equiv r$ on ∂G_n. Let B_n be the Blaschke product with factors $\bar{a}_k/|a_k|\ (a_k - z)/(1 - \bar{a}_k z)$, $k \neq n$. Then $|B_n|$ never vanishes on G_n. Furthermore, for $z \in \partial G_n$,

$$|B_n(z)| \geqq |h(z)| = r .$$

It follows from the minimum principle that

$$|B_n(a_n)| \geqq r ,$$

and the lemma is proved.

Define the measure δ_z to be point mass at z. We have the immediate corollary; see [2].

COROLLARY 2.1. *The measure* $\nu = \sum \delta_{a_n} \cdot (1 - |a_n|^2)$ *is a Carleson measure.*

Let I be an arc on the unit circle with center $e^{i\theta}$ and length $|I|$. For $m > 0$, define mI to be the arc with center $e^{i\theta}$ and length $m|I|$. The next lemma enables us to compare μ to ν.

LEMMA 2.2. *Suppose condition* (ii) *of Theorem* 2.1 *is true. Then there are constants* $c_1 > 0$ *and* $m > 0$ *such that*

(1) $\mu(G_n) \leqq c_1(1 - |a_n|^2)$ *for all* n.

(2) *if* I *is an arc on* T *and* $R(I) \cap G_n \neq \emptyset$, *then* $G_n \subseteqq R(mI)$.

Accepting Lemma 2.2, for the moment, we show that condition (ii) implies condition (i). For I an arc on T we have

$$\begin{aligned}\mu(R(I)) &= \sum_n \mu(R(I) \cap G_n) \\ &\leqq \sum \mu(G_n) \\ &G_n \cap R(I) \neq \varnothing \ .\end{aligned}$$

Since Lemma 2.2 is in force, $G_n \cap R(I) \neq \varnothing$ implies $a_n \in R(mI)$. Thus

$$\begin{aligned}\mu(R(I)) &\leqq \sum_{a_n \in R(mI)} \mu(G_n) \\ &\leqq \sum_{a_n \in R(mI)} c_1 \cdot (1 - |a_n|^2) \\ &= c_1 \nu(R(mI)) \\ &\leqq c_1 \gamma(\nu) \cdot m \cdot |I| \ ,\end{aligned}$$

where $\gamma(\nu)$ is the Carleson constant for ν. Thus μ is a Carleson measure.

We now prove Lemma 2.2. Fix w. Since h is a 1:1 map of R_n onto the disk $\{w: |w| < t\}$ we may choose a branch of h^{-1} such that

$$g(z) = h^{-1}(z/t)$$

maps the unit disk onto R_n. By the Schwarz-Pick theorem, if z_1, $z_2 \in R_n$,

$$(*) \qquad \left|\frac{z_1 - z_2}{1 - \bar{z}_1 z_2}\right| \leqq \left|\frac{h(z_1)/t - h(z_2)/t}{1 - t^{-2}\overline{h(z_1)}h(z_2)}\right| \ .$$

If z_1 and z_2 are restricted to G_n we see that

$$(**) \qquad \left|\frac{z_1 - z_2}{1 - \bar{z}_1 z_2}\right| \leqq c < 1 \ ,$$

where c depends only on r and t. It is not difficult to see that for an m depending only on c, if $z_1 \in R(I)$ then $N(z_1, c) \subseteqq R(mI)$. This establishes (2) of Lemma 2.2.

Next, equation (**) yields a $\beta > 0$, independent of n, for which

$$\inf_{z_1, z_2 \in G_n} \frac{1 - |z_1|^2}{1 - |z_2|^2} \geqq \beta \ .$$

In particular, if $z \in G_n$, then

$$\frac{1}{1 - |z|^2} \geqq \frac{\beta}{1 - |a_n|^2} \ .$$

Now let $z_1 \to z_2 = z$ in equation (*) and use the last inequality to conclude that

$$|h'(z)| \geqq \frac{t(1-(r/t)^2)}{1-|z|^2} \geqq \frac{t\beta(1-(r/t)^2)}{1-|a_n|^2}$$

for all $z \in G_n$. Since

$$h'(z) = \frac{1-|z_0|^2}{(1-\bar{z}_0 s(z))^2} s'(z) ,$$

we see that for some constant $c_2 > 0$,

$$|s'(z)| \geqq \frac{c_2}{1-|a_n|^2}$$

for all $z \in G_n$.

Finally, by this last inequality and condition (ii) of Theorem 2.1,

$$\mu(G_n) \cdot \frac{c_2}{1-|a_n|^2} \leqq \int_{G_n} |s'(z)| d\mu \leqq c .$$

This proves (1) of Lemma 2.2, and the theorem in one direction.

To show that condition (i) implies condition (ii), observe that $\mu(C_{k,z}) \leqq \mu(G_n)$ for some n. By the proof of Lemma 2.2, if $z \in G_n$, then for a constant $c_3 > 0$,

$$\frac{2}{1-|z|^2} \leqq \frac{c_3}{1-|a_n|^2} .$$

Furthermore, $G_n \subseteqq N(a_n, c)$. If μ is a Carleson measure then for a constant c_4, depending on c_3,

$$\mu(N(a_n, c)) \leqq c_4 \cdot \gamma(\mu)(1-|a_n|^2) ,$$

where $\gamma(\mu)$ is the Carleson constant of μ. Thus

$$\int_{C_{k,z}} |s'(z)| d\mu \leqq \int_{G_n} |s'(z)| d\mu \leqq \int_{G_n} \frac{1-|s(z)|^2}{1-|z|^2} d\mu(Z)$$
$$\leqq \frac{c_3}{1-|a_n|^2} \cdot \mu(G_n) \leqq \frac{c_3}{1-|a_n|^2} \mu(N(a_n, c)) \leqq c_3 \cdot c_4 \cdot \gamma(\mu) .$$

This proves the theorem.

As an application, suppose φ is an inner function and D_ε is connected. Let $\varepsilon < \delta < 1$ and set $\Gamma = \{z: |\varphi(z)| = \delta\}$. Let μ be arclength measure on Γ. By Theorems 1.1 and 1.2 we may apply Theorem 2.1, with φ in place of s, to μ. Since for any $C_{n,z} \in \mathscr{C}$,

$$\int_{C_{n,z}} |\varphi'(z)|\,d\mu(z) = \int_{\Gamma\cap C_{n,z}} |\varphi'(z)|\,|dz| \leqq 2\pi\ ,$$

we have the following result.

COROLLARY 2.1. *Let φ be inner and D_ε connected. Then if $\varepsilon < \delta < 1$, arclength on $\{z: |\varphi(z)| = \delta\}$ is a Carleson measure.*

In §1 we showed that under the hypotheses of Corollary 2.1 $D_{1/\delta}$ was simply connected. In fact, more is true.

THEOREM 2.2. *Let φ be inner and D_ε connected. Then if $\varepsilon < \delta < 1$ and $|\varphi(0)| < \delta$, $\partial D_{1/\delta}$ is a rectifiable Jordan curve.*

Proof. We first prove that $\partial D_{1/\delta}$ is a Jordan curve.

Let R_δ be defined as in Theorem 1.1, and write $R_\delta = \bigcup_{n=1}^\infty \Omega_n$, where the Ω_n are the components of R_δ. Let $\gamma_n = \partial\Omega_n \backslash \bar{D}$. Then if $J_n = \Omega_n \cap T$ and $F = T\backslash \cup J_n$, we see that

$$\partial D_{1/\delta} = F \cup \bigcup_{n=1}^{\infty} \gamma_n\ .$$

Let $\alpha_n: \bar{J}_n \to \bar{\gamma}_n$ be a homeomorphism which fixes the endpoints of $\bar{J}_n$. Define the mapping of T onto $\partial\Omega_{1/\delta}$ by the formula

$$\alpha(e^{i\theta}) = \begin{cases} \alpha_n(e^{i\theta}), & \text{if } e^{i\theta} \in \bar{J}_n \\ e^{i\theta}, & \text{if } e^{i\theta} \in F\ . \end{cases}$$

We must show that α is continuous. It suffices to do this for $e^{i\theta} \in F$. This amounts to showing that if a sequence of arcs J_n approach $e^{i\theta}$, then the associated arcs γ_n must approach $e^{i\theta}$. If this fails to be the case then there is a cluster point of the arcs γ_n, z_0, such that $z_0 \neq e^{i\theta}$. If $|z_0| < 1$ then it follows that $|\varphi(z_0)| = \delta$, and $z_0 \in \Gamma = \{z: |z| < 1,\ |\varphi(z)| = \delta\}$. As in §1, φ' never vanishes on Γ. Thus there is a ball centered around z_0 which Γ divides into two regions; on one of those regions $|\varphi| > \delta$, and on the other $|\varphi| < \delta$. This contradicts the assertion that z_0 is a cluster point of the arcs γ_n.

If there is no z_0 with $|z_0| < 1$, and $z_0 \neq e^{i\theta}$, then it is easy to see that

$$\varliminf_{r\to 1} |\varphi(re^{ix})| \leqq \delta$$

for all e^{ix} on an arc connecting z_0 to $e^{i\theta}$. Since φ is inner, this is impossible. Thus α is continuous at $e^{i\theta}$, and $\partial\Omega_{1/\delta}$ is a Jordan curve.

Turning to the rectifiability, it isn't hard to see that α has total variation

$$||d\alpha|| = \sum_{n=1}^{\infty} |\gamma_n| + |F| ,$$

where $|\gamma_n|$ denotes the length of the arc γ_n and $|F|$ denotes the measure of F. Since $\infty \notin \partial D_{1/\delta}$, by Corollary 2.1, $||d\alpha|| < \infty$. This proves Theorem 2.2.

REMARK. It follows from the rectifiability of $\partial D_{1/\delta}$ and Theorems VIII 30 and 31 in [8], that $|F| = 0$. Thus arclength measure on $\partial \Omega_{1/\delta}$ is equivalent to arc length measure on $\partial\Omega_{1/\delta}\backslash T$.

3. In this section we characterize Carleson measures for $(\varphi H^2)^\perp$ in the case that D_ε is connected.

For $\xi \in D$, define the function

$$K_\xi(z) = \frac{1 - \overline{\varphi(\xi)}\varphi(z)}{1 - \bar{\xi}z} .$$

Then $K_\xi \in M^\perp$ and

$$|| K_\xi ||_2^2 = \frac{1 - |\varphi(\xi)|^2}{1 - |\xi|^2} .$$

See [1], page 194 for the proofs. Let μ be a measure on $\bar{D}$ which assigns zero mass to K. Let μ_δ be the restriction of μ to D_δ. Then if $0 < \delta < 1$,

$$\int |K_\xi(z)|^2 \frac{1 - |\xi|^2}{1 - |\varphi(\xi)|^2} d\mu_\delta(z) \geqq (1 - \delta)^2 \int \frac{1 - |\xi|^2}{|1 - \bar{\xi}z|^2} d\mu_\delta(z) .$$

Suppose μ is a Carleson measure for $M^\perp$. Then the last inequality yields

$$\frac{c}{(1 - \delta)^2} \geqq \int \frac{1 - |\xi^2|}{|1 - \bar{\xi}z|^2} d\mu_\delta ,$$

where c is independent of ξ. It follows that μ_δ is a Carleson measure for D. Conversely, if μ_δ is a Carleson measure for D, then μ_δ is a Carleson measure for $M^\perp$. We have proven the following lemma.

LEMMA 3.1. *The following properties are equivalent:*
(i) *μ is a Carleson measure for $M^\perp$.*
(ii) (a) *μ_δ is a Carleson measure for D and*
(b) *$\mu - \mu_\delta$ is a Carleson measure for $M^\perp$.*

We turn, therefore, to the problem of characterizing Carleson measures for $M^\perp$ which assign zero mass to $K \cup D_\delta$.

Assume that $0 < \varepsilon < \delta < 1$. Then $D_{1/\delta}$ is simply connected and we may choose a conformal map $\sigma: D \to D_{1/\delta}$. Let $\psi = \sigma^{-1}$.

Suppose μ is a measure on $\bar{D}$ which assigns zero mass to $K \cup D_\delta$ and set $\mu_1 = |\psi'|\mu$. Then if $E \subseteq D$, the equation

$$\nu(E) = \mu_1(\sigma(E))$$

defines a measure on D. We prove the following theorem.

THEOREM 3.1. *The following properties are equivalent*:

(i) *The measure ν is a Carleson measure.*

(ii) *The measure μ is a Carleson measure for $M^\perp$.*

(iii) *There is a constant $c > 0$ such that*

$$\int_{\sigma(C_{n,z})} |\varphi'(z)|\, d\mu(z) \leqq c$$

for all sets $C_{n,z} \in \mathscr{C}$, where $\mathscr{C}$ is the collection defined in Theorem 2.1 *with $s = \delta\varphi(\sigma)$ and $\varepsilon \cdot \delta$ in place of ε.*

Proof. We show first that (i) implies (ii). If $f \in M^\perp$, then it is well known that f has a holomorphic extension to $D_{1/\delta}$. See [4]. We need an explicit expression for $f(z)$ when $|z| > 1$. Since $f \in M^\perp$,

$$\int_T \bar{\varphi} f \cdot \bar{b}\, d\theta = 0$$

for all $b \in H^\infty$. Thus

$$\bar{\varphi} f = e^{-i\theta}\bar{h} \text{ a.e. } [d\theta]$$

where $h \in H^2$. For all z, $|z| \geqq 1$ define

$$F(z) = \varphi(z)\frac{1}{z}\bar{h}(1/\bar{z}) \ . \tag{1}$$

Then $F(e^{i\theta}) = f(e^{i\theta})$ for $e^{i\theta} \notin K$ and it follows that $F(z) = f(z)$ for all $z \notin K \cup K^*$. Equation (1) will imply that f is well behaved on $D_{1/\delta}$.

To make this precise, let T_n be the circle of radius $1 - 1/n$ centered at 0, and set $C_n = \sigma(T_n)$. Then $E^2(D_{1/\delta})$ is the class of analytic functions defined on $D_{1/\delta}$ which satisfy the condition

$$\lim_{n\to\infty} \int_{C_n} |f(z)|^2 |dz| < \infty \ .$$

The space E^2 is closely related to $H^2(D)$. In fact $f \in E^2(D_{1/\delta})$ if and only if

$$f(\sigma(w))\sigma'(w)^{1/2} = g(w)$$

for some $g \in H^2(D)$. For a full discussion, see [5], pages 168–169.

Since $\partial D_{1/\delta}$ is a rectifiable Jordan curve, $\sigma' \in H^1(D)$, and the measure on T given by $|\sigma'(w)||dw|$ is arclength measure for $\partial D_{1/\delta}$.

Since g and σ' both have radial limits a.e. $[d\theta]$, it follows that $\lim_{r\to 1} f(\sigma(re^{i\theta}))$ exists a.e. $[d\theta]$. Thus we may write

$$\begin{aligned}\|g\|_2 &= \int_T |g(w)|^2|dw| = \int_T |f(\sigma(w)|^2|\sigma'(w)||dw| \\ &= \int_{\partial D_{1/\delta}} |f(z)|^2|dz| .\end{aligned}$$

Thus E^2 is a Hilbert space with norm defined by the equation

$$\|f\|_{E^2}^2 = \int_{\partial D_{1/\delta}} |f(z)|^2|dz| ,$$

and $g \to f$ is an isometry of H^2 onto E^2. Recall that $F = \partial D_{1/\delta} \cap T$ is a set of measure 0. Thus

$$\|f\|_{E^2}^2 = \int_{\partial D_{1/\delta}\setminus T} |f(z)|^2|dz| .$$

These observations and equation (1) are the key to the next lemma.

LEMMA 3.1. *If $f \in M^\perp$ then the extension of f to $D_{1/\delta}$ belongs to $E^2(D_{1/\delta})$. Furthermore,*

$$\|f\|_{E^2}^2 \leqq c\|f\|_2^2 ,$$

where c is independent of f.

Proof. Suppose $f \in M^\perp \cap H^\infty$. If f and h are related by the equation

$$f = \varphi e^{-i\theta}\bar{h} \text{ a.e. } [d\theta] ,$$

then $\|h\|_\infty = \|f\|_\infty$. Thus equation (1) shows that f is bounded on $D_{1/\delta}$. Since $\sigma' \in H^1$, $f \in E^2$. We calculate $\|f\|_{E^2}^2$ using the fact that f is continuous off the singular support of φ, and the fact that arclength on $\Gamma = \{z: |\varphi(z)| = \delta\}$ is a Carleson measure. Thus

$$\begin{aligned}\|f\|_{E^2}^2 &= \int_{\partial D_{1/\delta}\setminus T} |f(z)|^2|dz| = \int_{\partial D_{1/\delta}\setminus T} |\varphi(z)|^2|h(1/\bar{z})|^2\frac{|dz|}{|z|^2} \\ &= 1/\delta^2\int_\Gamma |h(w)|^2|dw|^2 \leqq \frac{\gamma}{\delta^2}\|h\|_2^2 = \frac{\gamma}{\delta^2}\|f\|_2^2 ,\end{aligned}$$

where γ depends only on the Carleson constant of $|dw|$ on Γ. This shows that the conclusion of the lemma is valid for $M^\perp \cap H^\infty$. Since linear combinations of the functions K_ξ are dense in $M^\perp$, $M^\perp \cap H^\infty$ is dense in $M^\perp$. A standard argument proves the lemma for all of $M^\perp$.

We complete the proof that condition (i) of Theorem 3.1 implies condition (ii). Since $g \in H^2$ if and only if $g(w) = f(\sigma(w))\sigma'(w)^{1/2}$ for $f \in E^2$, it follows that

$$\int |f(z)|^2 d\mu(z) = \int |g(w)|^2 d\nu(w) .$$

If ν is a Carleson measure, then from the last equation,

$$\int |f(z)|^2 d\mu \leqq \gamma(\nu)\,\|g\|_2^2 = \gamma(\nu)\,\|f\|_{E^2}^2 \leqq \gamma(\nu)c \cdot \|f\|_2^2 .$$

Thus $d\mu$ is a Carleson measure for $M^\perp$.

We next show that condition (iii) implies condition (i). Let $s(w) = \delta\varphi(\sigma(w))$. Then by the results of §1, $s: s^{-1}(A_{\varepsilon\delta,1}) \to A_{\varepsilon\delta,1}$ is a covering map. Observe that ν assigns zero mass off $\{w: \delta^2 \leqq |s(w)| \leqq \delta\}$. By Theorem 2.1 ν is a Carleson measure if and only if for some $c > 0$,

$$\int_{C_{n,z}} |s'(w)|\,d\nu \leqq c$$

for all $C_{n,z} \in \mathscr{C}$. (Here, the "ε" of Theorem 2.1 is replaced by "$\varepsilon\delta$".)

But

$$\int_{C_{n,z}} |s'(w)|\,d\nu = \int_{\sigma(C_{n,z})} \delta \cdot |\varphi'(z)|\,d\mu .$$

Thus (iii) implies (i).

All that remains is to prove (ii) implies (iii). We must find some constant c such that

$$\int_{\sigma(C_{n,z})} |\varphi'|\,d\mu \leqq c .$$

Recall that μ assigns zero mass to $K \cup D_\delta$. Let $N_{n,z} = \sigma(C_{n,z}) \cap \{\xi: \delta \leqq |\varphi(\xi)| \leqq 1\}$. Thus $N_{n,z}$ is a component of $\varphi^{-1}(R)$, where R is the intersection of the closed annulus $\{w: \delta \leqq |w| \leqq 1\}$ and the open ball $B(\delta^{-1}z, (1-\varepsilon\delta)/2\delta)$. It is enough to show that

$$\int_{N_{n,z}} |\varphi'|\,d\mu \leqq c .$$

We need the following lemma.

LEMMA 3.2. *Let* $C_{n,z_0} \in \mathscr{C}$. *Suppose* $\xi \in N_{n,z_0}$, $\arg \varphi(\xi) = \arg z_0$, *and* $|\varphi(\xi)| = \delta$. *Then there is a constant* c_1, *independent of* C_{n,z_0}, *such that*

$$|\varphi'(z)| \leqq c_1 |K_\xi(z)|^2 \frac{1-|\xi|^2}{1-|\varphi(\xi)|^2}$$

for all $z \in N_{n,z_0}$.

Proof. Let φ_ξ^{-1} denote the branch of φ^{-1} for which $\varphi_\xi^{-1}(\varphi(\xi)) = \xi$. Set T_ε equal to the circle of radius ε centered at the origin and let α be the radial projection of $\varphi(\xi)$ onto T_ε. Suppose Ω is the simply connected region bounded by the unit circle and the line tangent to T_ε at α. Let $g: \Omega \to D$ be a conformal map of Ω onto the disk such that $g(\varphi(\xi)) = 0$. Then $f = \varphi_\xi^{-1} \circ g^{-1}$ maps the disk into itself. By the Schwarz-Pick theorem,

$$|g(\varphi(z))| \geqq \left|\frac{\xi - z}{1-\bar{\xi}z}\right|$$

for $z \in \varphi_\xi^{-1}(\Omega)$. Thus

$$1 - |g(\varphi(z))|^2 \leqq \frac{(1-|\xi|^2)(1-|z|^2)}{|1-\bar{\xi}z|^2}$$

and

$$\frac{|1-(g(\varphi(z)))|^2}{1-|\varphi(z)|^2} \cdot \frac{1-|\varphi(z)|^2}{1-|z|^2} \leqq \frac{2}{1-\delta} \cdot \left|\frac{1-\overline{\varphi(\xi)\varphi}(z)}{1-\bar{\xi}z}\right|^2 \cdot \frac{1-|\xi|^2}{1-|\varphi(\xi)|^2}$$

for $z \in \varphi_\xi^{-1}(\Omega)$. If z is restricted to N_{n,z_0} then $(1-|g(\varphi(z))|^2)/(1-|\varphi(z)|^2)$ is bounded away from zero by a constant independent of z_0. Since $|\varphi'(z)| \leqq (1-|\varphi(z)|^2)/(1-|z|^2)$, the lemma is proved.

To complete the proof of Theorem 3.1 observe that (ii) implies that for some constant c_2,

$$\int \frac{1-|\xi|^2}{1-|\varphi(\xi)|^2} |K_\xi(z)|^2 d\mu(z) \leqq c_2$$

for all $\xi \in D$. Choose $C_{n,z_0} \in \mathscr{C}$ and ξ as in Lemma 3.2. Then

$$\int_{N_{n,z_0}} |\varphi'(z)| d\mu(z) \leqq \int_{N_{n,z_0}} c_1 \frac{1-|\xi|^2}{1-|\varphi(\xi)|^2} |K_\xi(z)|^2 d\mu(z) \leqq c_1 c_2 = c \,.$$

This completes the proof.

We complete this section by characterizing Carleson measures for $M^\perp$ in terms of the growth of the function

$$h(\xi) = \int \frac{1-|\xi|^2}{|1-\bar{\xi}z|^2} d\mu(z) \,.$$

THEOREM 3.2. *Let* φ *be an inner function and suppose* D_ε *is* *connected. If μ is a measure on $\bar{D}$ which assigns zero mass to K, then the following properties are equivalent:*

(i) *μ is a Carleson measure for $M^{\perp}$.*

(ii) *There is a constant c such that*

$$h(\xi) \leqq \frac{c}{1 - |\varphi(\xi)|}$$

for all $\xi \in D$.

(iii) (a) *μ_δ is a Carleson measure for D, where $\varepsilon < \delta < 1$, and* (b) *There is a constant c such that*

$$\int_{N_{n,z_0}} |\varphi'(z)| \, d\mu \leqq c$$

for all N_{n,z_0}.

Proof. We have already shown that (i) and (iii) are equivalent. That (i) implies (ii) follows easily from the inequality

$$\int |K_\xi(z)|^2 d\mu(z) \leqq c \cdot \frac{1 - |\varphi(\xi)|^2}{1 - |\xi|^2} .$$

We turn to the proof that (ii) implies (iii) (a).

Let I be an arc on T of length $|I|$. We must find c such that

$$\mu(R(I) \cap D_\delta) \leqq c|I| .$$

For $\xi \in D$, $\xi \neq 0$, define I_ξ to be the arc on T with center $\xi/|\xi|$ and length $2(1 - |\xi|)$. There is a constant γ such that

$$\frac{1 - |\xi|^2}{|1 - \bar{\xi} z|^2} \geqq \gamma(1 - |\xi|)^{-1} \tag{2}$$

for all $z \in R(I_\xi)$, and all $\xi \in D$.

Let $S_1 = R(I) \cap D_\delta$ and set $\alpha_1 = \max_{\xi \in S_1}(1 - |\xi|)$. Choose $\xi_1 \in S_1$ such that $1 - |\xi_1| \geqq 7/8\ \alpha_1$. Proceeding inductively, suppose S_1, $S_2, \cdots, S_n$ and $\xi_1, \xi_2, \cdots, \xi_n$ have been chosen. Let $S_{n+1} = S_n \backslash R(I_{\xi_n})$ and set $\alpha_{n+1} = \max_{\xi \in S_{n+1}}(1 - |\xi|)$. Choose ξ_{n+1} such that

$$1 - |\xi_{n+1}| \geqq 7/8\, \alpha_{n+1} .$$

In this fashion we obtain a sequence $\{\xi_n\}$ such that

$$S_1 \subseteqq \bigcup_{u=1}^{\infty} R(I_{\xi_n})$$

and

$$\sum_{n=1}^{\infty} 1 - |\xi_n| \leqq c_1 |I| ,$$

where c_1 is a constant independent of I.

Condition (ii) and inequality (2) yield

$$\mu(S_1) \leqq \sum_n \mu(R(I_{\xi_n})) \leqq \frac{1}{2\gamma} \sum_{n=1}^{\infty} \int_{R(I_{\xi_n})} \frac{(1-|\xi_n|)^2}{|1-\bar{\xi}_n z|^2} d\mu(z)$$
$$\leqq \frac{c}{\gamma} \sum_{n=1}^{\infty} 1 - |\xi_n| \leqq \frac{c \cdot c_1}{\gamma} |I| .$$

Thus μ_δ is a Carleson measure.

To show that (ii) implies (iii) (b), observe that with ξ and N_{n,z_0} related as in Lemma 3.2,

$$\frac{1-|\xi|^2}{|1-\bar{\xi}z|^2} \leqq c \cdot |\varphi'(z)|$$

for all $z \in N_{n,z_0}$. Thus (iii) (b) is an easy consequence of property (ii). This completes the proof.

4. Perhaps the most representative example occurs when $\varphi(z) = \exp(-(1+z)/(1-z))$. In this case, D_ε is a disk tangent to T at the point 1, and Theorems 3.1 and 3.2 are in force.

One calculates that

$$|\varphi'(z)| = \frac{2|\varphi(z)|}{|1-z|^2} .$$

Suppose μ is a measure on T which assigns zero mass to $\{1\}$. It follows from Theorem 3.2 that μ is a Carleson measure for $M^\perp$ if and only if

$$\int_{I_n} |\varphi'| \, d\mu \leqq c$$

for all arcs I_n of the form

$$I_n = (e^{i\pi}/n + 1,\ e^{i\pi}/n) ,$$

where $n = \pm 1, \pm 2, \cdots$.

Simple estimates show that this is the case if and only if

$$\int_{I_n} d\mu \leqq \frac{c}{n^2} .$$

This leads to the following result. For $f \in M^\perp$,

$$\sum_{n=-\infty}^{\infty} \max_{z \in I_n} |f(z)|^2 \cdot \frac{1}{n^2} \leqq c \cdot \|f\|_2^2 .$$

This may be regarded as a generalization of a theorem of Clark; see [3], pages 176–177.

More generally, let E be a closed compact subset of D with zero capacity. Let $\varphi: D \to D \backslash E$ be an analytic universal covering map. Then φ is an inner function; see [6]. If $E \subseteqq \{z: |z| < \varepsilon\}$, then it is not hard to show that D_ε is connected. Thus Theorems 3.1 and 3.2 apply to this class of inner functions.

Now suppose that $\varphi(z) = z^n$. Then $M^\perp$ is the span of the functions $1, z, z^2, \cdots, z^{n-1}$. Let $u \in L^1(T)$, $u \geqq 0$, and suppose u has the Fourier expansion $\sum_{-\infty}^{\infty} c_n e^{in\theta}$. If $f(z) = \sum_{m=0}^{n-1} a_m z^m$, then $f \in M^\perp$ and

$$\int_T |f(z)|^2 u(z) |dz| = \sum_{k,m}^{n-1} c_{m-n} a_k \bar{a}_m .$$

The expression on the right is a finite section Toeplitz operator. If we take the supremum over all $\{a_0, a_1, \cdots, a_{n-1}\}$ such that $\sum |a_k|^2 = 1$, then we obtain the largest eigenvalue of the form. On the other hand,

$$\sup_{f \in M^\perp, \|f\|_2 = 1} \int_T |f|^2 u d\theta$$

is the "Carleson constant" for the Carleson measure for $M^\perp$, $ud\theta$. Observe that for any ε, $0 < \varepsilon < 1$, $\{z: |z|^n < \varepsilon\}$ is connected. If we choose $\varepsilon = 1/4$ and $\delta = 1/2$, then applying Theorem 3.1 we see that if $\mathscr{I}_n$ is the collection of arcs I,

$$I = (e^{i\theta\pi}, \ e^{i\theta\pi + i\pi/n})$$

then for a constant c, independent of n,

$$\int_T |f|^2 u d\theta \leqq c \cdot \gamma_n$$

where $\gamma_n = \sup_{I \in \mathscr{I}_n} \int_I u \cdot n d\theta$ and f ranges over all $(z^n H^2)^\perp$ functions with norm less than 1.

Thus we obtain order of magnitude estimates for the largest eigenvalue of finite section Toeplitz operators. These results can be compared with the asymptotic estimates, in the case where u satisfies more restricted hypotheses, found on page 72 of Grenander and Szegö, [7].

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