MIXED MODULES OVER VALUATION RINGS

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1. Introduction. A p-primary abelian group is a module over the p-adic integers; thus Ulm's theorem can be viewed as a classification of reduced countably generated torsion modules over the p-adic integers, or, more generally, over a complete discrete valuation ring. It is with this point of view that Kaplansky and Mackey [4] generalized Ulm's theorem to cover mixed modules of rank 1. In this paper their result is generalized in various ways, sometimes to modules of finite rank, sometimes to modules over possibly incomplete rings. The structure theorems obtained are applied to solve square-root, cancellation, and direct summand problems.

The main idea is to squeeze as much information as possible from the proof of Ulm's theorem in [4]. In order to understand our procedure, we sketch that proof. Order, once for all, generating sets of the modules T and T': $t_1, t_2, \cdots; t'_1, t'_2, \cdots$ The plan is to build an isomorphism stepwise up these lists. The crucial point is then, given a height-preserving isomorphism $f: S \to S'$, S finitely generated, to extend f to a height-preserving isomorphism of $\{t_i, S\}$ and a suitable submodule of T' containing S'. In order to construct this extension it is necessary to normalize t_i in two ways:

(i) assume $pt_i \in S$;

(ii) assume that t_i has maximal height in the coset $t_i + S$. If T is torsion, both of these normalizations are always possible. Now the possibility of extra generality arises precisely at these two points. If T is mixed and (ii) is satisfied, then the proof will go through if T/S is torsion; this is what Kaplansky and Mackey did in their paper. In this paper, we define a class of modules in which (ii) can always be satisfied, and it is this class of modules which we shall consider.

2. Definitions. A *discrete valuation ring* (DVR) is a principal ideal domain R with a unique prime ideal (p). $\bigcap_{n=1}^{\infty}(p^n) = (0)$. Hence if $r \in R$ is non-zero, there is a maximal n, depending on r, such that $r \in (p^n)$. Define $|r| = e^{-n}$; define $|0| = 0$. $|\ |$ is a norm which satisfies the strong triangle inequality: $|r + r'| \leq \max |r|, |r'|$. This norm induces a metric on R. R is a *complete* DVR if it is complete in this metric. If R is incomplete, we may form its completion R^*, and R^* is a complete DVR. The p-adic integers is a complete DVR; it is also compact as a metric space.

Let Q be the quotient field of R. We define the *rank* of a module M (often called the 'torsion-free rank') to be the dimension of the Q

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vector space $Q \otimes M$. Thus if M is torsion, rank $M = 0$. The rank can also be defined as the cardinality of a maximal independent subset of M. Note that every element in an independent subset has infinite order.

The word module will mean unitary module over a DVR. All abelian group-theoretic notions can be found in [2; 3; 5].

3. *KM* **modules.** In this section we shall define a certain family of modules and determine some members of this family.

DEFINITION. A *semi-KM module* is a reduced countably generated module of finite rank.

DEFINITION. A module M *has the coset property* if the coset $x+S$ has an element of maximal height whenever S is a finitely generated submodule of M.

DEFINITION. A *KM module* is a semi-*KM* module with the coset property.

The coset property is the crucial part of the definition of a *KM* module; for later use, we now give a characterization of this property.

DEFINITION. Let S be a submodule of M; if $x \in M$, let x^* denote the image of x in M/S under the natural homomorphism. S is *copure* if any $x^* \in M/S$ has a pre-image x such that $h(x^*) = h(x)$. ($h(x)$ denotes the height of the element x).

LEMMA 3.1. *S is copure in M if and only if every coset of S has an element of maximal height.*

Proof. Induction on $h(x)$ that $h(x) = h(x^*)$ if x has maximal height in $x + S$.

COROLLARY 3.2. *M has the coset property if and only if every finitely generated submodule is copure.*

LEMMA 3.3. *If R is complete, a reduced module M with no elements of infinite height has the coset property.*

Proof. Let $S = \{y_1, \cdots, y_s\}$. It must be shown that $x + S$ contains an element of maximal height. We may assume that $x \notin S$, otherwise 0 has maximal height in $x + S$. Under this assumption we show by induction on s that $y + S$ contains only finitely many distinct heights.

Let $s = 1$. If $h(x + a_n y) = \alpha_n$ is strictly increasing, then $h(b_n y) = \alpha_n$, where $b_n = a_{n+1} - a_n$. Hence $h(b_{n+1}y) > h(b_n y)$. Let $(p^{m(n)})$ be the smallest ideal containing b_n. Then $m(n+1) > m(n)$, i.e., $m(n) \to \infty$, and so $b_n \to 0$. Hence $\{a_n\}$ is a Cauchy sequence and $a_n \to a$, since R is complete. Now $x + ay = x + a_n y + (a - a_n)y$. If $h((a - a_n)y) \geq \alpha_n$ for all n, then $x + ay$ has infinite height and is thus 0, contradicting $x \notin S$. Therefore we may assume that $h((a - a_n)y) = h((a - a_m)y)$ for all $m \geq n$. But then $a - a_n$ and $a - a_m$ are associates, contradicting $a - a_m \to 0$. Hence $\{\alpha_n\}$ cannot be strictly increasing, i.e., there can only be a finite number of heights in the coset.

For the general case, suppose $h(x + a_1^n y_1 + \cdots + a_s^n y_s) = \alpha_n$ is strictly increasing. Suppose further that each coordinate sequence $\{a_i^n\}$ is Cauchy, and so $a_i^n \to a_i$ for each i. Then

$$\begin{aligned} x + a_1 y_1 + \cdots + a_s y_s = (x &+ a_1^n y_1 + \cdots + a_s^n y_s) \\ &+ (a_1 - a_1^n)y_1 + \cdots + (a_s - a_s^n)y_s \,. \end{aligned}$$

The height of the first term on the right is α_n while the height of the remaining terms gets arbitrarily large. Hence $x + a_1 y_1 + \cdots + a_s y_s$ has infinite height and so must be 0, contradicting $x \notin S$. Hence $\{\alpha_n\}$ cannot be strictly increasing, i.e., there are only a finite number of heights.

Therefore we may assume $\{a_1^n\}$ contains no Cauchy subsequence, and so we may assume further that it consists of incongruent units. Now

$$h(a_1^{n+1}(x + \Sigma a_j^n y_j) - a_1^n(x + \Sigma a_j^{n+1} y_j)) = \alpha_n = h((a_1^{n+1} - a_1^n)x + \Sigma b_k^n y_k) \,,$$

where

$$b_k^n = a_1^{n+1} a_k^n - a_1^n a_k^{n+1}$$

and $k \geq 2$. Since $a_1^{n+1} - a_1^n$ is a unit, and since multiplication by a unit does not alter heights, we may assume it is 1. But there are only $s - 1$ y's occurring, and so the inductive hypothesis applies. Hence there can only be a finite number of heights, and so $\{\alpha_n\}$ cannot be strictly increasing. Thus $x + S$ contains only finitely many distinct heights.

LEMMA 3.4. *If R is compact and M is a reduced module of rank 2, then M has the coset property.*

Proof. Let S be a finitely generated submodule with $x \notin S$. By the method of [4], it suffices to consider the case when S is generated by two elements of infinite order, y and z. Moreover, we may assume $h(x + a_n y + b_n z) = \alpha_n$, where $\{\alpha_n\}$ is strictly increasing. Since R is compact, we may assume that $a_n \to a$ and $b_n \to b$. $x + ay + bz = (x + a_n y + b_n z) + ((a - a_n)y + (b - b_n)z)$. Now the height of the first term on the right is α_n. If the other term has height $\geq \alpha_n$ for all n, then $h(x + ay + bz) \geq \alpha_n$ for all n and $x + ay + bz$ is the desired element. Hence we may suppose that $h((a - a_n)y + (b - b_n)z) = \beta < \alpha_n$. This equation must hold for all $m \geq n$. If a sequence $\{c_n\}$ converges to c, there is a subsequence $\{c_{n_i}\}$ such that $c - c_{n_i}$ and $c_{n_{i+1}} - c_{n_i}$ are associates, In our case, there are units u_n and v_n such that $(a - a_n)y = u_n(a_{n+1} - a_n)y$ and $(b - b_n)z = v_n(b_{n+1} - b_n)z$. (We have assumed, for notation, that $\{a_n\}$ and $\{b_n\}$ are the subsequences). Hence

$$(a - a_n)y + (b - b_n)z = u_n(x + a_{n+1}y + b_{n+1}z)$$
$$- u_n(x + a_ny + b_nz) + (v_n - u_n)(b_{n+1} - b_n)z \, .$$

Hence $h((v_n - u_n)(b_{n+1} - b_n)z) = \beta$ for large n. Therefore, $(v_n - u_n)(b_{n+1} - b_n)$ are associates, and non-zero since $\beta < \alpha_n < \infty$. Hence there must be a maximal power of p dividing any of them, contradicting the fact that $b_{n+1} - b_n \to 0$.

LEMMA 3.5. *If R is complete and M is reduced of rank* 1, *then M has the coset property.*

Proof. Kaplansky and Mackey [4].

To this point, all modules with the coset property have been modules over a complete DVR. We shall now exhibit modules over a possibly incomplete ring which have the coset property. For this purpose we consider tensor products. All tensor products will be taken over the ring R.

LEMMA 3.6. *Let R be a* DVR *with completion R^*. Any R-module M can be imbedded as a pure R-submodule in $R^* \otimes M$; moreover, the torsion submodule T of M coincides with $R^* \otimes T$, which is the torsion submodule of $R^* \otimes M$.*

Proof. R^* is a torsion-free R-module, and R is a pure submodule [3]. Further, if $\delta + R \in R^*/R$, there is an $r \in R$ such that $\delta - r = p\delta'$, $\delta' \in R^*$. Therefore $\delta + R = p\delta' + R$ and so $p(R^*/R) = R^*/R$. Hence R^*/R is torsion-free and divisible.

Exactness of the sequence $0 \to R \to R^* \to R^*/R \to 0$ induces exactness of $\operatorname{Tor}(R^*/R, M) \to R \otimes M \to R^* \otimes M \to (R^*/R) \otimes M \to 0$. $R \otimes M = M$ and, since R^*/R is torsion-free, $\operatorname{Tor}(R^*/R, M) = 0$. Thus $x \to 1 \otimes x$ is an imbedding of M into $R^* \otimes M$. But the sequence also implies that $(R^* \otimes M)/M \approx (R^*/R) \otimes M$. Since R^*/R is torsion-free and divisible, we have $(R^*/R) \otimes M$ torsion-free. Hence M is pure in $R^* \otimes M$ and contains the torsion submodule of $R^* \otimes M$. We already know that $x \to 1 \otimes x$ is a monomorphism; this last remark shows it is an epimorphism when restricted to T. Thus $T \approx R^* \otimes T$, which is the torsion submodule of $R^* \otimes M$.

LEMMA 3.7. *If R is a* DVR *with completion R^*, and if M is an R-module of rank* 1 *with no elements of infinite height, then $R^* \otimes M$ has no elements of infinite height.*

Proof. Suppose $z = \Sigma \delta_i \otimes m_i \in R^* \otimes M$ has infinite height. By the preceding lemma, z has infinite order. Let $x \in M$ have infinite order. Since rank $M = 1$, there is an n such that for all i, $p^n m_i = r_i x$, $r_i \in R$. $p^n z = \Sigma \delta_i r_i \otimes x$. As any element in R^*, $\Sigma \delta_i r_i$ can be expressed as γp^k, where γ is a unit. But then $h(z) < h(p^n z) = h(\gamma \otimes p^k x) = h(1 \otimes p^k x) = h(p^k x)$ which is finite. This contradiction completes the proof.

LEMMA 3.8. *If M is an R-module of rank* 1 *with no elements of infinite height, then M has the coset property.*

Proof. Let S be a finitely generated submodule of M, and let $x \notin S$. Then $R^* \otimes S$ is a finitely generated R^*-submodule of $R^* \otimes M$. We now show $1 \otimes x \notin R^* \otimes S$. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S & \longrightarrow & M & \xrightarrow{\alpha} & M/S & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} & & \\ 0 & \longrightarrow & R^* \otimes S & \longrightarrow & R^* \otimes M & \xrightarrow{\beta} & R^* \otimes (M/S) & \longrightarrow & 0 \end{array}$$

where the downward maps are $y \to 1 \otimes y$. Then

$$\beta(1 \otimes x) = \beta i(x) = j\alpha(x) = j(x + S) .$$

But

$$\gamma\colon (R^* \otimes M)/(R^* \otimes S) \longrightarrow R^* \otimes (M/S)$$

defined by

$$\gamma(r^* \otimes m + R^* \otimes S) = \beta(r^* \otimes m)$$

is an isomorphism. In particular,

$$\gamma(1 \otimes x + R^* \otimes S) = \beta(1 \otimes x) = j(x + S) .$$

Since $x \notin S$, $x + S \neq 0$. Since j is a monomorphism, by Lemma 3.6, $j(x + S) \neq 0$. Therefore $1 \otimes x + R^* \otimes S \neq 0$, i.e., $1 \otimes x \notin R^* \otimes S$. Hence $1 \otimes x + R^* \otimes S$ contains only finitely many distinct heights, by Lemma 3.3. Therefore the pure subset $x + S$ of $1 \otimes x + R^* \otimes S$ car contain only finitely many distinct heights, and so it has an element of maximal height.

We now sum up the results of this section in the following theorem.

THEOREM 3.9. *A semi-KM module is a KM module if any of the following conditions hold:*

(i) *R is complete and M has no elements of infinite height;*

(ii) *R is compact and rank $M = 2$;*

(iii) *R is complete and rank $M = 1$;*

(iv) *rank $M = 1$ and M has no elements of infinite height.*

It is an open question whether these are all the semi-*KM* modules with the coset property. Later we shall give an example of a module of rank 2 with no elements of infinite height over an incomplete ring which does not have the coset property.

4. The Structure theorem.

The main result of this section is the classification of all *KM* modules.

DEFINITION. A *strand* is a function from the cartesian product of s copies of R into the ordinals and the symbol ∞, where R is a DVR and s is finite.

DEFINITION Two strands f and $g\colon R \times \cdots \times R \to$ ordinals and ∞ are *equivalent*, denoted $f \equiv g$, in case there is an s by s non-singular matrix A over R and non-negative integers m and n such that $g(p^{m+n}(r_1, \cdots, r_s)) = f(p^n(r_1, \cdots, r_s)A)$ for all $r_i \in R$. The argument of f is obtained by regarding $(r_1, \cdots, r_s)$ as a 1 by s matrix.

It is easy to verify that $f \equiv g$ is an equivalence relation. If M is a reduced module of finite rank s, then any ordered independent set of elements $x_1, \cdots, x_s$ determines a strand f by $f(r_1, \cdots, r_s) = h(\Sigma r_i x_i)$. f is *the strand determined by the* x's. It is straightforward to see that two strands determined by different ordered maximal independent subsets of M are equivalent in the above sense. Thus M determines an equivalence class of strands, which we denote $S(M)$. Clearly $S(M)$ is an invariant of M.

LEMMA 4.1. *Let M and M' be KM modules. Let S and S' be finitely generated submodules of M and M' respectively, let f be a height-preserving isomorphism of S onto S', and let $x \in M$ with $px \in S$. Then f can be extended to a height-preserving isomorphism between $\{x, S\}$ and a suitable submodule of M' which contains S'.*

Proof. Exactly as in [4].

LEMMA 4.2. *Let M and M' be KM modules with $S(M) = S(M')$. Then there are maximal independent subsets in M and in M' which determine the same strand.*

Proof. Let $y_1, \cdots, y_s$ be independent in M with strand f; let $y'_1, \cdots, y'_s$ be independent in M' with strand g. Since $S(M) = S(M')$, $f \equiv g$. Hence there are non-negative integers m and n and a non-singular matrix (a_{ij}) over R such that

$$g(p^{m+n}(r_1, \cdots, r_s)) = f(p^n(r_1, \cdots, r_s)(a_{ij})) ,$$

i.e.,

$$h(p^{m+n}\Sigma r_i y'_i) = h(p^n \Sigma\Sigma r_i a_{ij} y_j) .$$

Set $x_i = p^n \Sigma a_{ij} y_j$ and set $x'_i = p^{m+n} y'_i$.

THEOREM 4.3. *Let M and M' be KM modules. M and M' are isomorphic if and only if they have the same Ulm invariants and $S(M) = S(M')$.*

Proof. By Lemma 4.2, there are maximal independent subsets $x_1, \cdots, x_s$ in M, $x'_1, \cdots, x'_s$ in M' such that $h(\Sigma r_i x_i) = h(\Sigma r_i x'_i)$ for all $r_i \in R$. Let S be the submodule of M generated by the x's and let S' be the submodule of M' generated by the x''s. Define $f: S \to S'$ by $f(x_i) = x'_i$. Since S and S' are free on generators x_i, respectively x'_i, f is a well-defined isomorphism. Moreover, our choice of generators makes f height-preserving. This isomorphism is now extended stepwise to an isomorphism of M and M' by Lemma 4.1. To ensure catching all of M and M', we take fixed countable sets of generators for each and alternate between adjoining an element of M and an element of M'. Since the elements of M and M' have finite order modulo S and S' respectively, we can suppose that at each step we are adjoining an element x such that px lies in the preceding submodule. This is precisely the situation of Lemma 4.1.

COROLLARY 4.4. *Let M and M' be isomorphic KM modules. Then any height-preserving isomorphism between finitely generated submodules S and S' of M and M' respectively (rank S = rank M) can be extended to an isomorphism of M with M'.*

As first applications of the structure theorem, we now solve a square-root problem and a cancellation problem.

THEOREM 4.5. *Let M and M' be KM modules of rank 1 with $M \oplus M \approx M' \oplus M'$. Then $M \approx M'$.*

Proof. It is a corollary of Ulm's theorem that the above is true when M and M' are torsion. Hence the torsion submodules of M and M' are isomorphic. Let $x \in M$ have infinite order. Then there is an element $(a, b) \in M' \oplus M'$ such that $h(rx) = h(ra, rb)$ for all $r \in R$. Since rank $M' = 1$, there are non-negative integers m and n such that $p^m a = p^n ub = y$, where u is a unit in R. We assume $m \geq n$. Thus, for large k, we have $h(p^k x) = h((p^{k-m}y, p^{k-n}u^{-1}y)) = h(p^{k-m}y)$. Hence $S(M) = S(M')$. Therefore, $M \approx M'$ by the structure theorem.

I have been unable to prove the analogous result in the case of higher rank, and I conjecture it is false.

THEOREM 4.6. *Let M and M' be KM modules, and let T be a reduced countably generated torsion module such that $U_\alpha(T)$ is finite for all α, where $U_\alpha(T)$ is the αth Ulm invariant of T. Then $T \oplus M \approx T \oplus M'$ implies $M \approx M'$.*

Proof.

$$S(M) = S(T \oplus M) = S(T \oplus M') = S(M') .$$

By Ulm's Theorem, we may cancel T to obtain that the torsion submodules of M and M' are isomorphic. By the structure theorem, $M \approx M'$.

$S(M)$ is a rather cumbersome invariant. We make the following definition in order to rephrase Theorem 4.4.

DEFINITION. Two modules M and M' are *almost isomorphic* if there exist torsion modules T and T' such that $T \oplus M \approx T' \oplus M'$.

THEOREM 4.7. *Two KM modules M and M' are isomorphic if and only if they are almost isomorphic and they have the same Ulm invariants.*

Proof. The necessity is obvious. For sufficiency, note that if M and M' are almost isomorphic, then $S(M) = S(M')$. Since M and M' have the same Ulm invariants, $M \approx M'$ by 4.3.

5. Modules over incomplete rings. At present we have a structure theorem for KM modules, and the only KM modules over incomplete rings that we know are those of rank 1 with no elements of infinite height. In Lemmas 3.6, 3.7, and 3.8, however, we saw that we could obtain information about a module M by examining $R^* \otimes M$, which we henceforth denote M^*. We now investigate this situation more closely.

LEMMA 5.1. *The rank of M as an R-module = the rank of M^* as an R^*-module.*

Proof. Rank $M \geq \operatorname{rank} M^*$, for if $x_1, \cdots, x_s$ is a maximal independent subset of M, then $1 \otimes x_1, \cdots, 1 \otimes x_s$ is a maximal independent subset of M^*. For the other inequality, let S be a free submodule of M with rank $S = \operatorname{rank} M$. Since R^* is torsion-free, exactness of $0 \to S \to M$ implies exactness of $0 \to S^* \to M^*$. Since tensor product commutes with direct sums, rank $M \leq \operatorname{rank} M^*$

LEMMA 5.2. *Let W be an R^*-module of finite rank s, with torsion submodule T. Let M and M' be R-modules of rank s contained in W satisfying*:

(i) $T \subset M \cap M'$;

(ii) *there is an independent subset $x_1, \cdots, x_s$ in $M \cap M'$;*

(iii) *if f is the strand determined by the x's in M, and if g is the strand they determine in M', then $f = g$. Under these conditions, $M = M'$.*

Proof. Let $x \in M$. Since rank $W = s$, $p^k x = \Sigma c_i x_i$, $k \geq 0$, and $c_i \in R^*$. But each $c_i \in R$, lest $\Sigma c_i x_i, x_1, \cdots, x_s$ are $s + 1$ independent (over R) element in M, contradicting rank $M = s$. Hence $p^k x \in M \cap M'$. In M, $h(\Sigma c_i x_i) \geq k$. By (iii), $h(\Sigma c_i x_i) \geq k$ in M' Thus there is a $y \in M'$ such that $p^k y = \Sigma c_i x_i$. Hence $p^k(x - y) = 0$, and so $x - y \in T$. Thus $x = y + (x - y) \in M'$. The other inclusion is proved similarly.

LEMMA 5.3. *Let M and M' be reduced R-modules; let $x_1, \cdots, x_s$ be a maximal independent subset in $M, x'_1, \cdots, x'_s$ a maximal independent subset of M' such that $h(\Sigma r_i x_i) = h(\Sigma r_i x'_i)$ for all $r_i \in R$. If $c_i \in R^*$, then $h(\Sigma c_i \otimes x_i) = h(\Sigma c_i \otimes x'_i)$ if either is finite; also, if one of these heights is infinite, so is the other.*

Proof. We shall be done if we can prove $h(\Sigma c_i \otimes x_i) \geq k$ implies $h(\Sigma c_i \otimes x'_i) \geq k$, for any finite k. Choose $r_i \in R$ such that $c_i - r_i \in p^k R^*$. Then $\Sigma c_i \otimes x_i = \Sigma(c_i - r_i) \otimes x_i + \Sigma r_i \otimes x_i$ Hence $h(\Sigma r_i \otimes x_i) \geq k$. By Lemma 3.6, $h(\Sigma r_i \otimes x_i) = h(\Sigma r_i x_i) = h(\Sigma r_i x'_i) = h(\Sigma r_i \otimes x'_i)$. Hence $h(\Sigma r_i \otimes x'_i) \geq k$. But $\Sigma c_i \otimes x'_i = \Sigma(c_i - r_i) \otimes x'_i + \Sigma r_i \otimes x'_i$. Thus $h(\Sigma c_i \otimes x'_i) \geq k$.

DEFINITION. Let M be a module with no elements of infinite height. M is *taut* if length M = length M^*; otherwise M is *slack*.

Note that length ([3, page 26]) may be defined for not necessarily reduced modules. Thus M is taut if and only if the reduced part of M^* has no elements of infinite height. It is an open question whether slack modules exist; it is easy, however, to give an example in which M has no elements of infinite height while M^* has a proper divisible submodule. Let M be an indecomposable torsion-free R-module of rank 2 of the type exhibited in [3, page 46]. M is reduced (and so has no elements of infinite height, being torsion-free), but $M^* \approx R^* \oplus Q^*$, Q^* being the quotient field of R^*.

LEMMA 5.4. *Any direct sum of taut modules is taut.*

LEMMA 5.5. *Any module of rank* 1 *with no elements of infinite height is taut.*

Proof. Lemma 3.7.

DEFINITION. A module is *completely decomposable* if it is the direct sum of modules of rank 1.

COROLLARY 5.6. *Any completely decomposable module with no elements of infinite height is taut.*

LEMMA 5.7. *Any reduced torsion-free module is taut.*

Proof. There is a unique solution to the equation $py = x$.

THEOREM 5.8. *Let M and M' be taut semi-KM modules. Then M and M' are isomorphic if and only if they have the same Ulm invariants and $S(M) = S(M')$.*

Proof. Since M and M' have isomorphic torsion submodules, so do M^* and M'^*, by 3.6. By 4.2 there are maximal independent subsets $x_1, \cdots, x_s$ in M, $x'_1, \cdots, x'_s$ in M' such that $h(\Sigma r_i x_i) = h(\Sigma r_i x'_i)$ for all $r_i \in R$. Since both M and M' are taut, the reduced parts of M^* and M'^* have no elements of infinite height. By 5.3, $\{1 \otimes x_i\}$ and $\{1 \otimes x'_i\}$ determine the same strand. In particular, the divisible parts of M^* and M'^* have the same rank and hence are isomorphic, since they are torsion-free. By 4.3, $M^* \approx M'^*$. By Corollary 4.4, there is an isomorphism $f: M^* \to M'^*$ such that $f(1 \otimes x_i) = 1 \otimes x'_i$ for all i. But now M' and $f(M)$ satisfy the conditions of 5.2. Hence $M' = f(M)$. Thus M and M' are isomorphic.

Theorem 5.8 suggests that taut modules have the coset property. We now exhibit a counter-example.

EXAMPLE 5.9. There exist taut modules which do not have the coset property.

Proof. Let M be an indecomposable torsion-free R-module of rank

2, where R is (necessarily) incomplete. M is taut, by 5.7. Let S be a pure submodule of rank 1. Since M is reduced, S must be cyclic. Further, $M/S \approx Q$. Thus S cannot be copure. Hence M does not have the coset property, by 3.2.

6. Completely decomposable modules. We begin this section with the study of the simplest completely decomposable modules: those of rank 1. We have already seen that if we assume no elements of infinite height, modules of rank 1 are taut. Using results of the last section, we can now prove a cancellation law.

THEOREM 6.1. *Let M and M' be semi-KM modules of rank* 1 *with no elements of infinite height. Then $M \approx M'$ if and only if $M^* \approx M'^*$.*

Proof. By 3.6. $M^* \approx M'^*$ implies that the torsion submodules of M and M' are isomorphic. If x has infinite order in M, x' has infinite order in M', then the strands determined by $1 \otimes x$ and $1 \otimes x'$ are equivalent. But equivalence for modules of rank 1 is via two non-negative integers and a one-by-one matrix over R^*, i.e., an element of R^*. But any element of R^* has the form up^k where u is a unit. Since multiplication by a unit does not alter heights, we may assume that the one-by-one matrix lies in R. But then we are calculating equivalence over R. The purity of the imbedding of M into M^* yields $S(M) = S(M')$. Hence $M \approx M'$.

If rank $M = 1$, then $S(M)$ has a representative $f\colon R \to$ ordinals and ∞, where $f(r) = h(rx)$ for some element x of infinite order. But we know that if r and r' are associates in R, then $f(r) = f(r')$. Hence f is completely determined by its values at $p^k, k = 0, 1, 2, \cdots$ Thus $S(M)$ can be looked upon as an equivalence class of sequences of ordinals. Indeed, these ordinal sequences are the extra invariant Kaplansky and Mackey discovered in their paper.

DEFINITION. A sequence of ordinals $\{\alpha_n\}$ *has a gap at* α_n if $\alpha_{n+1} > 1 + \alpha_n$.

LEMMA (Kaplansky). *If* $\{\alpha_n\}$ *is the Ulm sequence of* x ([3, page 57]), *and if* $\{\alpha_n\}$ *has a gap at* α_n, *then the* α_nth *Ulm invariant of* $M \neq 0$.

Proof. Since $h(p^n x) = \alpha_n$ and $h(p^{n+1}x) = \alpha_{n+1} > 1 + \alpha_n$, there is a $y \in M$ such that $h(p^n y) \geq \alpha_n$ and $p^{n+2}y = p^{n+1}x$. Set $t = p^{n+1}y - p^n x$. Then t has order p and height α_n. Thus the α_nth Ulm invariant of M is non-zero.

Suppose we are given a monotone increasing sequence of non-negative integers and a torsion module T. Is there a module of rank 1 possessing these as invariants? Kaplansky's lemma provides a link between these two objects, and the following theorem shows it is the only restriction.

THEOREM 6.2. *Let T be a countaly generated torsion module with no elements of infinite height; let $\{\alpha_n\}$ be a strictly increasing sequence of non-negative integers such that $\{\alpha_n\}$ has a gap at α_n implies $U_{\alpha_n}(T)$ is non-zero. Then there exists a KM module M of rank 1 whose torsion submodule is isomorphic to T and such that $S(M)$ is the equivalence class of $\{\alpha_n\}$.*

Proof. In this proof we often denote p^k by $\exp k$. If $\{\alpha_n\}$ has only a finite number of gaps, equivalence allows us to assume that $\alpha_n = n$ for all n. Then $M = T \oplus R$ is the desired module. Therefore we may assume $\{\alpha_n\}$ has an infinite number of gaps. Let $\{\alpha_{n_i}\}$ be the subsequence of gaps. The conditions on T imply T is the direct sum of cyclic modules. Further the compatibility condition tells us that T has a cyclic summand C_i of order $(\exp(\alpha_{n_i} + 1))$; let a_i be a generator of C_i. There is a B such that $T \approx B \oplus \Sigma C_i$. We first construct a certain submodule M' of ΠC_i.

Define $x = \{u_i a_i\}$ where $u_i = \exp(\alpha_{n_i} - n_i)$. x has infinite order; for $p^m x = 0 \Longleftrightarrow p^m u_i a_i = 0$ for all $i \Longleftrightarrow \exp(m + \alpha_{n_i} - n_i)a_i = 0$ for all $i \Longleftrightarrow m + \alpha_{n_i} - n_i \geq \alpha_{n_i} + 1$ for all $i \Longleftrightarrow m \geq n_i + 1$ for all i. This is impossible since $n_i \to \infty$. We claim that if $p^k u_i a_i \neq 0$, then $p^k u_i \in (\exp \alpha_k)$. In other words, if $k + \alpha_{n_i} - n_i < \alpha_{n_i} + 1$, then $k + \alpha_{n_i} - n_i \geq \alpha_k$. Equivalently, if $n_i \geq k$, then $\alpha_{n_i} - \alpha_k \geq n_i - k$. But $\alpha_{n_i} - \alpha_k = (\alpha_{n_i} - \alpha_{n_i - 1}) + \cdots + (\alpha_{k+1} - \alpha_k) \geq n_i - k$. Thus for each k we may define an element x_k with the property that $(\exp \alpha_k)x_k = p^k x$: set $x_k = \{u_i^k a_i\}$, where $u_i^k = 0$ if $k \geq n_i + 1$, while $u_i^k = \exp(-\alpha_k + k + \alpha_{n_i} - n_i)$ otherwise.

Set $M' =$ the submodule of ΠC_i generated by the x_k's. Note that $h(p^k x) = \alpha_k$ in M'. It can be no greater, since the height of an element of ΠC_i is the smallest power of p which occurs in one of its coordinates. Hence $h(p^k x) = \alpha_k$ in ΠC_i, and so can be no larger in the submodule M'.

We still must determine the torsion submodule T' of M'. Given any two x_k's, multiplication of each by a suitable power of p makes their coordinates equal from some point on. Hence any element of finite order in M' cannot have an infinite number of non-zero coordinates. But it may be verified that for all i,

$$a_i = \exp(\alpha_{n_{i+1}} - \alpha_{n_i} - n_{i+1} + n_i)x_{n_{i+1}} - x_{n_i} .$$

Hence $T' = \Sigma C_i$. Thus $M = M' \oplus B$ is the module we seek, where B

is the module we originally found satisfying $T' \oplus B \approx T$.

We now prove the existence of minimal modules possessing an element of a given Ulm sequence.

COROLLARY. *Let $\{\alpha_n\}$ be a strictly increasing sequence of non-negative integers, and let $\{\alpha_{n_i}\}$ be its subsequence of gaps. Let T be the direct sum of cyclic modules C_i of order* $(\exp(\alpha_{n_i} + 1))$. *Then there exists a KM module* 1 *with torsion submodule T and which contains an element x such that $h(p^n x) = \alpha_n$. Further, M is a direct summand of any KM module M' of rank* 1 *which contains an elements whose Ulm sequence is $\{\alpha_n\}$.*

Proof. We need only prove the last statement, since the existence of M with the prescribed invariants follows immediately from Theorem 6.2. Let T' be the torsion submodule of M'. By Kaplansky's lemma, the α_{n_i}th Ulm invariant of T' is non-zero. Hence there are cardinals U_n such that $U_n(T') = U_n + U_n(T)$. Let V be the torsion module with Ulm invariants given by U_n. By Ulm's theorem, $T' \approx V \oplus T$. The KM module $V \oplus M$ has torsion submodule $V \oplus T$ and $S(V \oplus M) = S(M')$. Hence $V \oplus M$ and M' are isomorphic, by the structure theorem.

Thus there is an uncountable number of non-isomorphic KM modules of rank 1 with no elements of infinite height. In particular we have exhibited modules of rank 1 which do not split.

DEFINITION. $x_1, \cdots, x_s$ is a *decomposition set for M* if it is a maximal independent subset of M and $h(\Sigma r_i x_i) = \min h(r_i x_i)$ for all $r_i \in R$. A *subdecomposition set* is a not necessarily maximal independent subset satisfying the above condition on heights.

DEFINITION. A decomposition set has *k gaps at level n* if k of its elements have Ulm sequences which have a gap at n.

LEMMA 6.3. *Let $X = x_1, \cdots, x_s$ be decompostion set with k gaps at level n. Then the the nth Ulm invariant of $M \geq k$.*

Proof. If $x_1, \cdots, x_s$ is a decompostion set for M, so is $r_1x_1, \cdots, r_sx_s$ where $r_i \neq 0$ for all i. Hence we may assume that $h(x_i) = n$ and $h(px_i) > n + 1$ for $i \leqq k$. Thus there are elements y_i, $i \leq k$, such that $h(py_i) \geq n + 1$ and $p^2 y_i = px_i$. Set $t_i = py_i - x_i$. We now have k elements of order p and of height n. It remains to prove that they are independent over $R/(p)$. Suppose $\sum_{i=1}^{k} r_i t_i = 0$, where r_i is either 0 or a unit in R. By the definition of the t_i, $\Sigma r_i(py_i - x_i) = 0$ which implies that $p\Sigma r_i y_i = \Sigma r_i x_i$. Since X is a decomposition set, $h(\Sigma r_i x_i) = \min h(r_i x_i) = n$ or ∞. But $h(p\Sigma r_i y_i) \geq n+1$. Hence $\Sigma r_i x_i = 0$. The independence of the x's implies that each $r_i = 0$; hence the t_i are independent over $R/(p)$.

THEOREM 6.4. *Let M be a taut semi-KM module. M is completely decomposable if and only if M contains a decomposition set.*

Proof. If M is completely decomposable, the assertion is trivial. Suppose M contains a decomposition set $x_1, \cdots, x_s$. Define functions U_i: non-negative integers $\rightarrow$ cardinals $\leq \aleph_0$, $i = 1, 2, \cdots, s$ as follows: $\sum_{i=1}^{s} U_i(n) = n$th Ulm invariant of M; if the Ulm sequence of x_i has a gap at n, then $U_i(n) \neq 0$. By Lemma 6.3, the Ulm invariants of M are sufficiently large to allow this construction. Let T_i be the torsion module with Ulm invariants given by U_i. By Theorem 6.2, there exists a KM module of rank 1, M_i, having torsion submodule T_i and with $S(M_i)$ the equivalence class of the Ulm sequence of x_i. Consider ΣM_i. Since Ulm invariants are additive, the first condition in the definition of the U_i coupled with Ulm's theorem yields the fact that the torsion submodules of M and of ΣM_i are isomorphic. Further, $S(M) = S(\Sigma M_i)$. By Corollary 5.6, ΣM_i is a taut semi-KM module. By Theorem 5.8, $M \approx \Sigma M_i$.

LEMMA 6.5. *Let M and M' be taut semi-KM modules of rank* 1 *such that $S(M) = S(M')$. Then M and M' are almost isomorphic.*

Proof. Let T and T' be the torsion submodules of M and M' respectively. Then $M \oplus T'$ and $M' \oplus T$ are isomorphic, by 5.8.

We now prove a technical lemma which will allow us to obtain our first direct summand theorem.

LEMMA 6.6. *Let M be a reduced module of finite rank s. Let $x_1, \cdots, x_s$ be a decomposition set such that each x_i has the same Ulm sequence. Suppose also that $x_i = w_{i1}a_1 + \cdots + w_{is}a_s$, and, for all i, $|w_{i1}| \leq |w_{11}|$. Under these conditions, $y_i = w_{i1}x_1 - w_{11}x_i$, $i \geq 2$, is a subdecomposition set and each y_i is in A, the submodule generated by $a_2, \cdots, a_s$.*

Proof. Rank $M = s$ while rank $A \leq s-1$. Hence not all the w_{i1} are 0 lest we have s independent elements $x_1, \cdots, x_s$ lying in A. Thus w_{11} is non-zero.

First we show the y_i's are independent. Suppose $\Sigma r_i y_i = 0$. Then $0 = (\Sigma r_i w_{i1})x_1 - \Sigma r_i w_{11} x_i$ which implies $r_i w_{11} = 0$ for all $i \geq 2$, since the x's are independent. Since $w_{11} \neq 0$, we must have $r_i = 0$ for all $i \geq 2$;

hence the y's are independent.

Next we show that the y's satisfy the required condition on their heights.

$$h(\Sigma r_i y_i) = h(\Sigma r_i w_{i1} x_1 - \Sigma r_i w_{11} x_i) = \min h(\Sigma r_i w_{i1} x_1), h(r_i w_{11} x_i) .$$

But

$$|\Sigma r_i w_{i1}| \leq \max |r_i w_{i1}| = \max |r_i||w_{i1}| \leq \max |r_i||w_{11}| = \max |r_i w_{11}| .$$

Hence there is an i such that $|\Sigma r_i w_{i1}| \leq |r_i w_{11}|$. Therefore, $h(\Sigma r_i w_{i1} x_1) \geq h(r_i w_{11} x_i)$ for that i. Hence $h(\Sigma r_i y_i) = \min h(r_i w_{11} x_i)$. On the other hand, $\min h(r_i y_i) = \min h(r_i w_{i1} x_1 - r_i w_{11} x_i) = \min h(r_i w_{i1} x_1), h(r_i w_{11} x_i)$. But for all i, $|r_i w_{i1}| \leq |r_i w_{11}|$. Therefore, $h(r_i w_{i1} x_1) \geq h(r_i w_{11} x_i)$. Hence $\min h(r_i y_i) = \min h(r_i w_{11} x_i)$. Hence $h(\Sigma r_i y_i) = \min h(r_i y_i)$.

THEOREM 6.7. *Let M be a completely decomposable semi-KM module with no elements of infinite height. Let $M = \Sigma M_i$, all the M_i isomorphic and of rank* 1. *If $M = A \oplus B$, then B is completely decomposable. In fact, B is almost isomorphic to a direct sum of copies of M_i.*

Proof. We first prove that any two elements in M of infinite order have equivalent Ulm sequences. Let $x_i \in M_i$ have infinite order. Clearly these x's form a decomposition set. Further, since all the M_i are isomorphic, we may assume that all the x_i's have identical Ulm sequences. Let $z \in M$ have infinite order. There is an $m \geq 0$ such that $p^m z = \Sigma r_i x_i$. Suppose $|r_i| \leq |r_1|$. Then $h(p^{m+k} z) = h(p^k \Sigma r_i x_i) = h(p^k r_1 x_1)$ for any non-negative k.

Choose $a_1, \cdots, a_{s-k}$ independent in $A, a_{s-k+1}, \cdots, a_s$ independent in B. We are now in the situation of the lemma. Applying the lemma k times (after each application, we must normalize the y's obtained so that they have identical Ulm sequences), we obtain $s - k$ independent elements in $\{a_{s-k+1}, \cdots, a_s\} \subset B$ which is a subdecompostion set of M. By the purity of B, and since rank $B = s - k$, these elements constitute a decomposition set for B. By Theorem 6.4, B is completely decomposable. Hence $B = \Sigma B_j$, and our initial remarks imply that $S(B_j) = S(M_i)$ for all i and j. By Lemma 6.5, B_j and M_i are almost isomorphic. Hence B is almost isomorphic to a direct sum of copies of M_i.

I have been unable to discover the truth of Theorem 6.7 in the event all the M_i are not isomorphic to each other.

COROLLARY 6.8. *Let $M = \sum M_\alpha$ (α in some index set), each M_α a semi-KM module of rank* 1 *with no elements of infinite height. If all the M_α are isomorphic, any direct summand B of M of finite rank is almost isomorphic to a direct sum of copies of M_α's.*

Proof. Let $x_\alpha \in M_\alpha$ have infinite order. Let $y_1, \cdots, y_s$ be a maximal independent subset of B. There is a finite subset $x_{\alpha_1}, \cdots, x_{\alpha_k}$ of the x_α's such that $p^m y_i$ lies in the submodule they generate, for all i. Let B' be the submodule of M generated by B and $M_{\alpha_1}, \cdots, M_{\alpha_k}$. Since B' is countably generated and of finite rank, $B' = \sum M'_{\alpha_j}$, where $S(M'_{\alpha_j}) = [Ux_{\alpha_j}]$. (If $x \in M$, Ux is its Ulm sequence and $[Ux]$ is the equivalence class of Ux). Hence all the $S(M'_{\alpha_j})$'s are the same. Since B is a direct summand of M, it is a direct summand of B'. By 6.7, B is completely decomposable. Since all elements of infinite order have equivalent Ulm sequences, B is almost isomorphic to a direct sum of copies of M_α.

We are now in a position to consider uniqueness of a decomposition of a module into the direct sum of modules of rank 1. The unpredictability of the torsion submodules does not allow one to find pairs of isomorphic summands from two different decompositions. For example, if C is cyclic of order (p) and $M = R \oplus C \oplus C \oplus R$, different associations yield different decompositions of M as a direct sum of modules of rank 1 whose terms are not pairwise isomorphic. However, the two decompositions do have isomorphic refinements.

THEOREM 6.9. *Let $M = \sum_{i=j}^n M_i$, M_i a KM module of rank 1, all the M_i isomorphic. Any two decompositions of M into summands of rank 1 have isomorphic refinements.*

Proof. We saw in the proof of Theorem 6.7 that any two elements in M of infinite order have equivalent Ulm sequences. Hence if $M = \sum_{i=1}^n M'_i$, all the M'_i of rank 1, then $S(M_i) = S(M'_i)$ for all i. By the existence theorem, there are modules N_i of rank 1 such that:

(i) $N_i \oplus T_i \approx M_i$, $N_i \oplus T'_i \approx M'_i$ for some torsion T_i, T'_i;

(ii) if W_i is the torsion submodule of N_i, then the Ulm invariants of W_i are 0's and 1's. Now $\sum W_i \oplus \sum T_i \approx \sum W_i \oplus \sum T'_i$. By Ulm's Theorem and condition (ii), we may cancel and obtain $\sum T_i \approx \sum T'_i$. Since any two decompositions of a module which is the direct sum of cyclic modules have isomorphic refinements, $\sum T_i$ and $\sum T'_i$ have isomorphic refinements. This completes the proof.

As a corollary, we have another proof of the square root problem, Theorem 4.5.

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