Kuroda, Sigekatu
Osaka Math. J.
11 (1959), 213–248.

An Investigation on the Logical Structure of Mathematics (*X*)[1] *Concepts and Sets*

To Zyoiti SUETUNA on his 60th Birthday

By Sigekatu KURODA

Introduction

In this Part (X) we shall make an attempt to analyse the concepts*[2] of concepts and of sets. The problem is: What dependent variables in UL can be considered as concepts and what as sets? Whether a dependent variable in UL is to be considered as a concept or as a set is not an intrinsic property of the dependent variable itself but an extrinsic one which varies depending on the environment in which the dependent variable is involved. A dependent variable should be considered as a set in some UL-proofs while as a concept in some others, or more in detail, it will be used as a concept in some "proof components" (§5), and as a set in some others, of the same UL-proof. It will be used as a set of some theories while as a concept of other theories. It may well be accepted to say that the norm of distinguishing sets from concepts consists in being freely dealt with as they were objects. On the other hand, what is conceived logically may well be looked upon as an object if it can be freely used as an instance of universally or existentially quantified independent variables.

---

1) This part was first intended to be included in Part (II). Because of the difficulty of the problem it was separated, and its first draft has recently been entirely revised. Nevertheless, the words "concepts" and "sets" used frequently in previous Parts in referring to this Part (X) coincide with the "$\mathcal{E}$-concepts" and "$\mathcal{E}$-sets", respectively, defined at the end of §1 of this Part. This Part presupposes only §1–19 of Parts (I) and (II), Hamb. Abh. **22** (1958), pp. 249–266 and **23** (1959), pp. 206–221. However, Parts (III) and (IV), Nagoya Math. J. **13** (1958); Parts (VI), (VII), and (VIII), ibid. **14** (1959); and Part (IX), this J. **11** (1959) are occasionally referred to in this Part.

This Part has not yet, I believe, taken its final form and contains many problems which should be treated in other occasions. But the writer dare publish this belated Part in this present form.

2) The asterisk attached to the word concept in this Part means that the word "concept*" belongs to the meta-metalanguage, while "concept" without asterisk to the metalanguage of which the object-language is UL.

Assuming that the above norm would be accepted, a dependent variable occurring in a UL-proof P can surely be considered as a concept in P if it is substituted nowhere in P for bound variables. Such a variable, if it is used in the proof only as right-hand variables of $\in$, is clearly serving in the proof as an abbreviation of the property that is expressed by the definiens of the dependent variable, so that the dependent variable can be eliminated from the proof by replacing definiendum by definiens[3].

However, there are cases in which a concept is regarded as a concept of other concepts so that they are likely to be substituted, as its elements, for the element variable of its defining formula. In such cases also there must be some appropriate elimination procedures of such a concept in order that it may be regarded as a concept. So in § 1 we state a general definition of "sets and concepts" in a UL-proof as well as of a theory with respect to an unspecified species of "elimination transformations" which is assumed to be given in some way or other, while in the rest of this Part (X), except in the case of general consideration, we confine ourselves to dealing with a specified and very simple species $\mathcal{E}$ of elimination transformations, defined at the end of § 1 by (A) and (B). Thus, our concern in this Part is mainly the "$\mathcal{E}$-sets" and "$\mathcal{E}$-concepts" in a UL-proof as well as of a theory.

In § 2 "notions" in a UL-proof are defined as constants that occur in the UL-proof and fulfil the conditions (α_0), (β_0), and (γ_0) stated in § 2 and it is proved (Theorem 1) that a notion in a UL-proof is an $\mathcal{E}$-concept. The proof of Theorem 1 is referred to repeatedly afterwards so that the proof is divided into paragraphs indicated by numbers.

Notions are constants. In § 3 the conditions (α_0), (β_0), and (γ_0) are so weakened that the similar elimination procedures are applicable to dependent variables which are not constants. So we define in § 3 by (α_1), (β_1), and (γ_1) the species of "notional variables"[4] and it is proved (Theorem 4) that a species of notional variables in a UL-proof is a "species of $\mathcal{E}$-concepts".

3) In this Part not only the formula F^u in the defining formula $\forall x. u \in p \equiv F^u$ of p is called the definiens of p but also formulas F^m and $\neg F^m$ with any m are called the definiens of the definiendum $m \in p$ and $m \notin p$ respectively.

4) In applying an analytic method to a certain problem we have to begin with the precision of the definitions of the concepts* we are concerned with, no matter whether these concepts* may have some other precise definitions or may be conceived *distinctly* but *without* any precise definitions. When such a concept* undergoes analysis, it may split into a finite or an infinite number of different concepts* or it may come to be distorted. The concept* of "notions" defined in § 2 coincides fairly well with the concept* of notions we are acquainted with in logic. The concept* of "notional variables" is a technical generalization of the concept* of notions.

We have a great deal of option in choosing the species of elimination transformations as well as in selecting the conditions which are sufficient for a species of dependent variables to be eliminable by the species of elimination transformations. The species $\mathcal{E}$ of elimination transformations and the conditions given in § 2 and § 3 are only instances of a number of such options.

In § 4 "set variables" and "concept variables" are, in an unspecified way, defined independently of a UL-proof, and it is proved (Theorem 5) that under the condition (α_1) all the non-set (concept) variables are $\mathcal{E}$-eliminable from a UL-proof "relative to set variables", that is to say, $\mathcal{E}$-eliminable if we allow any set variables to enter anew the proof.

In § 5 "coarser and finer coherent components" of a UL-proof are defined in order to be able to apply our $\mathcal{T}$-elimination procedures not only to the whole proof but also to a suitably selected part of a proof, namely to a "$\mathcal{T}$-elimination (finer) component".

In § 6 "coherent cycles" in a UL-proof are defined because of the importance of the bearing on elimination of variables. Namely, in weakening the condition (α_1) further to (α_2) the existence of coherent cycles in a UL-proof gives a necessary and sufficient conditions for some dependent variables to be eliminated from a proof by $\mathcal{E}$ (Theorem 6, 7 and Corollary to Theorem 7).

In § 7 the repetition of elimination procedures is discussed. The eliminations of $\mathcal{T}$-concepts from a UL-proof are to be executed by successive application of suitably selected procedures in order to fit our purpose. There is no definite way to treat the elimination for all cases, as if there were no definite method of assuring limes processes in analysis. We have only to prepare a number of procedures which are practical to our various purposes. Among others, Theorem 8 concerning cut is effectively used during the repetition of elimination procedures.

Since the coherent cycles are closely related to predicativity and impredicativity, we are thus automatically led to the problem of clarifying impredicativity. Our system UL has a kind of universal variables as independent variables so that at the beginning there is no distinction between impredicative and predicative set- (or concept-) formations. For instance, the totality of natural numbers is given by an impredicative defining formula from the type theoretical point of view. We are thus not in a position to distinguish predicative and impredicative definitions.

The concepts* of impredicativity may be, roughly speaking, looked upon as a logical circle, if not vicious, which the defining procedures give rise to *with respect to bound variables.* There is no fear of vicious circles[5] to enter the defining procedures of dependent variables since these procedures are recursively defined. But circles would exist, if we, for the moment, made it a principle that the quantified variables in a defining formula of a dependent variable should not range over the dependent variable or dependent variables which are defined by using itself. Even when we would avoid such circles our recursive way of defining procedures would be allowed as they are. Circles in the above sense would enter really, only when in a UL-proof a dependent variable is substituted for a bound variable of the defining formula of a dependent variable in spite of the latter being presupposed by the former in the defining procedures. Interpreting the impredicativity as occurrences of such circles, we define a "Γ-impredicative proof" in § 8, when a species Γ of dependent variables participates in such circles. A Γ-impredicativity is "$\mathfrak{T}$-eliminable" or "$\mathfrak{T}$-essential" according as it is eliminable by a species $\mathfrak{T}$ of elimination transformations or not.

Our definition of impredicativity is based on the occurrence of circles of the above mentioned kind and this will be a natural, if very weak, interpretation of the usual concept* of impredicativity by our system. But, it is entirely another thing what significance, with respect to the problem of truth, this definition of impredicativity has. It seems that, no matter how we might define the concept* of impredicativity with the view of clarifying the circles which might be caused by the bound variables in the manner above alluded to, some kinds of impredicativity concern contradictions, and other kinds, such as impredicative mathematical inductions (§ 10 (vii)), turn out to be legitimate. The bearing of the coherent cycles on the problem of truth is just the same as in the case of impredicativity, which is shown by examples given in § 10.

The concept* that has a direct clear significance with respect to the problem of truth is rather simply the $\mathfrak{T}$-eliminability and non-$\mathfrak{T}$-eliminability. It might have been better, therefore, if we aimed only at adapting ourselves faithfully to the problem of truth, to use $\mathfrak{T}$-impredicativity as a synonym of non-$\mathfrak{T}$-eliminability. But this is a way too much deviated from the traditional usage of the concept* of impredicativity.

On the contrary, the predicativity is a concept which has been used in having a direct close relation to truth. The predicative mathmatics can be regarded as founded by its constructive method. Some formal systems have been proved, as is well known, to be consistent as far as it is predicative in the sense of type theory. So we idendify (§ 8) the

5) By vicious circles are meant here only those which are caused when a dependent variable p is defined by using a dependent variable which presupposes p itself.

$\mathfrak{T}$-eliminability with $\mathfrak{T}$-predicativity[6], including $\mathfrak{T}$-eliminable impredicativity.

In § 9 the relation of elimination of variables with consistency is discussed and in § 10 some examples and applications are given.

It is to be noted that in the metalogical discussions we use intuitive logic or mathematics as in previous Parts, although some notations and sentences are expressed for simplicity in a formal style.

In Appendix two fragments on intuitive knowledge and on defining formulas are added.

1. Concepts and sets in a UL-proof

We shall first define the two metalogical concepts*, i. e. "concepts" and "sets" in a UL-proof. They are defined with respect to a metalogically given species $\mathfrak{T}$ of transformations which are applicable not only to a UL-proof but also to a figure which itself is not necessarily a UL-proof but a result obtained from a UL-proof by a successive application of a finite number of transformations of $\mathfrak{T}$: so the terminologies "$\mathfrak{T}$-concepts" or "$\mathfrak{T}$-sets" in a UL-proof.

The species $\mathfrak{T}$ consists of two kinds of transformations: ***principal*** transformations and *auxiliary* ones. A sequence of transformations $\rho_1, \cdots, \rho_k, \tau_1, \cdots, \tau_l$, consisting of a non-empty sequence $\rho_1, \cdots, \rho_k$ $(k \geq 1)$ of principal transformations of $\mathfrak{T}$ followed by a possibly empty sequence $\tau_1, \cdots, \tau_l$ $(l \geq 0)$ of auxiliary transformations of $\mathfrak{T}$, is called for the moment ***$\mathfrak{T}$-normal*** with respect to a UL-proof P, if $P^{\rho_1 \cdots \rho_k \tau_1 \cdots \tau_l}$ is a UL-proof, which we denote simply by P', and further if the top sequence $\neg\sigma'$, H' of P' is related to the top sequence $\neg\sigma$, H of P in the following way. The conclusion H' of P' is the formula obtained from the conclusion H of P by the transformation $\rho_1 \cdots \rho_k \tau_1 \cdots \tau_l$. Any defining formula D' in σ' is the formula obtained by the transformation $\rho_1 \cdots \rho_k \tau_1 \cdots \tau_l$ applied to a defining formula D in σ; the definiens of D may remain the same as that of D' or may be affected by the the transformation. Some defining formulas in σ may disappear by the transformation. The formula

$$\text{(I)} \qquad \forall xyz.\ x=y \wedge x \in z \rightarrow y \in z\,,$$

if contained in σ, either remains the same in σ' or disappears by the transformation, or else (I) may be added in σ' when it is lacking in σ. The sequence of auxiliary transformations $\tau_1, \cdots, \tau_l$ has the effect of changing the figure $P^{\rho_1 \cdots \rho_k}$ into a correct UL-proof $P^{\rho_1 \cdots \rho_k \tau_1 \cdots \tau_l}$.

6) This definition of $\mathfrak{T}$-predicativity, or more in detail, $\mathfrak{T}$-Γ-predicativity (§ 8) seems to be in accord with the usual usage of the concepts* of predicativity, if $\mathfrak{T}$ and Γ are determined suitably.

A dependent variable p in a UL-proof p is called *$\mathcal{T}$-eliminable* from P, if P is transformed, by successive applications to P of a finite number of $\mathcal{T}$-normal sequences of transformations, to a UL-proof, say P^*, such that (i) p does not occur in P^* and (ii) any dependent variable, different from p and occurring in P with a definiens G, does not occur in P^* or does occur in P^* with a definiens G^* which is proved to be equivalent to G by using as premises only the premises of P and eventually the formula (I). When p is $\mathcal{T}$-eliminable from p, p is called *properly* (*improperly*) *$\mathcal{T}$-eliminable* from P if the definiens of no (some) dependent variables remaining in P^* are affected by the transformation of P to P^*.

A dependent variable p in a UL-proof P is called a *proper* (*improper*) *$\mathcal{T}$-concept* in P, if p is properly (improperly) $\mathcal{T}$-eliminable from P. Proper and improper $\mathcal{T}$-concepts together are called *$\mathcal{T}$-concepts* in P.

A dependent variable p in a UL-proof P is called a *$\mathcal{T}$-set* in P if it is metalogically known that p is not $\mathcal{T}$-eliminable from P. An improper $\mathcal{T}$-concept in P is called also an *auxiliary $\mathcal{T}$-set* in P, if the elimination of the $\mathcal{T}$-concept affects the definiens of at least a $\mathcal{T}$-set in P. A $\mathcal{T}$-set is called a *proper $\mathcal{T}$-set*, if necessary.

A species Π of dependent variables in a UL-proof P is called *simultaneously $\mathcal{T}$-eliminable* from P, if by an application of a finite number of $\mathcal{T}$-normal sequences of transformations the proof P is transformed to a UL-proof P^* in which no Π-variables occur and the definiens of any dependent variable remaining in P^* is proved to be equivalent to the definiens in P of the variable by using only the premises of P and perhaps (I) as premises. Π is called *simultaneously properly $\mathcal{T}$-eliminable* from P if Π is simultaneously $\mathcal{T}$-eliminable from P and if thereby the definiens of any dependent variables which remain in P^* are not affected by the transformation; *simultaneously improperly $\mathcal{T}$-eliminable* if some of them are affected. A species Π of dependent variables in a UL-proof P is called a *species*, a *proper species*, or an *improper species*, of *$\mathcal{T}$-concepts* in P if Π is simultaneously $\mathcal{T}$-eliminable, simultaneously properly or improperly $\mathcal{T}$-eliminable from P, respectively. It may happen that Π and Π' are species of $\mathcal{T}$-concepts (of proper $\mathcal{T}$-concepts) in P but the union $\Pi \cup \Pi'$ is not. A species Σ of dependent variables in a UL-proof P is called a species of $\mathcal{T}$-sets in P, if for any Σ-variable p it is known that p is a $\mathcal{T}$-set in P. A species $\mathcal{T}$ of transformations which is metalogically defined so as to have the above described properties is called a species of *elimination transformations*.

As the principal transformations of a most fundamental species $\mathcal{E}$ of elimination transformations, we use the following two replacements (A) and (B):

(A) Replacement of a formula of form $m \in p$ by F^m;

(B) Replacement of a formula of form $p \in m$ by $\exists x.\ x = p \wedge x \in m$[7].

In (A) the variable p is a dependent variable, m any variable, and

$$(\mathrm{D}p) \qquad \forall u.\ u \in p \equiv F^u$$

is the defining formula of p. In (B) the variables p and m are dependent or independent variables.

The auxiliary transformations of $\mathcal{E}$ will be defined later in such a way that by these auxiliary transformations the proof properties which are lost by the successive applications of principal transformations of $\mathcal{E}$ can be restored in a natural manner.

Now we shall investigate various conditions for a dependent variable to be an $\mathcal{E}$-concept in a proof.

2. Notions in a proof

A *constant* p occurring in a UL-proof P is called a *notion* in the proof if the following conditions (α_0), (β_0) and (γ_0) are fulfilled.

(α_0) *p is substituted in P for no bound variabes except the variable z of the formula* (I)[8].

(β_0) *If a P-constituent $\overline{\neg G^m}$ is associated with a P-formula $\neg \forall x G^x$, no dependent variable q^x occurs in G^x such that q^m is p.*

(γ_0) *In the definiens F^u of p occurs no dependent variable which depends on the element variable u.*

By (α_0) the substitution of p for a bound variable is not allowed in P except the association of a P-constituent

$$[p\mathrm{I}] \qquad \overline{\neg \forall xy.\ x = y \wedge x \in p \rightarrow y \in p}$$

with the P-premise $\neg$(I). We call the proof constituent $[p\mathrm{I}]$ the *extensionality principle with respect to the variable p.*

Now we shall prove the following theorem.

THEOREM 1. *A notion p in a proof P is an $\mathcal{E}$-concept in P.*

Proof. The principal transformations we can use are the transformations (A) and (B) stated in §1. The auxiliary transformations are formulated below at the end (1·7) of the proof of Theorem 1.

(1·1) If p occurs in P as left-hand variables we replace by (B) all the subformulas of P-formulas of form $p \in m$ by $\exists x.\ x = p \wedge x \in m$. Let P' be the figure thus obtained from P.

7) In order to prove the equivalence of $p \in m$ with $\exists x.\ x = p \wedge x \in m$ the formula (I) is necessary.

8) This assures that the notion p is considered extensionally. See below.

(1·2) The figure P' is a UL-proof. This is seen by the fact that the occurrences of p in P as left-hand variables are by virtue of (α_0) and (β_0) only possible when such occurrences are inherited from the occurrences in the definiens of some dependent variables, in the conclusion of P, or in some cut formulas. (Therefore, such occurrences of p in P are not essential, that is to say, we obtain a correct UL-proof from P if we replace all the p's in P as left-hand variables by any arbitrarily fixed variable which depends on no variable bound in P)

In order to see precisely that P' is a UL-proof, we should verify the following properties of P', each of which is evident by the above remark : (i) association property, (ii) cancelling property, (iii) independent variable restriction, (iv) dependent variable restriction, (v) that a cut in P changes into a cut in P', (vi) that the conclusion of P changes into that of P', (vii) that the premise of P changes into that of P', in particular, the invariance of the top formula $\neg$(I), if any, and the closedness of the premise of P'.

Some definiens in P may be affected by the above replacements.

(1·3) P' preserves obviously the conditions (α_0) (β_0), and (γ_0) and p does not occur in P' as left-hand variables.

(1·4) By (A) we replace all the subformulas of P'-formulas of form $m \in p$ by F^m. Let Q be the figure thus obtained from P'. Since p occurs in P' only as right-hand variables, p does not occur in Q.

(1·5) We shall amend Q so as to restore the lost proof properties.

(1·5·1) Without loss of generality we can assume that P' had the primitive cancelling property[9].

(1·5·2) It is evident that the cancelling pair of any string of P' changes into a (not necessarily primitive) cancelling pair of the corresponding string of the figure Q. The formula (I), if there is in P', remains invariant in Q. It is also evident that any cut in P' changes into a cut in Q.

(1·5·3) The defining formula (Dp) of p changes in Q into

$$\forall u.\ F^u \equiv F^u. \tag{1}$$

We erase the top formula $\neg$(1) from Q. Since (Dp) is not primitive, the cancelling property of Q is not destroyed by our assumption (1·5·1).

(1·5·4) The association property is surely preserved in Q except the places where the association of P-constituent $\overline{\neg G^l}$ with some l with a P-formula of form $\neg \forall x G^x$ took place in P.

9) Even when P has the primitive cancelling property, P' may lose it because of the replacement (B). But P' restores the primitive cancelling property if we add, in necessary cases, the successive decomposition of imprimitive cancelling pairs.

(1·5·5) By (α_0) and (β_0) the associations of the above mentioned form are also preserved in Q except the places where in P' either one of the P'-constituents

$$[p\mathrm{A}] \qquad \overline{m \notin p \qquad F^m},$$

$$[p\mathrm{N}] \qquad \overline{m \in p \qquad \neg F^m}$$

was associated with the defining formula of p and except the places where in P' was used the extensionality principle

$$[p\mathrm{I}] \qquad \overline{\neg \forall xy.\ x=y \wedge x \in p \rightarrow y \in p}$$

with respect to p.

(1·5·6) In the former case the constituents $[p\mathrm{A}]$ and $[p\mathrm{N}]$ change in Q into the cuts $\overline{\neg F^m \quad F^m}$ and $\overline{F^m \quad \neg F^m}$ respectively and the defining formula of p, which becomes (1) in Q, has been erased by (1·5·3) from Q. Hence in these places the association property is restored in Q.

(1·5·7) As for the latter, the extensionality principle $[p\mathrm{I}]$ changes in Q into

$$(2) \qquad \overline{\neg \forall xy.\ x=y \wedge F^x \rightarrow F^y}\,.$$

In order to recover the association property in these places of Q, we replace every Q-constituent (2) by the cut

$$(3) \qquad \overline{\forall xy.\ x=y \wedge F^x \rightarrow F^y \quad \neg \forall xy.\ x=y \wedge F^x \rightarrow F^y}$$

and place under the right cut-formula of (3) the part of Q which is under (2). We place further under the left cut-formula of (3) the proof of the cut formula, which runs as follows:

(4)

–	$\neg \forall xy.\ x=y \wedge F^x \rightarrow F^y$
$(r.s)$	
1	$r \neq s$
2	$\neg F^r$
3	F^s
	⇡

where r and s are eigen variables. In the part denoted by ⇡ we decompose successively the formulas 2 and 3 in (4) as follows. If F^s is of form $\forall x G^{s,x}$ we place under F^s in (4) the constituents $\overline{G^{s,w}}$ and $\overline{\neg G^{r,w}}$ associated with 3 and 2 in (4) respectively. If F^s is of form $A^s \wedge B^s$ we place under F^s in (4) the constituents $/\neg A^r, \neg B^r$ and $\overline{A^s \quad B^s}$ associated with 2 and 3 respectively. If F^s is of form $\neg \forall x G^{s,x}$ or $\neg .\ A^s \wedge B^s$, then we treat symmetrically with respect to r and s. Proceeding in this way successively, some strings become to contain cancelling pairs[10], or else we arrive at the primitive constituents of the following form:

$$(5)\quad \frac{\overline{r\notin m}}{s\in m} \qquad\qquad (5')\quad \frac{\overline{s\notin m}}{r\in m}$$

$$(6)\quad \frac{\overline{m\notin r}}{m\in s} \qquad\qquad (6')\quad \frac{\overline{m\notin s}}{m\in r},$$

where m is an independent or dependent variable. Note the condition (γ_0). For the symmetric reason we consider only (5) and (6). The formula $s\in m$ in (5) and the formula $m\in s$ in (6) are inherited from a subformula of form $s\in m'$ and $m'\in s$ of F^s respectively where m and m' are isological each other since in the above decomposition of F^s and $\neg F^r$ only eigen variables are substituted for bound variables. In order to complete the proof (4) we go on as follows in each case:

(1·5·7·1) Under the figure (5) we have only to use the extensionality principle

$$[m\mathrm{I}] \qquad \overline{\neg\forall xy.\ x=y\wedge x\in m\rightarrow y\in m}$$

with respect to m.

(1·5·7·2) Under the figure (6) we have only to use the association with the formula 1 in (4):

$$(7)\qquad \frac{\overset{(1)}{\overline{\qquad\qquad}}\quad -\ \neg\forall x.\ x\in r\equiv x\in s}{m\in r\not\equiv m\in s}\quad .$$

(1·5·7·3) In $[m\mathrm{I}]$ and (7) we have substituted m for a bound variable, namely for z of (I) and for x of the equality 1 in (4), respectively. Only these m's are the dependent variables perhaps substituted in the proof (4). Note that m is different from p.

(1·5·7·4) Thus the association property is restored in Q at the problematic places where $[p\mathrm{I}]$ took place in P.

(1·6·1) Let P^* be the figure thus obtained. The association property is preserved everywhere in P^*. P^* has also the cancelling property, since the cancelling property of Q is not destroyed by the transformation of Q to P^*. P^* preserves evidently the independent variable restriction.

(1·6·2) The dependent variable restriction is preserved in P^* in particular by virtue of (γ_0). For, first, in the replacement of $m\in p$ by F^m

10) Namely when one of A^s and B^s has come not to depend on s.

the dependent variables occurring in F^m occur already in P, namely m occurs naturally in $m \in p$ and those different from m occur in the definiens F^u of p by virtue of the assumption (γ_0) so that these variables have their defining formulas in P^*-top sequence, and second, no new dependent variables enter the proof (4) of the left cut-formula of the cut (3), as is already mentioned in (1·5·7·3).

(1·6·3) The closedness of the premise of P^* is clear, although some definiens occurred in the original proof P are perhaps affected. The affected definiens are proved by virtue of (γ_0) to be equivalent to the corresponding original ones only by using the premises of P and eventually the formula (I).

(1·7) The transformations used as auxiliary transformations of $\mathcal{E}$ during the process of the above proof are as follows:

(i) Erasing a formula.

(ii) Replacement of a constituent $\overline{\neg K}$ carrying a formula by the cut $\overline{K \neg K}$ when the cut is ordinary and placing under K the proof of the cut formula K.

(1·8) Thus the $\mathcal{E}$-eliminability of p from P is proved under the assumptions (α_0), (β_0), and (γ_0), and the proof of Theorem 1 is complete.

As for the simultaneous $\mathcal{E}$-elimination of notions from a proof we have the following theorems:

THEOREM 2. *Let P be a* UL*-proof. If $p_1, \cdots, p_k$ are notions in P, then $p_1, \cdots p_k$ are simultaneously $\mathcal{E}$-eliminable from P.*

From Theorem 2 immediately follows:

THEOREM 3. *All the notions in a* UL*-proof P are simultaneously $\mathcal{E}$-eliminable from P.*

Remark: The principal transformations (A) and (B) of $\mathcal{E}$ in §1 are stated with respect to p. If we denote these transformations by (A; p) and (B; p) respectively, then the principal transformations needed in proving Theorem 2 are (A; p_i) and (B; p_i) for $1 \leq i \leq k$.

Theorem 2 is a special case of Theorem 4 proved in §3, so we omit the proof of Theorem 2.

3. Notional variables in a UL-proof

We use in this §3 following terminologies concerning dependent variables, some of which are defined in Part (I). Let T be the tree of variables for a dependent variable p. A dependent variable occurring in T (including the top variable p of T) is called *subordinate* to p. The order of p was defined there as the length of the tree T. If p is homological to q, then p and q have the same order.

Assume that p depends on an indepenent variable x : $p=p^x$. In this case an upper part S' of a T-string S is called an *x-string* of T, if all the variables in S' depend on x and moreover either if S' is the whole string S of T or if no variable situated in T directly under the bottom variable of S' depends on x. (Note that the tree of variables of a dependent variable p is constructed from the recursive description of p.)

Let now Π be a species of dependent variables occurring in a UL-proof P. The species Π is called a species of *notional variables* in P if the following conditions (α_1), (β_1), and (γ_1) are fulfilled.

(α_1) *Any variable of* Π *is substituted in* P *for no bound variables except the variable* z *of the premise* (I).

(β_1) *Let a P-constituent* $\overline{\neg G^l}$ *be associated with a P-formula* $\neg\forall x G^x$ *and* q^x *be a dependent variable occurring in* G^x *and depending on* x. *Then either both* q^x *and* q^l *do not belong to* Π *or both do. In the latter case for any dependent variable* k^x *in any x-string of the tree of variables for* q^x *either both* k^x *and* k^l *do not belong to* Π *or both do.*

(γ_1) *No dependent variable occurring in the definiens of a dependent variable belonging to* Π *depends on its element variable.*

THEOREM 4. *Let* Π *be a species of notional variables in a* UL-*proof* P. *All the* Π-*variables are simultaneously* $\mathcal{E}$-*eliminable from* P.

Proof. The proof of Theorem 1 with some changes gives that of Theorem 4, if in the proof of Theorem 1 the conditions (α_0), (β_0), and (γ_0) are replaced by the conditions (α_1), (β_1), and (γ_1) respectively and the variable p is regarded as any arbitrary Π-variable. We shall describe the necessary changes with some additional remarks. The numbers ($4\cdot n$) in the following correspond to the paragraph ($1\cdot n$) of the proof of Theorem 1.

($4\cdot1$)-($4\cdot3$) The same as in ($1\cdot1$)-($1\cdot3$). It is only to be noted that we need only the first part of (β_1) in ($4\cdot2$).

($4\cdot4$) The transformation of P to Q goes on, instead of ($1\cdot4$), as follows. Let $p_1, \cdots, p_k$ be the sequence of all the Π-variables, occurring in P and not isological each other. For any one j ($1\leq j\leq k$) we replace all the occurrences in P' of form $m\in p_j^*$ by F_j^m, where p_j^* is a variable isological to p_j and F_j^u is the definiens of p_j^*. Let P'' be the figure thus obtained from P'. In P'' the variables isological to p_j do not occur since m in $m\in p_j^*$ is not a Π-variable and no dependent variable in F_j^u depends on u by (γ_1). Some of the definiens of p_i ($i\neq j$, $1\leq i\leq k$)

remaining in P'' may be affected but the condition (γ_1) is fulfilled for these p_i and the variables isological to p_i occur in P'' also only as right-hand variables. No new dependent variables enter P''. For any remaining p_i we proceed in the same way, getting a figure P''' from P''. Proceeding sucessively in this way we get a series of figures $P', P'', \cdots, P^{(k+1)}$. Put $Q=P^{(k+1)}$. No Π-variables occur in Q.

(4·5) The procedures of transforming Q into a UL-proof are quite parallel to (1·5·1)-(1·5·7·4). We note only the following. The association in P' of a P'-constituent $\overline{\neg G^l}$ with a P'-formula $\neg\forall x G^x$ may be destroyed in the intermediate $P^{(i)}$. But these possible destructions are finally restored in $Q=P^{(k+1)}$ by virtue of (β_1).

In this connection it is also to be noted that after a [pA] or [pN] for some Π-variable p changes into a cut, the cut remains as cut in the later stage of transformations since both cut formulas are affected in the same way. This remark is also available for any cut in P' and for any cancelling pair in P'.

No Π-variables occur in the formulas F^x and F^y in (2) which occurs in Q and which is the result of transforming a [pI] in P' with respect to a Π-variable p. Hence the transformation of Q to a correct UL-proof is achieved just in the same way as in (1·5) and the rest of the proof of Theorem 4 runs in the same way as in the proof of Theorem 1.

Remark 1. The species Π of all the notional variables in P is a proper species of $\mathcal{E}$-concepts in P, exactly if no Π-variables occur in the definiens of non-Π-variables.

Remark 2. By the above proof it is seen that the $\mathcal{E}$-elimination of Π-variables in Theorem 4 can be achieved regardless of the orders of eliminations of Π-variables.

4. Set variables and concept variables

Let m be any variable and p^x be a dependent variable which depends on x. We use, as usual, the notation p^m in order to denote the dependent variable p^x_m which we obtain when we substitute m for x in p^x. A species Σ of dependent variables is called a *set species*[10a)], if for any m, p^x, and p^m it holds that p^m belongs to Σ, exactly if both m and p^x belong to Σ. If Σ is a set species, the species of all dependent variables which do not belong to Σ is called the *concept species* with respect to Σ. When a decomposition of the species of all the dependent variables into a set species Σ and a concept species Π is given, Σ-variables and Π-variables

10a) This has, of course, no relation to Brouwer's Mengenspezies (spread species).

are called respectively *set variables* and *concept variables* with respect to this decomposition[11].

Hereafter, we assume that a set species Σ is given and Π is the concept species with respect to Σ, and we regard the species Π in the conditions (α_1), (β_1), and (γ_1) as representing this concept species so that Π is defined independently of a given UL-proof P. Then, the condition (β_1) follows from (α_1). For, if a P-constituent $\neg G^l$ is associated with a P-formula $\neg \forall x G^x$ and q^x is a dependent variable occurring in G^x, then l is by (α_1) a set variable or an independent variable, so that by the definition of Π the variables q^x and q^l are both set variables or both concept variables. In the latter case the same reasoning shows that the latter condition of (β_1) is also fulfilled. Thus from Theorem 4 follows that if Π is the concept species with respect to a set species Σ and if (α_1) and (γ_1) are fulfilled for a UL-proof P and the species Π, then all the Π-variables occurring in P are $\mathcal{E}$-eliminable from P. Further, we shall drop the assumption (γ_1) and prove Theorem 5 below. Before doing this we shall consider what will happen when we drop the assumption (γ_1).

Let p_0 be a Π-variable occurring in P and $\forall u_0 . u_0 \in p_0 \equiv F_0^{u_0}$ be the defining formula of p_0. When (γ_1) is not assumed, a dependent variable, say p_1, which depends on the element variable u_0 of $F_0^{u_0}$ may occur in the definiens of p_0, so that the defining formulas of p_0 and p_1 are, if these occurrences are explicitly written, as follows:

$$(8) \qquad \forall u_0 . u_0 \in p_0 \equiv F_0(u_0 ; p_1^{u_0}),$$

$$(9) \qquad \forall u_1 . u_1 \in p_1^{u_0} \equiv F_1(u_1 ; u_0 ; p_2^{u_1, u_0}),$$

where in (9) another possible occurrence in F_1 of a dependent variable $p_2^{u_1, u_0}$ depending on u_1 and u_0 is indicated.

---

11) As is remarked at the end of Introduction, we use always intuitive logic in the metalogical consideration. So the species Σ is assumed to be defined recursively by using the defining procedure of dependent variables, so that for a given dependent variable it is intuitively decidable whether it is a Σ-variable or non-Σ-variable. Thereby, we do not take the extensionality of dependent variables into consideration but only the procedures by which they are defined. (For instance, the species consisting of V, 0, N and all the elementary sets generated by V, 0, and N (i. e. the species of sets of the theory $T_0(N)$, see Part (VIII), p. 138) is a set species.) In fact, a Σ-variable may be proved to be coextensional to a Π-variable in some subsystems of UL. When we speak later about the elimination of Π-variables from a proof, we do not mean the elimination of every variable which is coextensional to a Π-variable in some subsystem of UL but the elimination of those variables by whose defining procedure it is known that they are Π-variables. The union $\Sigma \cup \Pi$ coincides intuitively with the species of all the dependent variables. But the species $\Sigma \cup \Pi$ and the universal constant V in UL are quite different objects. The former is a notation belonging to our metalanguage while the latter to our object language.

When Σ is a set species, it is not excluded that a non-Σ-variable occurs in the definiens of a Σ-variable.

It may happen that a formula of form $m_0 \in p_0$ with some variable m_0 occurs in a proof P, and if $m_0 \in p_0$ is replaced by $F_0(m_0\,;\, p_1^{m_0})$, it may also happen that $m_1 \in p_1^{m_0}$ with some m_1 occurs in $F_0(m_0\,;\, p_1^{m_0})$, and again $m_2 \in p_2^{m_1, m_0}$ may occur in $F_1(m_1\,;\, m_0\,;\, p_2^{m_1, m_0})$, and so on. The variables $m_0, m_1, m_2, \cdots$ may also depend on element variables or on other variables, such as $m^{m_0}, m^{m_1, m_0}, \cdots$ or $m_0^x, m_1^{m_0}, m_2^{m_1, m_0}, \cdots$ and so on. In such cases the degrees of variables $p_1^{m_0}, p_2^{m_1, m_0}, \cdots$ are successively increasing and these variables may not be defined in the premise of P.

We introduce some terms. Let Γ and Δ be any species of dependent variables. We denote by $\Gamma(\Delta)$ the species of dependent variables *generated by* Δ *with basis* Γ, namely the species of all the dependent variables $p^{m_1, \cdots, m_n}$ where $m_1, \cdots, m_n$ are independent variables or Δ-variables and $p^{x_1, \cdots, x_n}$ is a Γ-variable with $x_1, \cdots, x_n$ as complete system of variables. In particular $p^{m_1, \cdots, m_n}$ is a variable generated by $m_1, \cdots, m_n$ with basis $p^{x_1, \cdots, x_n}$. Now, when eliminating Π-variables from P, we allow new variables to enter the transformed proof so long as these variables are set variables, and we define $\mathcal{T}$-elimination of Π-variables from a UL-proof P *relative to* Σ as $\mathcal{T}$-transformation of P into a UL-proof P^* in which Π-variables do not occur but any Σ-variables may occur, no matter whether they occurred in the original proof P or not. The means of proof of equivalence of affected definiens to the original ones should also be so extended in a natural manner to the case of elimination of Π-variables from P relative to Σ that the equivalence proof shall be executed by using only the defining formulas of any Σ-variables and of $\Pi_0(\Sigma)$-variables and perhaps the formula (I), where Π_0 is the species of Π-variables occurring in P.

We define one more term. Let p be a dependent variable and T the tree of variables for p. We erase every Σ-variable occurring in T under p, together with the whole part of T that is under the Σ-variable. Thus we get a tree $T(\Sigma)$ from T which consists exclusively of Π-variables, perhaps except p. The length of $T(\Sigma)$ is called the *order of* p *relative to* Σ. A Σ-variable is of order 0 relative to Σ. It is noted that the order relative to Σ of a Π-variable p is possibly smaller than that of a Π-variable which is subordinate to p.

Now we formulate Theorem 5 as follows.

THEOREM 5. *Let* Σ *be a set species,* Π *the concept species with respect to* Σ*, and* P *a* UL-*proof. Assume that* Π *and* P *fulfil the condition* (α_1). *Then all the* Π-*variables occurring in* P *are simultaneously* $\mathcal{E}$-*eliminable relative to* Σ.

Proof. The proof runs on again quite similar to those of Theorems 1 and 4. As is remarked before, the condition (β_1) is fulfilled by virtue of (α_1) and of the fact that Π is the concept species with respect to Σ. So we have only to examine the preservability of dependent variable restriction, the equivalence proof of the affected definiens to the original ones, and the finiteness of the procedures.

As for the first we should add to the premises of each figure P', P'', $\cdots$ (notations as in the proofs of Theorems 1 and 4), obtained at each step of transformations, the defining formulas of dependent variables which entered anew at each step of transformations. If all the Π-variables, together with those added newly in this way, could, after all, be eliminated, then all the dependent variables remaining in the possible final figure Q would be Σ-variables. So the elimination relative to Σ would be achieved.

As for the second we have extended above the means of equivalence proof so as to be able to execute this proof reasonably.

As for the last we refer to (4・4) of the proof of Theorem 4. In order to see the finiteness of the procedures we define an ordinal number $\nu(P')$ of the proof P' by

$$\nu(P')=\omega^M a_M+\omega^{M-1}a_{M-1}+\cdots+a_0 \tag{10}$$

where M is the maximum of the orders relative to Σ of the dependent variables occurring in P and $a_i(0\leq i\leq M)$ is the number of non-isological dependent variables occurring in P' of relative order i. The entrance of new dependent variables into the transformed figures P', P'', $\cdots$ are caused, as is explained above, by the possible existence of dependent variables such as $p_1^{u_0}$, $p_1^{u_0,u_1}_2$, $p_3^{u_0,u_1,u_2}$, $\cdots$ depending on element variables. Owing to the closedness of premises (Part (I), § 11), these variables $p_1^{u_0}$, $p_2^{u_0,u_1}$, $\cdots$ are defined in the premise of P while $p_1^{m_0}$, $p_2^{m_0,m_1}$, $\cdots$ are the variables perhaps entering anew P', P'', $\cdots$ in replacing $m_0\in p_0$, $m_1\in p_1^{m_0}$, $\cdots$ by $F_0(m_0, p_1^{m_0})$, $F_1(m_1; m_0; p^{m_1,m_0})$, $\cdots$ respectively. Since in P', P'', $\cdots$ there is no occurrence of Π-variables as left-hand variables, these $m_0, m_1, \cdots$ are Σ-variables or independent variables. Note the case where $m_1=l_1^{m_0}$, $m_2=l^{m_0,m_1}$, $\cdots$. In this case $l_1^{u_0}\in p_1^{u_0}$ occurs in $F_0(u_0; p_1^{u_0})$ of (8). Since $l_1^{u_0}$ is, as left-hand variable in P, a Σ-variable and since m_0 is also a Σ-variable or an independent variable, $l_1^{m_0}=m_1$ is a Σ-variable, because Σ is a set species. The same reasoning shows that $m_2=l^{m_0,m_1}$ is also a Σ-variable, and so on. Therefore the order of $p_r^{m_{r-1},\cdots,m_0}$ relative to Σ is equal to the relative order of $p_r^{u_{r-1},\cdots,u_0}$.

In passing from $P^{(i)}$ to $P^{(i+1)}$ by eliminating a Π-variable p_j from $P^{(i)}$ the relative order of dependent variables perhaps newly entering $P^{(i+1)}$ is by the above reason at least one smaller than that of p_j. Therefore the coefficient of ω^k of $\nu(P^{(i+1)})$, k being the relative order of p_j, becomes by one smaller than that of $\nu(P^{(i)})$, the coefficients of ω^{k-1}, ω^{k-2}, $\cdots$, ω^0 may increase, while those of ω^M, $\cdots$, ω^{k+1} remain the same. Hence $\nu(P^{(i+1)}) < \nu(P^{(i)})$, and our procedures come to an end after a finite number of steps.

Another attention should be paid to the transformation of the principle of extensionality to cuts. Namely, in the final figure Q the P-constituent $[p\text{I}]$ with Π-variable p is changed into (2) of which F^x and F^y do not contain Π-vriables. But in the present case it may happen that in the proof (4) we arrive at, besides (5), (5′), (6), and (6′), a primitive constituent of the following form (see figure (4)):

$$(11)\quad \frac{\overline{4\ \ m^r \notin l^r}}{5\ \ m^s \in l^s} \qquad\qquad (11')\quad \frac{\overline{m^s \notin l^s}}{m^r \in m^s}$$

where m^r and l^r are Σ-variables, one of which does not perhaps depend on r or may be r itself.

We consider only (11) for the symmetric reason. If l^r is r we treat (11) as (6). So we assume that l^r is a dependent variable and let $G^{u;r}$ be the definiens of l^r. Instead of placing under 5 in (11) the proof constituent $[l^r\text{N}]$ with the left formula $m^r \in l^r$ directly, put the following figure under (11):

$$(12)\quad \begin{array}{l}
\overline{}^{\,(\text{I})} \\
-\qquad \neg\forall xy.\ x=y \wedge x\in l^s \rightarrow y \in l^s \\
\hline
\underset{(*)}{m^r = m^s} \quad 6\ \ m^r \in l^s \quad \text{Cut} \quad \underset{(5)}{m^s \notin l^s} \\
\qquad\qquad \underset{(**)}{l^r = l^s} \quad - \quad l^r \neq l^s \\
\qquad\qquad\qquad\qquad - \quad \neg\forall x.\ x \in l^r \equiv x \in l^s \\
\qquad\qquad\qquad\qquad\quad \underset{(4)}{m^r \in l^r} \qquad \underset{(6)}{m^r \notin l^s}
\end{array}$$

Under (∗) and (∗∗) we proceed as follows:

$$\begin{array}{lll}
- & (*) & \\
\hline
- & \forall x.\ x\in m^r \equiv x \in m^s & \\
\hline
(w) & w \in m^r & w \in m^s \\
 & w \notin m^s & w \notin m^r
\end{array}
\qquad\qquad
\begin{array}{lll}
- & (**) & \\
\hline
- & \forall x.\ x\in l^r \equiv x \in l^s & \\
\hline
(w)\ 4 & w \in l^r & w \in l^s \\
5 & \underset{(\dagger)}{w \notin l^s} & w \notin l^r
\end{array}$$

Now we use the defining formulas of m^x and l^x and place, for instance, under (†) the following figure:

$$(13) \qquad \cfrac{(4)\;\cfrac{(\dagger)}{G^{w\,;\,r}}}{(5)\;\neg G^{w\,;\,s}}$$
$$\Uparrow$$

The part in (13) indicated by $\Uparrow$ is again the decomposition of $\neg G^{w\,;\,r}$ and $G^{w\,;\,s}$. Since r, s, and w are independent variables, in $\neg G^{w\,;\,r}$ and $G^{w\,;\,s}$ possibly occur only dependent variables of absolute order less than that of l^x[12]. Hence, repeating the decomposition successively in this way, any string of the figure finally comes either to contain a cancelling pair or to arrive at a figure of form (5) or (6). In the last case we have only to proceed one step more just as in (1·5·7·1) or (1·5·7·2) of the proof of Theorem 1. Thus, the proof (4) of the left cut formula of cut (3) completes.

Thus, Theorem 5 is proved.

5. Coherent components of a proof

Let P be a UL-proof and for the sake of simplicity we assume that P has the primitive cancelling property and also the properties (a′), (b), and (c′) (see Part (II), p. 214).

Let $C(A, K)$ be the carrier (ibid. p. 220) in P of an affirmative subformula A of a P-formula K. Then each formula, say G, of $C(A, K)$ has at a fixed position a subformula, say A^*, such that A^* is the subformula of G that is derived from A by successive applications of associations perhaps in substituting variables for some bound variables occurring in A. Every such subformula A^* of each $C(A, K)$-formula, including A of K, is called a *carried formula* of the carrier $C(A, K)$.

We say that $C(A, K)$ is the carrier of any carried formula A^* of $C(A, K)$ and denote $C(A, K)$ also by $C(A^*, K)$, if no confusion be feared.

We consider in the sequel only the carriers $C(A, K)$ in P such that A is an affirmative primitive formula i. e. a formula of form $a \in b$. By virtue of the primitive cancelling property of P every bottom formula of such a carrier $C(a \in b, K)$ is of form $a^* \in b^*$ or $a^* \notin b^*$. The formula K

12) The above procedures enable us, by inserting cuts, to avoid the possibility of dependent variables of higher order entering the proof. Namely, $G^{w\,;\,r}$ and $\neg G^{w\,;\,s}$ in (13) will be $G(m^r\,;\,r)$ and $\neg G(m^r\,;\,s)$ respectively if the cut $\overline{l^r = l^s \;\; l^r \neq l^s}$ in (12) is not inserted. If we eliminate the inserted cut by Cut Theorem dependent variables of higher order may enter the proof[13]. Since in Q no Π-variables occur, we may place also under (11) the definiens of $m^r \notin l^r$ and $m^s \in l^s$ directly without using Π-variables. The above procedures show the finiteness of procedures by using the decreasing absolute orders.

13) See foot-note 19).

of the carrier $C(a\in b, K)$ considered in the sequel is either a top formula of P or a cut formula.

Two affirmative primitive subformulas $a\in b$ and $c\in d$ of P-formulas F and G, respectively, are called *coherent* in P if $a\in b$ and $c\in d$ are related in P in the following recursive manner.

(i) F and G belong to the same carrier in P, and $a\in b$ and $c\in d$ are carried formulas of the carrier.

(ii) $a\in b$ and $c\notin d$ constitute the cancelling pair of a P-string. (In this case $a=c$, $b=d$, $F=a\in b$, $G=c\notin d$.)

(iii) F and G are cut formulas of the same cut and $a\in b$ and $c\in d$ are situated at the corresponding positions in F and G, respectively. (In this case $F=\neg G$, $a=c$, $b=d$.)

(vi) $a\in b$ and $c\in d$ are the middle and the right-most formulas, respectively, of the P-top formula (I). (In this case $a=x$, $c=y$, $b=d=z$.)

(v) $a\in b$ is coherent in P to an affirmative primitive subformula of a P-formula which is coherent to $c\in d$.

A tree $\mathcal{C}$ in P consisting of affirmative primitive subformulas of P-formulas is called a *component* of P or *P-component,* if P is closed with respect to coherency, that is to say, if any affirmative primitive subformula of a P-formula which is coherent to a $\mathcal{C}$-formula belongs also to $\mathcal{C}$. All the affirmative subformulas of all the P-formulas constitute a component of P, which is called the *maximal P-component.* A P-component $\mathcal{C}$ is called a *coherent P-component,* if any P-component which has at least a formula common with $\mathcal{C}$ contains $\mathcal{C}$. The maximal P-component is divided into a finite number of mutually disjoint coherent P-components, in other words, any affirmative primitive subformula of any P-formula belongs exactly to one coherent P-component. A P-component which is not coherent is called a *composite P-component.* A P-component has its top affirmative primitive formulas in P-top formulas or cut formulas. Thereby, by a top formula of a P-component $\mathcal{C}$ is meant a formula which is not a derivative of a formula belonging to $\mathcal{C}$.

We now cut off the coherency of any affirmative subformula in the defining formula D of any dependent variable p to a subformula in any P-constituent $[p\mathrm{A}]$ or $[p\mathrm{N}]$ which is associated with $\neg D$. Assuming that the successive associations of P-constituents to the P-top formula $\neg$(I) are written in P without any abbreviation, we have in P, as derivatives of $\neg$(I), P-constitutents of form

$$[\mathrm{I}'] \qquad \overline{\neg .\ m=l \wedge m\in p \to l\in p}\,.$$

We cut off also the successive coherency of the primitive affimative subformulas in all $[\mathrm{I}']$ above up to those of (I).

After cutting off the coherency in P as described above, we define *finer coherent P-components* and *finer P-components* in the same way as before. A finer coherent P-component has its top affirmative primitive formulas in formulas attached to [pA] or [pN] for some p's or attached to [I′], or else in the conclusion of P or cut formulas. Any primitive affirmative subformulas of P-formulas, except the defining formulas and the derivatives of $\neg$(I) which are over [I′], belongs exactly to one finer coherent P-component. A coherent P-component whose top formulas lie in premises of P splits perhaps into a finite number of finer coherent P-components. A P-component, if it is necessary to distinguish it from a finer P-component, is called a *coarser P-component.*

Let now Π be a species of dependent variables and C be a finer P-component which has the following properties:

(i) C consists exclusively of affirmative primitive subformulas whose left-hand or right-hand variable is a Π-variable.

(ii) If the left formula of a [pA] or a [pN] belongs to C, then all the affirmative primitive formulas which occur in the right formula of the [pA] or [pN], respectively, and of which the right-hand or left-hand variable is a Π-variable belong also to C.

Such a finer P-component C is called an *$\mathcal{E}$-elimination P-component for* Π. By definition Π-variables may occur in P outside an $\mathcal{E}$-elimination P-component for Π. However, it is clear by the proof of Theorems 1-5 that the Π-variables occurring in C are $\mathcal{E}$-eliminable from C if Π and C, instead of Π and P, fulfil the assumptions of these Theorems[14]. Thereby the $\mathcal{E}$-elimination of Π from C means that the part of P which is outside C remains untouched during the $\mathcal{E}$-elimination, except possibly some premises. The component C is naturally extended during the transformations by introducing new cuts. It is also to be noted that after the elimination of Π-variables from C some defining formulas of Π-variables may possibly remain in P when Π-variables are found in some finer P-ccomponents which are outside C. Remarking this, we state this fact as follows.

THEOREM 6. *Theorems* 1-5 *hold for an $\mathcal{E}$-elimination P-component instead of for P.*

By Theorem 6 it is practically convenient to perform the $\mathcal{E}$-elimination of variables successively for a minimal $\mathcal{E}$-elimination P-components.

---

14) Thereby we mean that a variable m is substituted for x in C if $\overline{\neg G^m}$ is associated in P with a formula $\neg\forall x G^x$ and a primitive subformula $A(m)$ of G^m which is obtained from a primitive subformula $A(x)$ of G^x by this substitution belongs to C.

6. Coherent cycles in a UL-proof

For a dependent variable p occurring in a UL-proof P we denote by $[p, \pi]$ a P-constitutent $[p\mathrm{A}]$ or $[p\mathrm{N}]$ which is situated at the place π in P. By $\mathrm{L}[p, \pi]$ we denote the affirmative subformula of the left formula of $[p, \pi]$, and by $\mathrm{R}[p, \pi]$ an affirmative primitive subformula of the right formula of $[p, \pi]$.

Let $[p_0, \pi_0]$ and $[p_1, \pi_1]$ be P-constituents where possibly $p_0 = p_1$ and $\pi_0 \neq \pi_1$, or $\pi_0 = \pi_1$ and accordingly $p_0 = p_1$. When $\mathrm{L}[p_0, \pi_0]$ is coherent to an $\mathrm{R}[p_1\ \pi_1]$ we call $\langle \mathrm{L}[p_0, \pi_0], \mathrm{R}[p_1, \pi_1] \rangle$ a *coherent ordered segment* in P with $\mathrm{L}[p_0, \pi_0]$ as *initial*, and $\mathrm{R}[p_1\ \pi_1]$ as *end*, point. Let $\mathcal{O}_0$ be the species of all the coherent ordered segments in P. Let further $\mathcal{O}_1$ be the species of all the ordered segments $\langle \mathrm{R}[p_1, \pi_1], \mathrm{L}[p_1\ \pi_1] \rangle$ where $\mathrm{R}[p_1, \pi_1]$ is the end point of a member of $\mathcal{O}_0$. All the ordered segments belonging to $\mathcal{O}_0$ and $\mathcal{O}_1$ constitute an "oriented graph"[15] in P which we denote by $\mathcal{S}$. Let $\mathcal{S}_1, \cdots, \mathcal{S}_r$ be the "connected components" of $\mathcal{S}$.

Let C be a finer coherent P-component. If a formula A of C is a point of $\mathcal{S}_i$, we say that C *crosses* $\mathcal{S}_i$ at A. Since both formulas of a member α of $\mathcal{O}_0$ are coherent in P, if C crosses at a formula of α, then C crosses also at the other formula of α. In this case we say that C crosses $\mathcal{S}_i$ at α. We denote by $\mathcal{S}_i(C)$ the species of all members of $\mathcal{O}_0$ at which C crosses $\mathcal{S}_i$ and call $\mathcal{S}_i(C)$ the *cross-section* of $\mathcal{S}_i$ with C. Since any two formulas which belong to two different connected components of $\mathcal{S}$ and which belong to members of $\mathcal{O}_0$ can not be coherent in P each other, a finer coherent P-component can not cross more than one connected component of $\mathcal{S}$. On the other hand, for any member α of $\mathcal{O}_0$ there is a finer coherent P-component which crosses $\mathcal{S}$ at α. Thus, the species of all the members of $\mathcal{O}_0$ which belong to a connected component $\mathcal{S}_i$ of $\mathcal{S}$ decomposes into disjoint non-empty cross-sections $\mathcal{S}_i(C_1^{(i)})$, $\cdots$, $\mathcal{S}_i(C_s^{(i)})$, $(1 \leq i \leq r,\ s = s(i))$.

A "closed path" consisting of $2k$, $(k \geq 1)$, different members of $\mathcal{S}_i$ is called a *coherent cycle* of $\mathcal{S}_i$ of length $2k$[16], namely a sequence $\beta_0, \alpha_0, \beta_1, \alpha_1, \cdots, \beta_{k-1}, \alpha_{k-1}$ of $2k$ different members of $\mathcal{S}_i$ is a coherent cycle of $\mathcal{S}_i$ of length $2k$, if $\alpha_t = \langle \mathrm{L}[p_t, \pi_t], \mathrm{L}[p_{t+1}, \pi_{t+1}] \rangle$ for $t = 0, \cdots, k-2$, $\beta_t = \langle \mathrm{R}[p_t, \pi_t], \mathrm{L}[p_t, \pi_t] \rangle$ for $t = 0, \cdots, k-1$, and $\alpha_{k-1} = \langle \mathrm{L}[p_{k-1}, \pi_{k-1}], \mathrm{R}[p_0, \pi_0] \rangle$. A P-constituent $[p, \pi]$ is called *self-coherent* if there is an $\mathrm{R}[p, \pi]$ such that $\langle \mathrm{L}[p, \pi], \mathrm{R}[p, \pi] \rangle$, $\langle \mathrm{R}[p, \pi], \mathrm{L}[p, \pi] \rangle$ is a coherent cycle (of length 2).

---

15) Concerning graphs we use, as far as we can, the same terminologies as in Dénes König: Theorie der endlichen und unendlichen Graphen, Leibzig (1936). All the graphs considered below are finite.

16) The length of a coherent cycle is an even positive number.

Let now Σ be a set species and Π the concept species with respect to Σ. Let further C be a finer P-component of a UL-proof P which consists of all such finer coherent P-components that have non-empty cross-sections with a connected component $\mathcal{S}_i$ of the oriented graph $\mathcal{S}$ in P. We set the following conditions (α_2) which is weaker than (α_1).

(α_2) *Any* Π-*variable is substituted*[14] *in* C *for no bound variable except the element variables of defining formulas of* Π-*variables and the variable* z *of the formula* (I).

We shall examine the conditions of $\mathcal{E}$-eliminablity of Π-variables relative to Σ from C under the assumption (α_2).

Since it is allowed by (α_2) for Π-variables to be substituted, as left-hand variables, for element variables of defining formulas of Π-variables, the replacement (B) of eliminating left-hand Π-variables from C can not be applied. So we shall examine the elimination of left-hand Π-variables from C by the replacement (A).

Let $[p, \pi]$ be a P-constituent and assume that $L[p, \pi]=q\in p$ belongs to C where p and q are Π-variables. Then by virtue of the definition of set species and the assumption (α_2), all the left-hand and right-hand variables of formulas of the coherent finer P-component $C_0(\subseteqq C)$ to which $q\in p$ belongs are Π-variables.

We try to replace all the formulas $q^*\in p^*$ of C_0 by $F_*^{q^*}$ where F_*^u is the definiens of p^*. If there is a self-coherent P-constituent $[p_0, \pi_0]$ such that $\langle L[p_0, \pi_0], R[p_0, \pi_0]\rangle$ for some $R[p_0, \pi_0]$ belongs to the cross-section $\mathcal{S}_i(C_0)$, our replacement can not change the P-constituent $[p_0, \pi_0]$ into a cut. For, $L[p_0, \pi_0]$ and $R[p_0, \pi_0]$ are replaced simultaneously.

Assume therefore that $\mathcal{S}_i(C_0)$ has no segment common with a coherent cycle of length 2. Let $\langle L[p_0, \pi_0], R[p_1, \pi_1]\rangle$ be a segment of $\mathcal{S}_i(C_0)$. By our assumption neither $L[p_1, \pi_1]$ nor $R[p_0, \pi_0]$ belongs to C_0. There may be perhaps more than one $R[p_1, \pi_1]$ which belongs to C_0. If we apply the replacement (A) for all C_0-formulas in P, it is then clear that $[p_0, \pi_0]$ changes into a cut[17] and at least one, or possibly more, $R[p_1, \pi_1]$ belonging to C_0 are replaced in the right formula of $[p_1, \pi_1]$.

If $L[p_0, \pi_0]$ is a "boundary point" of the graph $\mathcal{S}_i$, the "boundary segment" $\langle L[p_0, \pi_0], R[p_1, \pi_1]\rangle$ of $\mathcal{S}_i$ is removed from $\mathcal{S}_i$ by our replacement. If $L[p_0, \pi_0]$ is an "inner point" of $\mathcal{S}_i$, let $\langle L[p_{-1}, \pi_{-1}], R[p_0, \pi_0]\rangle$ be a segment of $\mathcal{S}_i$ which is connected to $\langle L[p_0, \pi_0], R[p_1, \pi_1]\rangle$ through $\langle R[p_0, \pi_0], L[p_0, \pi_0]\rangle$. By our assumption $\langle L[p_{-1}, \pi_{-1}], R[p_0, \pi_0]\rangle$ can not belong to $\mathcal{S}_i(C_0)$, since otherwise $[p_0, \pi_0]$ would be self-

17) By this replacement not only the left formula $L[p_0, \pi_0]$ of $[p_0, \pi_0]$ but also the left formulas $L[p, \pi]$ of other $[p, \pi]$'s may be replaced. In this case these $[p, \pi]$'s are all changed into cuts, since our replacements are the replacements of definiendum by definiens.

coherent. Let C_{-1} be the finer coherent P-component to which $\langle L[p_{-1}, \pi_{-1}], R[p_0, \pi_0]\rangle$ belongs. Neither p_{-1} nor the left-hand variable of $L[p_{-1}, \pi_{-1}]$ is necessarily a Π-variable. Let $\overline{K \neg K}$ be the cut into which $[p_0, \pi_0]$ changes by our replacement. We denote by $[\bar{p}_1, \pi_1]$ the P-constituent into which $[p_1, \pi_1]$ changes, where $\bar{p}_1$ suggests the change of the right formula of $[p_1, \pi_1]$ by our replacement. Now, the segment $\langle L[p_{-1}, \pi_{-1}], R[p_0, \pi_0]\rangle$ of S_i disappears from S_i, since $[p_0, \pi_0]$ changes into a cut $\overline{K \neg K}$. Instead, through the intervention of subformulas of the cut formulas K and $\neg K$, $L[p_{-1}, \pi_{-1}]$ is connected directly to perhaps more than one $R[\bar{p}_1, \pi_1]$ so that the original path $\langle L[p_{-1}, \pi_{-1}], R[p_0 \pi_0]\rangle$, $\langle R[p_0, \pi_0], L[p_0, \pi_0]\rangle$, $\langle L[p_0, \pi_0], R[p_1, \pi_1]\rangle$ in S_i may be looked upon as splitting into a finite number of segments $\langle L[p_{-1}, \pi_{-1}], R[\bar{p}_1, \pi_1]\rangle$ of the new S_i. Thus, again the inner segment $\langle L[p_0, \pi_0], R[p_1, \pi_1]\rangle$ is removed from S_i.

Assume that there was in S_i a coherent cycle, say Γ, of length greater than two which runs through $\langle L[p_{-1}, \pi_{-1}], R[p_0, \pi_0]\rangle$ and $\langle L[p_0, \pi_0], R[p_1, \pi_1]\rangle$. By our replacements the cycle Γ perhaps splits into a finite number of coherent cycles. But the lengths of these coherent cycles are at least by two smaller than that of Γ——at least, because the cycle Γ may have other segments than $\langle L[p_0, \pi_0], R[p_1, \pi_1]\rangle$ common with $S_i(C_0)$. By the above procedures the cross-section $S_i(C_0)$ of the connected component S_i of S with C_0 is freed from substitutions of Π-variables for element variables of defining formulas. Let Γ^* be a coherent cycle which was born from the original cycle Γ. If in Γ^* there remains another segment $\langle L[p_0^*, \pi_0^*], R[p_1^*, \pi_1^*]\rangle$ of which the left-hand variable q_0^* of $L[p_0^*, \pi_0^*]$ and p_0^* are Π-variables. We can apply the same procedures on $S_i(C_0^*)$ where C_0^* is the finer coherent P-component to which $L[p_0^*, \pi_0^*]$ belongs, provided that $S_i(C_0^*)$ has no segment common with a coherent cycle of length 2. This last condition is not fulfilled, unless Γ^* is of length greater than 2. In order, therefore, to be able to repeat our procedures as long as Π-variables occur as left-hand variables in the coherent cycles $\Gamma, \Gamma^*, \Gamma^{**}, \cdots$ of decreasing length, it is necessary that at least one left-hand variable occurring in the original cycle Γ is a Σ-variable. If this condition is fulfilled for any coherent cycle of S_i, then it is seen from the above consideration that by the repetition of our procedures the connected component S_i of S can be freed from substitutions of Π-variables for element variables of defining formulas.

Assume that P^* is the figure obtained from P satisfying the above condition by removing the relevant substitution of Π-variables. In different finer coherent P^*-components the right formulas of P^*-constituents

$[\bar{p}, \pi_1]$, $[\bar{\bar{p}}, \pi_2]$, $\cdots$ which are the results obtained from the P-constituents $[p, \pi_1]$, $[p, \pi_2]$ with the same p may perhaps be different (note the above description where the notation $\bar{p}$ is introduced). If therefore we use, if necessary, different defining formulas in different finer coherent P^*-components for the same dependent variable p in P, then P^* is a correct UL-proof[18], and C^* into which C changes and Π fulfil the condition (α_1). Thus we get the following theorems.

THEOREM 6. *Let* Σ *be a set species,* Π *the concept species with respect to* Σ, P *a* UL-*proof,* $\mathcal{S}_i$ *a connected component of the directed graph* $\mathcal{S}$ *in* P, *and* C *the finer* P-*component consisting of all the finer coherent* P-*components which cross* $\mathcal{S}_i$. *Assume that* C *and* Π *fulfil the condition* (α_2) *and further that there is no coherent cycle* Γ *in* $\mathcal{S}_i$ *such that all the left-hand variables occurring in* Γ *are* Π-*variables. Then* C *is* $\mathcal{E}$-*transformable to a finer* P^*-*component* C^* *of a* UL-*proof* P^* *such that* C^* *and* Π *fulfil the condition* (α_1).

Remark: In C^* may occur coherent cycles containing Σ-variables and also Π-variables as right-hand variables.

THEOREM 7. Σ, Π P, *and* $\mathcal{S}$ *as in* Theorem 6. *Assume that* P *and* Π *fulfil the condition* (α_2) *and that there is no coherent cycle* Γ *in* $\mathcal{S}$ *such that all the left-hand variables occurring in* Γ *are* Π-*variables. Then all the* Π-*variables are* $\mathcal{E}$-*eliminable from* P *relative to* Σ.

If a connected component $\mathcal{S}_i$ of the directed graph $\mathcal{S}$ of P has no coherent cycle, $\mathcal{S}_i$ is called of finite length and the maximum of lengths of all "oriented paths" in $\mathcal{S}_i$ is called the *length* of the connected component $\mathcal{S}_i$. If all the connected components $\mathcal{S}_1, \cdots, \mathcal{S}_r$ of $\mathcal{S}$ are of finite lengths, the maximum of the lengths $\mathcal{S}_1, \cdots, \mathcal{S}_r$ is called the length of $\mathcal{S}$.

Remark: There may be in a UL-proof P a P-component $[p, \pi]$ of which the left formula $L[p, \pi]$ does not belong to the oriented graph $\mathcal{S}$ of P, that is to say, $L[p, \pi]$ is neither initial nor end point of any oriented segment in P. Such a P-formula $L[p, \pi]$ is called an *isolated*

18) Note that by condition (α_2) the Π-variables are not substituted in C for the variables x and y of (I), so that the formulas $[I']$ in P remain unchanged in P^* since the above replacements are performed for the primitive formulas both of whose left-hand and right-hand variables are Π-variables. The condition (α_2) assumes the principle of extensionality for Π-variables only with respect to the domain of Σ-variables. If we further weaken the condition (α_2) to the condition (α_3) by replacing its last clause "the variable z of the formula (I)" of (α_2) by "the variable x, y, and z of the formula (I)", then the principle of extensionality for Π-variables takes the general form. We do not enter here into details of the elimination problems under the condition (α_3).

point in P. Theorem 7 and Corollary to Theorem 7 are applicable when there are in P isolated points $L[p, \pi]$ for some Π-variables p.

Corollary. Σ, Π, P, *and* $\mathcal{S}$ *as in* Theorem 6. *Assume that* P *and* Π *fulfil the condition* (α_2). *If* $\mathcal{S}$ *is of finite length, then all the* Π*-variables in* P *are* $\mathcal{E}$*-eliminable from* P *relative to* Σ.

7. Repetition of elimination procedures

Given a UL-proof P, there is freedom of choice of what dependent variables in P we wish to eliminate from P by a certain species $\mathcal{T}$ of elimination transformations. Moreover, the species of some dependent variables may be eliminable from a $\mathcal{T}$-elimination P-component (coarser or finer) while not eliminable from others.

For instance, assume that we have eliminated all the notions in P by $\mathcal{E}$-transformations, getting a UL-proof P_1. Some dependent variables in P_1 which are not notions in P become possibly notions in P_1. We again eliminate from P_1 by $\mathcal{E}$-transformations all the notions in P_1, getting a UL-proof P_2. We can repeat these procedures successively until we arrive at a UL-proof in which no notions occur. The notions in $P, P_1, P_2, \cdots$ may well be called the *notions in* P *of* 0*-th, first, second,* $\cdots$ *order*, respectively, and the notions in P of any order the *notions in* P *in the generalized sense.* The situation is the same for notional variables in P.

We can also apply Theorem 6 first and remove the substitution of some dependent variables from a certain P-component $\mathcal{C}$. Some of these dependent variables may still remain in the transformed proof P_1 but some of them are possibly $\mathcal{E}$-concepts in P_1. After eliminating these $\mathcal{E}$-concepts from P_1, we can perhaps apply again Theorem 6 for some remaining dependent variables. A species of dependent variables in P which are $\mathcal{E}$-eliminable by such successive $\mathcal{E}$-elimination transformations is a species of $\mathcal{E}$-concepts by definition given in §1. During the course of $\mathcal{E}$-eliminations of $\mathcal{E}$-concepts from P we may apply Theorem 6 in taking different species Σ of dependent variables as set species.

In this way the choice of dependent variables to be eliminated, of P-components to which elimination procedures are to be applied, and of the order of eliminations, should be determined from case to case so as to fit in with our purpose under consideration. Thus the methods of eliminations obtained hitherto are, so to speak, "subroutines" which must be linked up in a suitable way in order to compose a "main routine" of a elimination procedure. We should prepare a number of subroutines for this purpose.

In this connection we consider the relation between elimination of dependent variables and that of cuts from a UL-proof P. Assume that P has the primitive cancelling property and the properties (a′), (b), and (c′). Assume further that P has a cut $\overline{C \neg C}$ and $\forall x F^x$ is a subformula of C. Then under one of the cut formulas, say C, an eigen variable, say w, is substituted for x of $\forall x F^x$ and under the other $\neg C$ an appropriately selected variable, say m, is substituted for x of $\neg \forall x F^x$. Assume that m depends on no variable which appears, over $\neg \forall x F^x$ and under the cut formula $\neg C$, as an eigen variable of any P-constituent associated with a derivative of $\neg C$ of form $\forall x G^x$. Then, as is seen from the proof of cut theorem (Part (II), pp. 215/6), when the cut is removed from P, the variable w is replaced by m wherever w occurs in P-formulas under the P-formula which is directly upon the P-constituent of which w is the eigen variable. If several m's are used in P this replacement is executed for each such m. The substituted variable m is a variable occurring in P. Moreover, in case there is in F^x a dependent variable, say p^x, depending on x, the variable p^m is also found in the P-formula $\neg F^m$ so that no new dependent variables enter the cut-eliminated proof as far as the substitution of m for w in F^w concerns. But if this eigen variable w is again substituted in P for the bound variable x of a P-formula of form $\neg \forall x K^x$ which is either a derivative of F^w, of a P-top formula, or else of other cut formula in P, then it may happen that a dependent variable of form q^w which occurs possibly in $\neg K^w$ is changed into q^m which is perhaps not defined in P. The same situation may occur when, instead of w, a dependent variable which depends on w is substituted for x in $\neg \forall x K^x$. These dependent variables which may enter newly the transformed proof are all those that are generated by some variables, like m above, with a dependent variable, like q^x above, that is defined in the premise of P, as basis.

Remark 1[19]. After eliminating a cut from a UL-proof P the defining formulas of some dependent variables which are generated by some variables with dependent variables defined in the premise of P as basis must be, if necessary, adjoined to the resulting figure.

Remark 2. A cut in a UL-proof may avoid some dependent variables to enter the proof.

On the other hand, it is seen from the proof of cut theorem that the following theorem holds.

19) This and the following remarks should have been placed after the proof of cut theorem. (Part (II), p. 214).

THEOREM 8. *Let P be a* UL*-proof with a cut $\overline{C\ \neg C}$. Let further $\forall xF^x$ be a subformula of C and a P-constituent $\overline{\neg F^m}$ be associated with the derivative $\neg\forall xF^x$ of C or $\neg C$. This variable m is freed from substitution for x in $\neg\forall xF^x$ when we eliminate the cut from P.*

Since in the course of $\mathcal{E}$-elimination procedures some [pA] and [pN] are changed into cuts, Theorem 8 is useful as a subroutine in order to make some variables free from substitution. However,

Remark 3. If the eigen variable w of the P-constituent $\overline{F^w}$ associated with $\forall xF^x$ in Theorem 8 or a dependent variable q^w depending on w is substituted under $\overline{F^w}$ for a bound variable x of a P-formula of form $\neg\forall xG^x$ which is not a derivative of the cut formula C or $\neg C$, then the substitution of m or q^m for x in $\neg\forall xG^x$ occurs in the proof after eliminating the cut $\overline{C\ \neg C}$ from P.

8. Predicativity and impredicativity

A dependent variable p is called *superior* to q, if there is a dependent variable q' such that a dependent variable generated with q' as basis occurs in the tree of variables for p and there is a dependent variable to which both q' and q are homological. It is then necessary and sufficient for p and q to be superior each other that there is a dependent variable to which both p and q are homological. Thus, the species of all the dependent variables are partly ordered with respect to the superiority relation modulo the homological relation mentioned above. If p is superior to q, then q is called *inferior* to p.

Let P be a UL-proof, $\mathcal{C}$ a finer P-component, and $p_0, \cdots, p_k$ a sequence of $k\,(\geq 0)$ dependent variables occurring in $\mathcal{C}$ which are not necessarily non-isological each other. The sequence $p_0, \cdots, p_k$ is called an *impredicative cycle* in $\mathcal{C}$ if p_k is substituted for a bound variable (including the element variable) of a derivative of the defining formula of p_0 and moreover if for each $i=1, \cdots, k$ either p_{i-1} is inferior to p_i or p_{i-1} is substituted in $\mathcal{C}$ for a bound variable of a derivative of the defining formula of p_i.

Remark 1. Since the superiority relation is an ordering relation between dependent variables, such a subsequence of an impredicative sequence $p_0, \cdots, p_k$ is an impredicative sequence in $\mathcal{C}$ that we obtain from $p_0, \cdots, p_k$ when we remove all the intermediate variables but the leftmost and the rightmost variables p_i and p_j $(0\leq i<j\leq k)$ from any consecutive subsequence $p_i, p_{i+1}, \cdots, p_j$ in $p_0, \cdots, p_k$ such that for each $\nu=i, i+1, \cdots, j-1$ p_ν is substituted for no bound variable of derivatives of the defining

formula of $p_{\nu+1}$ (so that by definition p_ν is necessarily inferior to $p_{\nu+1}$ for $\nu=i, i+1, \cdots, j-1$). The two extreme cases of impredicative sequences in C are thus: first, a sequence $p_0, \cdots, p_k$ such that for all $i=0, \cdots, k$ p_i is substituted in C for a bound variable of a derivative of the defining formula of p_{i+1} ($p_{k+1}=p_0$); and second, a sequence p_0 consisting of a single variable such that p_0 is substituted in C for a bound variable of a derivative of the defining formula of p_0.

Let Γ be a species of dependent variables. An impredicative cycle in C is called *Γ-impredicative* in C if there is at least a Γ-variable in the cycle. A *Γ-impredicative P-component* C or a *Γ-impredicative proof* P is one in which there is a Γ-impredicative cycle.

Let now $\mathcal{T}$ be a species of elimination transformations and C a $\mathcal{T}$-elimination P-component[20] of a UL-proof P. A Γ-impredicative P-component C (or a Γ-impredicative proof P) is called *$\mathcal{T}$-eliminably Γ-impredicative* if C (or P) is $\mathcal{T}$-transformable to a proof component (or a proof) which has no Γ-impredicative cycle. If it is known that there is no such $\mathcal{T}$-transformations, C (or P) is called *$\mathcal{T}$-essentially Γ-impredicative.* A theory T is called Γ-impredicative if there is a Γ-impredicative T-proof. A theory T is $\mathcal{T}$-elimiminably Γ-impredicative if it is known that any Γ-impredicative T-proof is $\mathcal{T}$-eliminably Γ-impredicative. A theory T is $\mathcal{T}$-essentially Γ-impredicative if there is a $\mathcal{T}$-essentially Γ-impredicative T-proof.

A proof P (or a $\mathcal{T}$-elimination P-component C) is called *$\mathcal{T}$-predicative relative to* Γ or simply *$\mathcal{T}$-Γ-predicative* if non-Γ-variables are simultaneously $\mathcal{T}$-eliminable from P (or C). If it is known that such an elimination is impossible, P (or C) is called *non-$\mathcal{T}$-Γ-predicative.* A theory T is called $\mathcal{T}$-Γ-predicative if it is known that any T-proof is $\mathcal{T}$-Γ-predicative. A theory T is non-$\mathcal{T}$-Γ-predicative if there is a non-$\mathcal{T}$-Γ-predicative T-proof.

In order to define a theory T in UL or a subsystem T of UL it is necessary and sufficient to define a species Γ of dependent variables as the *species of sets* Σ of the theory T and a rule of use of the Σ-variables in T.[21] A theory T is called an *extension* of a theory t if any t-proof is a T-proof.

Let T/t be an extension of a theory t and let $\Sigma(T)$ and $\Sigma(t)$ be the species of sets of the theory T and t respectively. Then $\Sigma(t) \subset \Sigma(T)$

20) For an unspecified $\mathcal{T}$ we understand by a $\mathcal{T}$-elimination P-component such a P-component within which we can execute reasonably $\mathcal{T}$-elimination procedures.

21) Generally the species $\Sigma(T)$ of sets of a theory T is a species of proper $\mathcal{T}$-sets of T for a suitably chosen species $\mathcal{T}$ of elimination transformations. For some species $\mathcal{T}'$ of elimination transformations a subspecies of $\Sigma(T)$ may be a species of $\mathcal{T}'$-concepts.

and the rule of use of $\Sigma(t)$-variables in t is contained in the rule of use of $\Sigma(T)$-variables in T.

An extension T/t of a theory t is called a ***$\mathcal{T}$-predicative extension*** if it is known that any T-proof is $\mathcal{T}$-transformable to a t-proof.

9. Elimination of variables and consistency

Let T/t be a $\mathcal{T}$-predicative extension of a theory t. Then any T-proof is $\mathcal{T}$-reducible to a t-proof so that in particular a T-proof of a contradiction, if any, is $\mathcal{T}$-reducible to a t-proof of a contradiction. Prefixing the species $\mathcal{T}$ of elimination transformations explicitly, we express this fact in saying that *a $\mathcal{T}$-predicative extension T/t of a theory t is $\mathcal{T}$-consistent relative to t*, or more strongly, T is *$\mathcal{T}$-reducible to t*. In particular, *a $\mathcal{T}$-predicative extension of a consistent theory is $\mathcal{T}$-consistent.* The stronger the species $\mathcal{T}$ of elimination transformations, the weaker the results about consistency which it gives rise to. As an extreme case, let $\mathcal{J}$ be the species of transformations which are required in eliminating Hilbert-Bernays' ι-symbol. The dependent variables in UL are expressed by ι-symbols. Hence, owing to Hilbert-Bernays' theory of elimination of ι-symbols, we see that UL *is $\mathcal{J}$-consistent*, or more strongly, UL *is $\mathcal{J}$-reducible to the predicate logic with a dyadic relation* $\in$. This assertion is stronger than the result of consistency of UL which follows directly from cut theorem.

If a UL-proof is $\mathcal{T}$-transformable to a proof in the predicate logic with a dyadic relation $\in$, the proof is called a *$\mathcal{T}$-tautology.* If any proof of a theory is a $\mathcal{T}$-tautology, the theory is called a $\mathcal{T}$-tautology. Thus, UL is an $\mathcal{J}$-tautology.

The elimination transformation $\mathcal{E}$ may well be regarded as weak enough to obtain more significant consistency results. It is, however, to be noted that, whatever species of elimination transformations we may use, the problem of consistency is endless in the following sense. Namely, assume that the consistency of a theory T with Σ as the species of sets of T is proved. The conclusion H of a T-theorem $\sigma \vdash H$ may have assumptions $A_1, \cdots, A_n$, namely H may be of form $A_1 \wedge A_2 \wedge \cdots \wedge A_n \rightarrow B$. Even if $\sigma \vdash H$ is a T-theorem, we do not yet know whether $\sigma \vdash \neg. A_1 \wedge \cdots \wedge A_n$ is also a T-theorem or not. Namely, the problem of the consistency of an assumption in T is still open. This problem is in particular significant when the assumption is a primary one. Even if some assumptions in T are proved to be consistent in T, there may be still other assumptions which are regarded as primary, and so on. However, we shall at present confine ourselves to the investigation into the problem: what kind of set formation will be consistent?

10. Examples and applications

When a formula proved in previous Parts is dealt with in the following examples, the number of the formula and the page of the Part where a proof of the formula is given are indicated. In each such case this proof of the formula is denoted by P.

(i) $\iota*3$ $\iota\in\mathrm{Un}$ (p. 48, Part (III)).

Un and ι are notions in P so that Un and ι are $\mathcal{E}$-eliminable from P by Theorem 2 (see also § 7). The other dependent variables occurring in P are those which are homological to $\langle ab\rangle$ and to those which are used to define $\langle ab\rangle$, i.e., $\{a\}$ and $\{a, b\}$. These variables are all notional variables in P so that they are $\mathcal{E}$-eliminable from P by Theorem 4.

The oriented graph[22] of P is shown in Fig. 1. The left and right formulas of a P-constituent are respectively indicated by the white and black circles in Fig. 1. A variable attached to a white circle is the dependent variable to which the P-constituent concerns. Since the oriented graph is of length 2 and no dependent variable is substituted in P for bound variables, except ι for element variable of Un, we see also by Corollary 2 of Theorem 7 that all the dependent variables in P are $\mathcal{E}$-eliminable from P. The formula $\iota\in\mathrm{Un}$ is an $\mathcal{E}$-tautology.

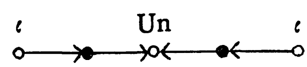

Fig. 1.

The constant Un is used as a notion in any proof given in previous Parts.

(ii) $\circ*2$ $\sigma\in\mathrm{Un}\wedge a\in\mathrm{D}_{\sigma}\rightarrow a^{\sigma\tau}=a^{\tau\circ\sigma}$ (pp. 36–37, Part (III)).

The dependent variables used in P are, up to isological variables, as follows: Un, D_{σ}, $a^{\sigma\tau}$, $a^{\tau\circ\sigma}$, $\langle ab\rangle$, $\langle a^{\sigma}b\rangle$, a^{σ}, $\langle aa^{\sigma}\rangle$, $\tau\circ\sigma$ and those which are required in defining these variables, i.e. $\{a\}$, $\{a, b\}$, $\{a^{\sigma}\}$, $\{a^{\sigma}, b\}$, and $\{a, a^{\sigma}\}$.

Not only the oriented graph of P but also those of any proofs given in Part (III) are of finite length.

Among the dependent variables mentioned above, the variables $\langle ab\rangle$, $\langle a^{\sigma}b\rangle$, $\langle aa^{\sigma}\rangle$, and a^{σ} are proper $\mathcal{E}$-sets in P. For, a^{σ} is substituted for the bound variable y in the P-constituent at the bottom on the right-hand side of p. 36, Part (III); $\langle ar\rangle$ and $\langle ar^{\sigma}\rangle$ are used in the proof of the cut formula Im$*3$ (see right below of the proof of Im$*3$, p. 30, Part

22) The graph is described only for the variables ι and Un. The graphs of other examples are also shown for the main variables of the proofs.

(III)) as substitutes for x and y of the principle of extensionality with respect to the independent variable σ; the same for $\langle bc\rangle$ and $\langle a^{\sigma}c\rangle$ at the bottom of the left-hand side of P (p. 37, Part (III)).

Un is a notion, and $\tau\circ\sigma$, $a^{\sigma\tau}$ and $a^{\tau\circ\sigma}$ are notional variables, in P, so that these variables are $\mathcal{E}$-eliminable from P by Theorem 1 and 4. After eliminating these variables, D_{σ} is $\mathcal{E}$-eliminable from P, since $L[D_{\sigma}N]$'s in P are isolated (see Remark following Theorem 7). The eliminated variables are all proper $\mathcal{E}$-concepts so that the definiens of the remaining dependent variables are not affected.

The variables $\langle ab\rangle$, $\langle a^{\sigma}b\rangle$, a^{σ}, and $\langle aa^{\sigma}\rangle$ are, thus, all the proper $\mathcal{E}$-sets in P and the variables required in defining these $\mathcal{E}$-sets in P are auxiliary $\mathcal{E}$-sets in P (definition in § 1).

(iii) Russell's condradiction. By using the defining formula $\forall u.\ u\in R \equiv u\notin u$ of Russell's self-contradictory set R we have the following proof of contradiction.

$$\cfrac{\cfrac{}{R\notin R \quad R\notin R}\quad R\in R \qquad R\in R \quad \cfrac{}{R\notin R \quad R\notin R}}{}$$

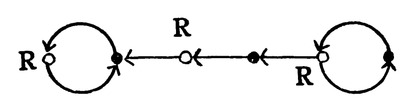

Fig. 2.

The oriented graph is shown in Fig. 2 which has two coherent cycles of length 2, namely the two bottom proof constituents are self-coherent. The constant R is a proper $\mathcal{E}$-set in the proof of Russell's contradiction.

(iv) $$\neg.\ \sigma\in \mathrm{Un}\wedge D_{\sigma}=a\wedge W_{\sigma}=\mathfrak{P}(a)$$ (p. 105, Part (VI)).

We define $L^{a,\sigma}$, as is done there, by

$$\forall u.\ u\in L^{a,\sigma}\equiv u\in a\wedge\neg\exists x.\ u\in x\wedge\langle ux\rangle\in\sigma\,.$$

A proof of the above formula runs as follows (in the proof $L^{a,\sigma}$ is abbreviated as L):

$$\begin{array}{ll} - & \neg.\ \sigma\in\wedge\mathrm{Un}\wedge D_{\sigma}=a\wedge W_{\sigma}=\mathfrak{P}(a) \\ \hline [11]- & \sigma\notin \mathrm{Un} \\ [8][5]- & D_{\sigma}\neq a \\ [9][10]- & W_{\sigma}\neq\mathfrak{P}(a) \\ \hline 1 & \neg\forall xyz.\ \langle xy\rangle\in\sigma\wedge\langle xz\rangle\in\sigma\to x=y \\ 2 & \neg\forall x.\ x\in D_{\sigma}\equiv x\in a \\ - & \neg\forall x.\ x\in W_{\sigma}\equiv x\in\mathfrak{P}(a) \\ [L] \hline - & L\in W_{\sigma}\not\equiv L\in\mathfrak{P}(a) \end{array}$$

[9]– $\mathrm{L}\notin \mathrm{W}_\sigma$

– $\neg\exists x.\ \langle x\mathrm{L}\rangle\in\sigma$

(w)

[2]3 $\langle w\mathrm{L}\rangle\notin\sigma$

(2)

– $w\in \mathrm{D}_\sigma \not\equiv w\in a$

[8]– $w\in \mathrm{D}_\sigma$ [5]4 $w\notin a$ (*)

– $\exists x.\ \langle wx\rangle\in\sigma$

$\langle w\mathrm{L}\rangle\in\sigma$ (3) [2]

[10]– $\mathrm{L}\in\mathfrak{P}(a)$

– $\mathrm{L}\subseteq a$

– $\forall x.\ x\in\mathrm{L}\rightarrow x\in a$

(w)

[7]– $w\notin\mathrm{L}$

[6]3 $w\in a$

– $\neg.\ w\in a\wedge\cdots$

Spf $w\notin a$ $\cdots$ (3) [6]

(*) Cut

[1]5 $w\notin\mathrm{L}$

(5)

– $\neg.\ w\in a\wedge\neg\exists x.\ w\in x\wedge\langle wx\rangle\in\sigma$

Spf $w\notin a$

– $\exists x.\ w\in x\wedge\langle wx\rangle\in\sigma$

[L]

– $w\in\mathrm{L}\wedge\langle w\mathrm{L}\rangle\in\sigma$

$w\in\mathrm{L}$ (5) [1] $\langle w\mathrm{L}\rangle\in\sigma$ (3) [2]

[1]5 $w\in\mathrm{L}$

[5]

– $w\in a\wedge\neg\exists x.\ w\in x\wedge\langle wx\rangle\in\sigma$

$w\in a$ (4) [5] – $\neg\exists x.\ w\in x\wedge\langle wx\rangle\in\sigma$

(W)

[3]6 $w\notin W$

[4]7 $\langle wW\rangle\notin\sigma$

[L](1)

$\langle wW\rangle\in\sigma$ (7) [4] $\langle w\mathrm{L}\rangle\in\sigma$ (3) [2] $W\neq\mathrm{L}$ (5, 6, =) [3][1]

We denote the above proof by P. There are in P eleven finer coherent P-components. The numbers in the brackets [ ] in P show these eleven P-components. These numbers are indicated only for primitive P-formulas and equalities. Namely, for a bottom P-formula the number is attached under the bottom P-formula and for a P-formula which is not a bottom formula to the left of the formula. For an equality the numbers of the left-hand and the right-hand primitive formulas are written side-by-side.

The dependent variable L is a proper $\mathcal{E}$-set in P since L is substituted in the three P-constituents indicated by [L] in P and in the two more P-constituents [W_σN] and [$\mathfrak{P}(a)$A] for element variables (these P-constituents are written as usual abbreviatedly in P).

The oriented graph, say $\mathcal{S}$, of P consists of two connected components: the one crosses the finer coherent components [1] and [11] and the other [10]. Outside $\mathcal{S}$ there are two isolated points which concerns W_σ and D_σ respectively. The graph $\mathcal{S}$ with two isolated points is shown in Fig. 3, which has a coherent cycle of length 2. The self-coherent P-component is situated directly under the left cut-formula of the singular cut $\overline{w \notin \mathrm{L} \quad w \in \mathrm{L}}$ in P. The finer P-components [8] and [9] cross the isolated points $\mathrm{L}[\mathrm{D}_\sigma \mathrm{A}]$ and $\mathrm{L}[\mathrm{W}_\sigma \mathrm{N}]$, respectively.

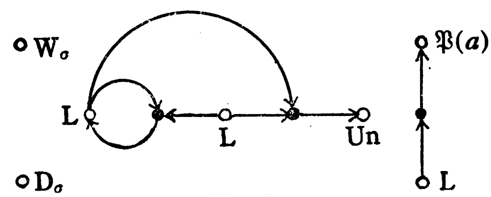

Fig. 3.

Besides L, the following dependent variables up to isological variables occur in P: Un, D_σ, W_σ, $\mathfrak{P}(a)$, $\langle wx \rangle$, $\langle w\mathrm{L} \rangle$, and those used in defining $\langle wx \rangle$ and $\langle w\mathrm{L} \rangle$. These variables are all $\mathcal{E}$-eliminable as follows.

Un is a notion in P. Eliminate Un first from the finer coherent P-component [11]. The connected component of $\mathcal{S}$ which crosses [11] is reduced to a graph with one coherent cycle of length two and a boundary segment (see the middle figure of Fig. 3). Eliminate then D_σ, W_σ, $\mathfrak{P}(a)$ from the finer P-components [8], [9], and [10], respectively. Remove by the replacement (B) all the left-hand occurrences of ordered pairs. Using the recursive definition of ordered pairs to the reversed direction, eliminate all the variables isological to $\langle wx \rangle$ and $\langle w\mathrm{L} \rangle$, together with those required in defining these variables. Erase unnecessary defining formulas. We get thus a UL-proof in which the single dependent variable $\mathrm{L}^{a,\sigma}$ occurs, the definiens of which is affected by our elimination transformations.

Thus, as stated in p. 105, Part (VI), the three variables $\mathrm{L}^{\mathrm{V},\iota}$, ι, and V lead to a contradiction which is nothing else than Russell's contradiction, as is stated there.

$\mathrm{L}^{a,\sigma}$ constitutes an $\mathcal{E}$-essentially impredicative cycle both in the original proof and in the transformed proof.

(v) $\quad \bigcap\text{-cl} \in \bigcap\text{-cl}$, $\quad$ where $\bigcap a$ is, as usual, defined by

$$(13) \qquad \forall u.\ u \in \bigcap a \equiv \overset{a}{\forall} x.\ u \in x\,,$$

and $\bigcap$-cl ($\bigcap$-closedness) by

$$(14) \qquad \forall u.\ u \in \bigcap\text{-cl} \equiv \overset{x \subseteq u}{\forall} x.\ \bigcap x \in u\,.$$

By using (13) and (14) the above formula $\bigcap\text{-cl} \in \bigcap\text{-cl}$ is proved, which shows an instance of a constant having itself as its element. The proof runs as follows:

$$[1]- \quad \bigcap\text{-cl} \in \bigcap\text{-cl}$$

$$- \quad \overset{x \subseteq \bigcap\text{-cl}}{\forall x.}\ \bigcap x \in \bigcap\text{-cl}$$

$(\mathcal{W})$

$$1 \quad \mathcal{W} \not\subseteq \bigcap\text{-cl}$$

$$[2]- \quad \bigcap \mathcal{W} \in \bigcap\text{-cl}$$

$$- \quad \overset{x \subseteq \bigcap \mathcal{W}}{\forall x.}\ \bigcap x \in \bigcap \mathcal{W}$$

(A)

$$2 \quad A \not\subseteq \bigcap \mathcal{W}$$

$$[3]- \quad \bigcap A \in \bigcap \mathcal{W}$$

$$- \quad \overset{\mathcal{W}}{\forall x.}\ \bigcap A \in x$$

(B)

$$[4]3 \quad B \notin \mathcal{W}$$

$$[5]4 \quad \bigcap A \in B$$

(1)

$$- \quad \neg \forall x.\ x \in \mathcal{W} \rightarrow x \in \bigcap\text{-cl}$$

$$\underset{\substack{(3)\\ [4]}}{B \in \mathcal{W}} \qquad [6]- \quad B \notin \bigcap\text{-cl}$$

$$- \quad \neg \overset{x \subseteq B}{\forall x.}\ \bigcap x \in B$$

$$- \quad A \subseteq B \qquad\qquad \underset{\substack{(4)\\ [5]}}{\bigcap A \notin B}$$

$$- \quad \forall x.\ x \in A \rightarrow x \in B$$

(w)

$$[7]5 \quad w \notin A$$

$$[8]6 \quad w \in B$$

(2)

$$- \quad \neg \forall x.\ x \in A \rightarrow x \in \bigcap \mathcal{W}$$

$$\underset{\substack{(5)\\ [7]}}{w \in A} \qquad - \quad w \notin \bigcap \mathcal{W}$$

$$- \quad \neg \overset{\mathcal{W}}{\forall x.}\ w \in x$$

$$\underset{\substack{(3)\\ [4]}}{B \in \mathcal{W}} \qquad \underset{\substack{(6)\\ [8]}}{w \notin B}$$

The above proof is denoted by P. In P there are eight finer coherent P-components indicated again by the numbers in brackets [ ]. The oriented graph of P is a connected graph of length 4 (Fig. 4).

The dependent variables occurring in P are $\bigcap$-cl and $\bigcap x$, up to isological variables. The substitution of these variables in P are: $\bigcap$-cl for the element variable of $\bigcap$-cl in the coherent P-component [1]; $\bigcap \mathcal{W}$

for that of $\bigcap$-cl in [2]; and $\bigcap A$ for that of $\bigcap \mathcal{W}$ in [3]. Since the oriented graph of P is of finite length, all the dependent variables are $\mathcal{E}$-eliminable from P by Corollary of Theorem 7. Hence $\bigcap$-cl$\in\bigcap$-cl is an analytical $\mathcal{E}$-tautology (definition of an analytical proof is given in Part (IV), p. 133).

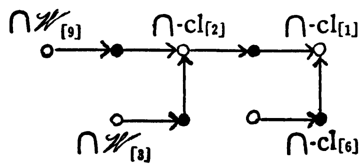

Fig. 4.

(vi) N$\notin$N (p. 125–126, Part (VII)).

The dependent variables occurring in P are: 0, N, s', and P. The oriented graph (Fig. 5) in P is connected and has a coherent cycle of length 4. The coherent cycle crosses two finer coherent P-components. In these components N and P are substituted for bound variables of the definiens of P and N, respectively. Hence, N and P constitute an $\mathcal{E}$-essentially impredicative cycle.

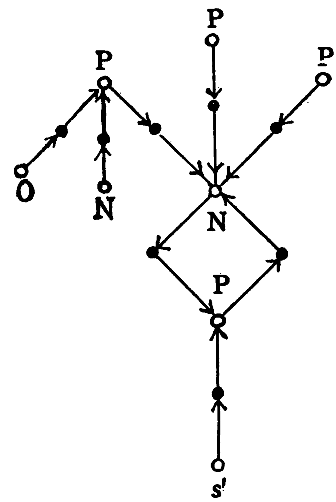

Fig. 5.

(vii) Mathematical Induction.

Let P be a UL-proof in which a negative P-constituent [NN] at a place π occurs. We denote this P-constituent by [Nπ]. If there is in P an impredicative cycle which concerns [Nπ] then [Nπ] is called an *impredicative mathematical induction*[23]. In the proof of N$\notin$N an impredicative mathematical induction is used which is $\mathcal{E}$-essential. If N occurs in the definiens of a set M for induction and if M is substituted for x of [Nπ] in a UL-proof, then [Nπ] is an impredicative mathematical induction, since the substituted variable M is superior than N. Such a mathematical induction is frequently used in the deductions in Part (IX). However, N is used in any proof in Part (IX) as a notion, so that N is $\mathcal{E}$-eliminable from the deductions in Part (IX). Moreover, the proper $\mathcal{E}$-sets used in these deductions, besides 0, V, N and elementary sets generated by 0, V, and N, are: ι, κ_σ, $\tau_{b,\sigma}$, ν, and λ_σ, defined in pp. 9, 36, and 39, Part (IX). Let Σ be the species of all these variables and of any sets for induction, and $T_1(N, \Sigma)$ be the natural number theory (definition in p. 134 Part (VIII)) with Σ as the species of sets. Since all the non-Σ-variables, such as Add$\restriction$N$\times$V, N$\times$V, D_{Add}, Add$\restriction$N$\times$N, N$\times$N,

23) Owing to the definition of predicativity in this Part, I delete "; otherwise *predicative*" from line 14, p. 14, Part (IX).

$W_{Add \upharpoonright N \times N}$, etc., are $\mathcal{E}$-eliminable from the deductions in Part (IX), these deductions remain within the $\mathcal{E}$-Σ-predicative extension of $T_1(N, \Sigma)$.

The proof of $N \notin N$ above is the single case of $\mathcal{E}$-essentially impredicative mathematical inductions used in previous Parts.

APPENDIX

1. On intuitive knowledge

In previous Parts we used the expression "intuitive knowledge" of which we shall give a definition. Let T be a theory, $\mathcal{T}$ a species of elimination transformations, and Σ a species of dependent variables. Assume that it is known that any dependent variable not belonging to Σ is $\mathcal{T}$-eliminable from any T-proof. Assume that I is a knowledge, independent of T and $\mathcal{T}$, by which we can decide for any Σ-constants m and p, which of $m \in p$ and $m \notin p$ holds, and which does not, and also, which of $m = p$ and $m \neq p$ holds, and which does not. Assume further that the $\mathcal{T}$-consistency of T is proved by using the *whole* knowledge I. Then the knowledge I is called the *intuitive knowledge with respect to T and* $\mathcal{T}$. Thus the intuitive knowledge I is a concept* correlative to a $\mathcal{T}$-consistent formal system T. Thereby I may contain some knowledge which is undecidable in T but I stands in harmony with T, as is mentioned before, in the sense that what is known to be false by I is T-unprovable. Thus the formal system T is founded by I as a consistent system and the validity of the intuitive knowledge I is assured by the consistency of T. The form and intuition give foundations mutually to each other.

2. On defining formula

The defining formula $\forall u.\ u \in p \equiv F^u$ of a dependent variable p does not determine the properties of p at all but it determines entirely what properties the variable p should possess when p is placed in a certain environment.

Mathematical Institute
Nagoya University

(Received November 18, 1959)