MODULES OVER BOUNDED DEDEKIND PRIME RINGS. II

Hidetoshi MARUBAYASHI

The main purpose of this paper is to generalize the theory of pure-injective abelian groups (equivalently, algebraically compact groups) to the case of modules over bounded Dedekind prime rings.

Throughout this paper, R denotes a bounded Dedekind prime ring with an identity and every (right) R-module is unitary.

In Section 1, we define the concepts of cocyclic and cofinitely generated modules and give characterizations of these modules. Cocyclic modules are used for a characterization of pure-injective modules in Section 2.

In Section 2, the concepts of pure-injective and algebraically compact modules will be introduced by the analogy of that of abelian groups. We show that these concepts are equivalent and that a pure-injective module is a direct summand of direct product of cocyclic modules (see Theorem 2. 6).

In Section 3, we generalize a result [10] on modules over commutative discrete valuation rings to the case of modules over non-commutative discrete valuation rings. In Theorem 3. 6, we show that every module over non-commutative discrete valuation rings has a basic submodule and that any basic submodules of the module are isomorphic. Using basic submodules, we determine, in Section 4, the structure of reduced, algebraically compact modules. In Theorem 4. 7, we show that the pure-injective envelope of an R-module M is isomorphic to the direct sum of the injective envelope of M^1 and $\hat{M}$, where $M^1 = \cap MA$ (A ranges over non-zero ideals of R) and $\hat{M}$ is the completion of M with respect to the R-adic topology.

In Section 5, we establish that there is a one-to-one correspondence between all divisible, torsion R-modules and all reduced, torsion-free, algebraically compact R-modules. This extends a result of Harrison [8] to modules over bounded Dedekind prime rings. "Submodule" and "homomorphism" without modifier mean always "R-submodule" and "R-homomorphism" respectively.

This paper is a continuation of [11]. A number of concepts and results are needed from [11].

1. Cofinitely generated modules

In this paper, a ring R is always a bounded Dedekind prime ring unless otherwise stated. Let P be a prime ideal of R. We denote, in this paper, the local ring of R with respect to P by R_P and its maximal ideal by P'. Furthermore, we denote the completion of R_P with respect to P' by $\hat{R}_P$ and its maximal ideal by $\hat{P}$. Since $R/P^n \cong R_P/P'^n$ by a natural correspondence, $\hat{R}_P$ is isomorphic to the completion of R with respect to P (see [7]). Let M be an R-module and let A be an ideal of R. We define $M[A] = \{m \in M \mid mA = 0\}$.

Let M be an R-module. Following [6], we shall call M *cocyclic* if there is an element $m \in M$ such that any homomorphism $\phi : M \to N$ with $m \notin \operatorname{Ker} \phi$ implies that ϕ is a monomorphism. In this case, m may be called a *cogenerator of M*. This concept is naturally dual of cyclic modules. Since every submodule is a kernel of a homomorphism, if M is cocyclic, then M is uniform and has a unique simple submodule. Conversely, if M is uniform and has a unique simple submodule S, then M is cocyclic, and any element $\neq 0$ in S is a cogenerator of M.

Proposition 1. 1. *A module M is cocyclic if and only if M is isomorphic to $e\hat{R}_P/e\hat{P}^n$ or is of type P^∞, where P is a prime ideal of R and e is a uniform idempotent in $\hat{R}_P$.*

Proof. If $M \cong e\hat{R}_P/e\hat{P}^n$, then M is uniform and has a unique simple module $e\hat{P}^{n-1}/e\hat{P}^n$ by Lemma 2. 3 of [11], because $e\hat{P}^{n-1}/e\hat{P}^n \cong e\hat{R}_P/e\hat{P}$. If M is of type P^∞, then by Theorem 3. 17 of [11], $M \cong \varinjlim e\hat{R}_P/e\hat{P}^n$. Therefore M is uniform and has a unique simple module $e\hat{R}_P/e\hat{P}$. Conversely, if M is cocyclic, then M has a unique simple module S. By Lemma 3. 15 of [11], $S \cong e\hat{R}_P/e\hat{P}$, where P is a prime ideal of R and e is a uniform idempotent in $\hat{R}_P$. Since M is uniform, it is clear that M is P-primary. If M is divisible, then M is of type P^∞ by Theorem 3. 17 of [11]. If M is reduced, then M is isomorphic to $e\hat{R}_P/e\hat{P}^n$ by Theorem 3. 24 of [11].

A module M is called *cofinitely generated*, if $\bigcap_\alpha M_\alpha = 0$, where M_α are submodules of M, implies that there exist finitely many $\alpha_1, \alpha_2, \cdots, \alpha_n$ such that $\bigcap_{i=1}^n M_{\alpha_i} = 0$. This concept is naturally dual of finitely generated modules. Cofinitely generated modules are characterized as follows:

Theorem 1. 2. *For an R-module M, the following conditions are equivalent:*
(i) *M is cofinitely generated;*
(ii) *the socle $S(M)$ of M is finitely generated and M is an essential extension of $S(M)$;*
(iii) *M is an essential extension of a torsion, finitely generated R-module;*
(iv) *M is a direct sum of a finite number of cocyclic modules;*

(v) *M is an artinian R-module.*

Proof. The equivalence of (i) and (ii) was proved for a module over a general ring by Onodera [13].

(ii) $\Rightarrow$ (iii): This is trivial.

(iii) $\Rightarrow$ (iv): Let N be the finitely generated torsion submodule of M. Since M is an essential extension of N, we obtain that M is also torsion and $\dim M = \dim N = n < \infty$. Now let $M = D \oplus C$, where D is the divisible part of M and C is the reduced part of M. Then D is a direct sum of a finite number of divisible cocyclic modules and C is an essential extension of $C \cap N$. Hence, by Theorem 3. 2 of [11], we may assume that M is a reduced P-primary module and N is a finitely generated P-primary module, where P is a prime ideal of R. Since $n = \dim M = \dim N$, we have $M[P] = N[P] = S_1 \oplus \cdots \oplus S_n$, where S_i are simple R-modules. Then the injective hull $E(M)$ of M is isomorphic to $E(S_1) \oplus \cdots \oplus E(S_n)$. By Theorem 3. 17 of [11] the submodules of $E(S_i)$ are only $e\hat{R}_P/e\hat{P}^n$, $n = 1, 2, \cdots$, where e is a uniform idempotent in $\hat{R}_P$ and thus $E(S_i)$ is an artinian module. Hence M is also artinian module and so $M = M_1 \oplus \cdots \oplus M_n$, where M_i is an indecomposable R-module. By Corollary 3. 26 of [11], M_i is a cocyclic module.

(iv) $\Rightarrow$ (v): This is trivial.

(v) $\Rightarrow$ (ii): This is trivial.

2. Pure-injective and pure-projective modules

A short exact sequence $0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$ of R-modules is said to be *pure-exact* if Image α is a pure submodule of M.

A module P is *pure-projective* if for any pure-exact sequence $0 \to L \to M \to N \to 0$, the natural homomorphism

$$\mathrm{Hom}\,(P, M) \to \mathrm{Hom}\,(P, N)$$

is surjective. Similarly, a module G is *pure-injective* if for any such sequence, the natural homomorphism

$$\mathrm{Hom}\,(M, G) \to \mathrm{Hom}\,(L, G)$$

is surjective.

In this section, we give characterizations of pure-projective and pure-injective modules.

Since a projective module is a direct summand of a direct sum of cyclic modules, by Corollary 3. 8, Theorem 3. 10 of [11] and Corollary 3 of [14], we have

Proposition 2. 1. *An R-module is pure-projective if and only if it is a direct*

summand of a direct sum of cyclic modules.

Lemma 2. 2. *For a submodule S of an R-module M, the following conditions are equivalent:*

(i) *S is pure in M;*

(ii) *S/SA is a direct summand of M/SA for every non-zero ideal A of R;*

(iii) *if $B \subseteq S$, where B is a submodule of S, such that S/B is of bounded order, then S/B is a direct summand of M/B.*

Proof. (i)$\Rightarrow$(ii): This follows from the same way as in Theorem 27. 10 of [6].

(ii)$\Rightarrow$(iii): Asumme that S/B is of bounded order. Then there exists an ideal $A \neq 0$ of R such that $SA \subseteq B$ and so, by the assumption, $M/SA = S/SA \oplus K/SA$, where K is a submodule of M. Then it immediately follows that

$$M/B = S/B \oplus (K+B)/B .$$

(iii)$\Rightarrow$(i): If $xc = s$, where $x \in M$, $s \in S$ and c is regular in R, then there exists an ideal $A \neq 0$ of R such that $Rc \supseteq A$, because R is bounded. By the assumption, $M/SA = S/SA \oplus K/SA$, where K is a submodule of M. We denote the image of an element m ($\in M$) in M/SA by $\bar{m}$. Now we write $\bar{x} = \bar{s}_1 + \bar{k}$, where $s_1 \in S$ and $k \in K$. Then $\bar{s}_1 c = \bar{s}$ and so $s_1 c - s \in SA \subseteq Sc$. Hence $s_1 c - s = s_2 c$ with $s_2 \in S$ and thus $(s_1 + s_2) c = s$, as desired.

By the validity of Lemma 2. 2, the proof of the following lemma is proceeded as in Theorem 29. 2 of [6].

Lemma 2. 3. *A short exact sequence $0 \to L \to M \to N \to 0$ is pure-exact if and only if for any bounded module G, the natural homomorphism $\mathrm{Hom}\,(M, G) \to \mathrm{Hom}\,(L, G)$ is surjective.*

Lemma 2. 4. (i) *Let m be a non-zero element in an R-module M and let B be a submodule of M that is maximal with respect to the property of excluding m. Then M/B is cocyclic.*

(ii) *Let N_α ($\alpha \in \Lambda$) be R-modules. Then $\Sigma_\alpha \oplus N_\alpha$ is pure in $\Pi_\alpha N_\alpha$, where Λ is an index set.*

Proof. (i) follows from the same way as in Proposition 25. 2 of [6].

(ii) immediately follows from the definition of pure submodules.

Lemma 2. 5. *Every R-module can be embedded as a pure submodule in a direct product of cocyclic modules.*

Proof. Let $\{G_\alpha\}$ be the family of cocyclic quotient modules and bounded quotient modules of an R-module M, and let $G = \Pi_\alpha G_\alpha$. The natural epimor-

phism $\theta_\alpha: M \rightarrow G_\alpha$ induces a homomorphism $\theta: M \rightarrow G$, where $\theta(m) = (\theta_\alpha(m))$ for every $m \in M$. Then, by Lemma 2.4, θ is a monomorphism. To prove that $\theta(M)$ is pure in G, we let $\theta(m) = gc$, where $g \in G$, $m \in M$ and c is regular in R. Then there exists a non-zero ideal A of R such that $Rc \supseteq A$. If $M/MA = 0$, then $M = Mc$ and so $m = m_1 c$ for some $m_1 \in M$. Hence $\theta(m) = \theta(m_1)c$. If $M/MA \neq 0$, then $m + MA = (m_\alpha + MA)c$ for some $m_\alpha \in M$, and so $m - m_\alpha c \in MA \subseteq Mc$. Hence $m = m_2 c$ for some $m_2 \in M$ and thus $\theta(m) = \theta(m_2)c$. If G_α is of bounded order, then G_α is a direct sum of cocyclic modules by Theorems 3.7 and 3.38 of [11] and thus G_α can be embedded as a pure submodule in a direct product of cocyclic modules by Lemma 2.4. Therefore M can be embedded as a pure submodule in a direct product of cocyclic modules.

Following [14], an R-module M is *compact* if there is a compact Hausdorff topology on M making it a topological group and such that the right multiplications by elements of R are continuous.

An R-module M is *algebraically compact* if M is a direct summand in every R-module G that contains M as a pure submodule (see [6]). Every divisible module and every bounded module are algebraically compact (see [11]).

It follows at once that a direct summand of an algebraically compact module is again algebraically compact, and a module is algebraically compact exactly if its reduced part is algebraically compact.

Theorem 2.6. *The following conditions on an R-module M are equivalent*:

(i) *M is pure-injective*;

(ii) *M is algebraically compact*;

(iii) *M is a direct summand of a direct product of cocyclic modules*;

(iv) *M is a direct summand of a compact R-modules*;

(v) *every finitely soluble family of linear equations over R in M has a simultaneous solution* (cf. Theorem 38.1 of [6]).

Proof. By Theorem 3.10 of [11] and Theorem 2 of [14], (i), (iv) and (v) are equivalent.

(i)$\Rightarrow$(ii): This is trivial.

(ii)$\Rightarrow$(iii): This follows immediately from Lemma 2.5.

(iii)$\Rightarrow$(i): Let $G = \Pi_\alpha G_\alpha$, where G_α are cocyclic, let M be a direct summand of G and let

$$M \xrightarrow{\rho} G \xrightarrow{\pi} M,$$

where ρ is the injection and π is the projection.

Now let $0 \rightarrow H \xrightarrow{i} K \rightarrow L \rightarrow 0$ be a pure-exact sequence and let $\phi: H \rightarrow M$ be any homomorphism. Since G_α is divisible or of bounded order, by Lemma 2.3, there exists $\psi_\alpha: K \rightarrow G_\alpha$ such that $\psi_\alpha i = \pi_\alpha \rho \phi$, where $\pi_\alpha: G \rightarrow G_\alpha$ is the projec-

tion.   We define a map $\psi\colon K\to G$ by $\psi(k)=(\psi_\alpha(k))$, where $k\in K$, i.e., $\pi_\alpha\psi=\psi_\alpha$. Then we can easily prove that $\phi=(\pi\ \psi)i$, as desired.

By using (iii) in Theorem 2. 6, the proof of the following Corollary is proceeded as in Corollary 38. 2 of [6].

Corollary 2. 7.   *Let M be a reduced module.   Then M is a pure-injective module if and only if M is a direct summand of a direct product of reduced cocyclic modules.*

3.   Basic submodules over g-discrete valuation rings

We shall study, in sections 4, 5, the structure of algebraically compact modules over bounded Dedekind prime rings.   For this purpose, we shall generalize, in this section, a result [10] on modules over commutative discrete valuation rings to the case of modules over non-commutative discrete valuation rings.

In commutative rings, a ring is a discrete valuation ring if it is a principal ideal domain with unique maximal ideal.   We now generalize, in a natural way, the concept of this to the case of non-commutative rings.   A ring R is called a *generalized discrete valuation ring* (for short: *g-discrete valuation ring*) if

(i) R is a prime ring,

(ii) R is a right and left principal ideal ring,

(iii) $J(R)$ is a unique maximal ideal of R,

(iv)   Idempotents modulo $J(R)$ can be lifted.

Furthermore, if R is a domain and $R/J(R)$ is a division ring, then we call R a *discrete valuation ring*.   Let P be a prime ideal of a bounded Dedekind prime ring R.   Then, by Lemmas 2. 2 and 2. 3 of [11], $\hat{R}_P$ is a g-discrete valuation ring.

Throughout this section R will be a fixed g-discrete valuation ring with unique maximal ideal P and Q will be the quotient ring of R.   Furthermore we denote the completion of R with respect to P by $\hat{R}$, and its maximal ideal by $\hat{P}$.

Lemma 3. 1.   (i) $R=(D)_k$, *where D is a discrete valuation ring with unique maximal ideal $P_0=p_0D=Dp_0$ $(p_0\in D)$.*

(ii) $\hat{R}$ *is a g-discrete valuation ring with maximal ideal $\hat{P}=p_0\hat{R}=\hat{R}p_0$ and $\hat{R}=(\hat{D})_k$, where $p_0\in D$ with $P_0=p_0D=Dp_0$.*

(iii)   *Let e, f be any uniform idempotents in R.   Then $eR\cong fR$.*

(iv) *Let e be any idempotent in R.   Then e is a uniform idempotent in R if and only if eR/eP is a simple R-module.*

Proof.   Since idempotents modulo P can be lifted, it is clear that $\hat{R}=(\hat{D})_k$. The other assertions follow from the same arguments as in Lemmas 2. 1, 2. 2 and 2. 3 of [11].

Let $R=(D)_k$.   Throughout this section, e_{ij} will denote the matrix with 1 in

the (i, j) position and zeros elswehere.

Lemma 3.2. *Let R, P, D and p_0 be as same as in Lemma 3.1. Then every proper R-submodule of $e_{11}Q$ containing $e_{11}R$ is of the form $e_{11}Rp_0^{-n}$ for some non-negative integer n.*

Proof. Since $P_0^n R = Rp_0^n$ for all n, it is clear that $e_{11}Rp_0^{-n}$ is an R-submodule of $e_{11}Q$ containing $e_{11}R$. Conversely, let K be an R-submodule of $e_{11}Q$ containing $e_{11}R$. First we shall prove that if $e_{11}rp_0^{-n} \in K$, where $e_{11}r \notin P$, then $e_{11}P_0^{-n} \in K$. We put

$$e_{11}r = \begin{pmatrix} r_{11}, \cdots, r_{1k} \\ 0 \end{pmatrix}, \text{ where } r_{1j} \in D \ (1 \le j \le k).$$

Since $e_{11}r \notin P$, there is a natural integer j $(1 \le j \le k)$ such that r_{1j} is a unit in D. Then

$$e_{11}rp_0^{-n}e_{j1} = \begin{pmatrix} r_{1j}p_0^{-n}, & 0.\cdots, 0 \\ 0 & 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since there is a unit u in D such that $r_{1j}p_0^{-n} = p_0^{-n}u$, we have $e_{11}p_0^{-n} = e_{11}rp_0^{-n}(e_{j1}u^{-1}e_{11}) \in K$, as desired. Since any element $e_{11}q$ in $e_{11}Q$ has the form $e_{11}ap_0^{-n}$, where $e_{11}a \notin P$, we may assume that $K \supseteq e_{11}p_0^{-n}$ and $K \not\supseteq e_{11}p_0^{-(n+1)}$ for some natural integer n. Then it is clear that $K \supseteq e_{11}Rp_0^{-n}$. Suppose that $K \supsetneq e_{11}Rp_0^{-n}$, then there is an element $k \in K$ such that $k \notin e_{11}Rp_0^{-n}$. We write $k = e_{11}ap_0^{-l}$, where $e_{11}a \notin P$ and $l > n$. Then we have $K \supseteq e_{11}Rp_0^{-l}$ and thus $K \supseteq e_{11}p_0^{-n-1}$, which is a contradiction. Hence $K = e_{11}Rp_0^{-n}$.

Let M be a torsion-free R-module. Then we can prove that the mapping $m \to m \otimes 1$, $m \in M$, is an R-monomorphism of M into the tensor product $M \otimes_R Q$, every element of MQ has the form mc^{-1}, where $m \in M$ and c is regular in R, and $M \otimes_R Q \simeq MQ$ under the correspondence $m \otimes q \to mq$, $q \in Q$. As usual, we consider M as a submodule of MQ and identify $m \in M$ with $m \cdot 1$. In similar fashion if N is a submodule of M, we have $N \subseteq N \otimes_R Q \simeq NQ = \{nc^{-1} \mid n \in N, \ c: \text{regular in } R\}$.

Lemma 3.3. *Let M be a torsion-free R-module with dim $M = 1$. Then M is either isomorphic to eQ or isomorphic to eR, where e is a uniform idempotent in R.*

Proof. (i) If M is divisible, then, by Lemma 3.16 of [11], $M \simeq eQ$ for any uniform idempotent e in R, because eQ is a minimal right ideal of Q.

(ii) If M is not divisible, then, clearly M is reduced and $Me_{11}R = M \neq 0$. Hence there exists an element $x \in M$ such that $xe_{11}R \neq 0$. By Theorem 2.4 of

[3], we have $e_{11}R \simeq xe_{11}R$ and so, by Proposition 4. 4 of [4], $MQ \simeq e_{11}Q$. So we may assume that $e_{11}R \subseteq M \subsetneqq e_{11}Q$. By Lemma 3. 2, $M = e_{11}Rp_0^{-n}$ for some non-negative integer n and thus $M = e_{11}Rp_0^{-n} = e_{11}p_0^{-n}R \simeq e_{11}R$, as desired.

Lemma 3. 4. *If M is not a divisible R-module, then M contains a non-zero pure cyclic uniform submodule.*

Proof. By Theorem 3. 19 of [11], we may assume that M is reduced.

(i) If M is torsion-free and if x is a non-zero element of M, then $xQ = S_1 \oplus \cdots \oplus S_n \subseteq MQ$, where S_i is a simple Q-module. Since MQ is an essential extension of M as an R-module, $N_i = S_i \cap M \neq 0$ for each i. Since $N_iQ = S_i$ and $N_i = N_iQ \cap M$, we have dim $N_i = 1$ and N_i is pure in M. By Lemma 3. 3, $N_i \simeq eR$ for any uniform idempotent e in R.

(ii) If M is not torsion-free, then the torsion module T of M is non-zero. By Theorems 3. 2 and 3. 24 of [11] T possesses a uniform cyclic direct summand N. Since T is pure in M, N is pure in M.

Lemma 3. 5. *Let M be an R-module and let S be a pure submodule such that M/S is not divisible. Then there exists an element $y \in M$ such that $S \cap yR = 0$ and $S \oplus yR$ is again pure.*

Proof. By Lemma 3. 4, there exists an element $y_0 \in M/S$ such that y_0R is a pure uniform submodule. Let $\sigma \colon M \to M/S$ be the canonical epimorphism and let N be the inverse image in M of y_0R. If y_0R is torsion-free, then, by Theorem 3. 1 of [11], y_0R is R-projective and so $N = S \oplus yR$ for some $y \in N$ with $\sigma(y) = y_0$. If y_0R is torsion, then, by Lemma 3. 5 of [11], $N = S \oplus yR$ for some $y \in N$ with $\sigma(y) = y_0$. Next we shall show that N is pure in M. Suppose that $xc = s + yr$, where $x \in M$, $s \in S$, $r \in R$ and c is regular in R. Then $\sigma(x)c = y_0r$. By the purity of y_0R, there exists an element y_0r_1 in y_0R, where $r_1 \in R$ with $\sigma(x)c = (y_0r_1)c$ and thus $xc - yr_1c \in S$. By the purity of S, there exists an element $s_1 \in S$ such that $(x - yr_1)c = s_1c$ and so $xc = (s_1 + yr_1)c$. Hence N is pure in M, as desired.

A submodule B of an R-module M is called a *basic submodule* if it satisfies the following conditions:

(i) B is a direct sum of cyclic modules,

(ii) B is pure in M,

(iii) M/B is divisible.

Theorem 3. 6. *Let R be a g-discrete valuation ring and let M be an R-module. Then*

(i) *M possesses a basic submodule.*

(ii) *Any two basic submodules of M are isomorphic.*

(iii) *Let S be a pure submodule of M and let B be a basic submodule of S.*

Then there exists a basic submodule of M which contains B as a direct summand.

Proof. (i) Let $\{x_i\}$ be a maximally pure independent subset of M, and let B be the submodule generated by the x's. Then B is certainly a direct sum of cyclic modules and, by Lemma 3. 5, M/B is divisible. Hence B is a basic submodule of M.

To prove (ii), let $R=(D)_k$, $J(R)=P$ and $P=p_0R=Rp_0$ be as in Lemma 3. 1. Let B be any basic submodule of M. By Theorems 3. 1, 3. 38 of [11] and Lemma 3. 3, B is a direct sum of uniform cyclic submodules and so B is a direct sum of a torsion-free module C and a torsion module E. First we shall prove that C is independent of the choice of B. By Lemma 3. 3, $C\cong\Sigma\oplus eR$ and $C/CP\cong\Sigma\oplus eR/eP$, where e is a uniform idempotent in R. By Lemma 3. 1, eR/eP is a simple R-module. Therefore by Lemma 3. 1 and Krull-Remak-Schmidt-Azumaya's theorem, it is enough to show that C/CP is an invariant for M. But this follows immediately from the same argument as in Lemma 21 of [10].

Next, we shall prove that the torsion component E of B is a basic submodule of T, where T is the torsion part of M. Suppose that T/E is not divisible. Then, by Lemma 3. 5, there exists an element $y\in T$ such that $E\cap yR=0$ and $E'=E\oplus yR$ is pure in T. Since $C\cap T=0$, the sum $B'=E'+C$ is direct. We shall prove that B' is pure in M. If $xc=t+z$, where $x\in M$, $z\in C$, $t\in E'$, $tp_0^l=0$ and c is regular in R, then $xcp_0^l=zp_0^l\in C$. By the purity of C, there exists an element $z_1\in C$ such that $zp_0^l=z_1cp_0^l$. Since C is torsion-free, $z=z_1c$ and thus $(x-z_1)c=t\in E'$. By the putity of E', there exists an element $t_1\in E'$ such that $(x-z_1)c=t_1c$ and thus $xc=(t_1+z_1)c$, as desired. The purity of B' implies that B'/B is pure in M/B. Since B'/B is of bounded order, B'/B is a direct summand of M/B by Theorem 3. 12 of [11] and thus B'/B is divisible, which is a contradiction. Hence E is a basic submodule of T. To prove that the submodule E is independent of the choice of B, by Theorem 3. 39 of [11], it is enough to show that the number of uniform cyclic summands of order P^n in E is an invariant for M. For $m>n$, it is equal to the number of uniform cyclic summands of order P^n in E/EP^m. Hence we may prove that E/EP^m is an invariant for M. Since E is a basic submodule of T, we have $TP^m+E=T$ and $TP^m\cap E=EP^m$. Hence $E/EP^m\cong(TP^m+E)/TP^m=T/TP^m$ and thus E/EP^m is an invariant for M for every m.

(iii) This follows from the same way as in (iii) of Lemma 21 of [10].

By Lemma 3. 14 of [11], a primary module over a bounded Dedekind prime ring is a module over a g-discrete valuation ring and so it has a basic submodule. Futhermore the concepts of quasi-basis and lower basic submodules in modules over g-discrete valuation rings can also be introduced by the analogy of that of abelian groups, and the results of Chapter V of [5] can be easily

carried over to that of modules over bounded Dedekind prime rings.

4. Complete modules and algebraically compact modules

For an R-module M we introduce two topologies as follows: The R-adic topology on M is defined by taking as neighborhoods of 0 the submodules MA (A are non-zero ideals of R). And the P-adic topology on M is defined by taking as neighborhoods of 0 the submodules MP^k ($k=1, 2, \cdots$), where P is a prime ideal of R. These topologies make M into a not necessarily Hausdorff topological R-module. A module M is called *complete* in a given topology if it is Hausdorff, and every Cauchy net in M has a limit in M. If $M = \Pi_\alpha M_\alpha$, then M is complete in the R-adic topology if and only if M_α is complete in the R-adic topology for each α, because R is noetherian.

If M is a pure submodule of an R-module G, then G is a *pure essential extension of* M if there are no non-zero submodules $S \subseteq G$ with $S \cap M = 0$ and the image of M is pure in G/S. A pure extension G of M is a *pure-injective envelope* if G is pure-injective and the extension is pure-essential. By Proposition 6 of [14], for any module, pure-injective envelopes exist and are unique up to isomorphism. Furthermore, by the same arguments as in abelian groups (see §41 of [6]), G is a pure-injective envelope of a module M if and only if G is a maximal pure-essential extension of M. Let M be an R-module. We denote the submodule $\cap MA$ by M^1, where A ranges over non-zero ideals of R.

Lemma 4. 1. *Let G be the pure-injective envelope of an R-module M and let $G = D \oplus C$, where D is divisible and C is reduced. Then $M^1 = D \cap M$. In particular, if $M^1 = 0$, then G is reduced.*

Proof. Since R is bounded, it is clear that $G^1 = \cap Gc$, where c ranges over regular elements in R. From this fact and Corollary 3. 11 of [11], we have

$$M^1 = \cap MA = \cap (GA \cap M) = G^1 \cap M = D \cap M .$$

If $M^1 = 0$, then $D \cap M = 0$. Clearly the image of M in G/D is pure and thus $D = 0$. Hence G is reduced.

For an abelian group M, the following three conditions are equivalent: (i) M is reduced, algebraically compact; (ii) M is complete in the Z-adic topology; (iii) $M = \Pi_p M_p$, where M_p is complete in the p-adic topology, where Z is the rational integers and p are prime numbers (cf. Theorem 39. 1 and Proposition 40. 1 of [6]).

For modules over bounded Dedekind prime rings, we have

Theorem 4. 2. *The following conditions on an R-module M are equivalent:*
(i) *M is reduced, algebraically compact;*

(ii) *M is complete in the R-adic topology*;

(iii) $M = \Pi_P M_P$, *where P ranges over non-zero prime ideals of R and each M_P is complete in the P-adic topology.*

In particular, the M_P are uniquely determined by M.

Proof. (i)$\Rightarrow$(ii): By Corollary 2.7, M is a direct summand of a direct product of reduced cocyclic modules. Since a reduced cocyclic module is complete in the R-adic topology, M is complete in the R-adic topology.

(ii)$\Rightarrow$(iii): Since M is Hausdorff, i.e., $M^1 = 0$, M is reduced and $0 = \cap_P \cap MP^k$ (P ranges over non-zero prime ideals of R and $k = 1, 2, \cdots$) by the same argument as in [1, p. 73, Proposition 6]. Let $M_P = \lim_{\leftarrow} M/MP^n$. Then the following definition makes M_P in a natural way into a module over the $\hat{R}_P$. Let $\hat{m} = (\cdots, m_i + MP^i, \cdots)$ be any element of M, where $m_i - m_j \in MP^i$ ($j \geq i$) and let $\hat{r} = (\cdots, r_i + P'^i, \cdots)$ be any element of $\hat{R}_P$, where $r_i - r_j \in P'^i$ ($j \geq i$). Since $R/P^n \cong R_P/P'^n$ by the natural correspondence and $P'^n \cap R = P^n$, we may assume that $r_i \in R$, $r_i - r_j \in P^i$ ($j \geq i$). We define $\hat{m}\hat{r} = (\cdots, m_i r_i + MP^i, \cdots)$. It is easily verified that with this definition M_P becomes an $\hat{R}_P$-module. It is clear that M_P is complete in the P-adic topology (equivalently, it is complete in the $\hat{P}$-adic topology). Let $\theta: M \to \Pi_P M_P$ be the diagonal homomorphism. Then it is clear that θ is a monomorphism. Using [1, p. 73, Proposition 6], we can prove that θ maps continuosly onto a dense submodule of ΠM_P and that the induced topology on $\theta(M)$ as a submodule is the same as the R-adic topology. Hence θ is an isomorphism.

Since $M_P P_i = M_P(P_i \neq P)$ and $0 = \cap_k M_P P^k$ ($k = 1, 2, \cdots$), we obtain that $M_P = \cap_{P_i \neq P} MP^k$ ($k = 1, 2, \cdots$). Hence the components M_P are uniquely determind by M

(iii)$\Rightarrow$(i): Since M_P is complete in the P-adic topology, it is in a natural way an $\hat{R}_P$-module, and so $M_P P_i = M_P$ for every non-zero prime ideal P_i ($\neq P$) of R. Hence M_P is complete in the R-adic topology. Thus M is complete in the R-adic topoloty. Now let G be an R-module such that G contain M as a pure submodule.

(a) If G is complete in the R-adic topology, then we may assume that $G = \Pi G_P$, where G_P is an $\hat{R}_P$-module, and complete in the P-adic topology, and that $G_P \supseteq M_P$. Let B_P be a basic submodule of M_P. By Theorem 3.6, we enlarge B_P to a basic submodule $B_P' = B_P \oplus C_P$ of G_P. Then we obtain that $G_P = \hat{B}_P = \hat{B}_P \oplus \hat{C}_P$ and that $M_P = \hat{B}_P$ Thus we have $G = M \oplus \Pi \hat{C}_P$.

(b) If G is not necessarily complete and $G^1 = 0$. Then, by Lemma 4.1, the pure-injective envelope $\bar{G}$ of G is also reduced, and hence $\bar{G}$ is complete. By case (a), we obtain that $\bar{G} = M \oplus K$, and thus $G = M \oplus (K \cap G)$.

(c) Let G be an arbitrary R-module and let $\bar{G} = G/G^1$. Then clearly $G^1 \cap M = 0$ and $\bar{M} = M \oplus G^1/G^1$ ($\cong M$) is pure in $\bar{G}$. Hence by case (b), we

obtain that $\bar{G}=\bar{M}\oplus\bar{K}$, where K is a submodule of G. Since $G=M+K$, and $M\cap K=(M\cap K)\cap G^1=(M\cap G^1)\cap K=0$, we have $G=M\oplus K$. Hence M is reduced, algebraically compact.

Using (iii) in Theorem 4. 2, the proof of the following Corollary is proceeded as in Corollary 40. 4 of [6].

Corollary 4. 3. *Every non-zero reduced, algebraically compact R-module contains a direct summand which is isomorphic to $e\hat{R}_P$ or $e/\hat{R}_P/e\hat{P}^n$ ($n=1, 2, \cdots$) for some P, where e is a uniform idempotent in $\hat{R}_P$.*

Corollary 4. 4. *Let R be a complete g-discrete valuation ring with quotient ring Q and let M be an indecomposable R-module. Then M is isomorphic to one of the following four modules: eR/eP^n, eR, eQ, or type P^∞, where e is a uniform idempotent in R and P is a unique maximal ideal of R.*

Proof. If M is reduced, torsion-free, then, by Lemma 3. 4, M contains a non-zero pure cyclic submodule which is isomorphic to eR. By Theorem 4. 2, it is a direct summand of M and thus $M\cong eR$. Now the assertion immediately follows from Corollary 3. 26 of [11].

Corollary 4. 5. *Let R be a complete g-discrete valuation ring and let M be a countably generated, torsion-free R-module. Then M is the direct sum of a divisible module and a reduced module E, where if E is finitely generated, then $E\cong eR \oplus\cdots\oplus eR$ for some uniform idempotent e in R and if E is not finitely generated, then E is a free R-module.*

Proof. By Theorem 3. 9 of [11], M is the direct sum of a divisible module and a reduced module E. If E is finitely generated, then, by Theorem 3. 1 of [11] and Lemma 3. 3, $E\cong eR\oplus\cdots\oplus eR$. If E is not finitely generated, then it is clear that M is expressed as the union of an ascending sequence of pure submodules $N_1\subseteq N_2\subseteq\cdots\subseteq N_n\subseteq\cdots$, where dim $N_n=n$. By induction, we assume that N_n is a direct sum of n number of copies of eR. Hence N_n is algebraically compact by Theorem 4. 2. Since $0\to N_n\to N_{n+1}$ is pure-exact and dim $N_{n+1}=n+1$, we have $N_{n+1}=N_n\oplus eR$. Hence E is a direct sum of an infinite number of copies of eR. Thus E is R-free by Theorem 2. 4 of [2].

Let $\{A_i|i\in I\}$ be the set of all non-zero ideals of R. We put $i\leq i$ for $i, j\in I$ to mean $A_i\supseteq A_j$. Thus the index set I is directed.

Now we put $\hat{M}=\varprojlim M/MA_i$ and $\mu: M\to\hat{M}$ is the natural homomorphism. Since M/MA is isomorphic to $\prod_i M/MP_i^{\alpha_i}$ for each ideal A of R ($A=P_1^{\alpha_1}\cdots P_k^{\alpha_k}$), we have $\hat{M}=\prod_P \hat{M}_P$, where $\hat{M}_P=\varprojlim M/MP^n$. It is clear that $\hat{M}_P$ is complete in the P-adic topology and that $\hat{M}_P\overleftarrow{A}=\hat{M}_P P^\alpha$, where P^α is the heighest power of P dividing A. Hence $\hat{M}_P$ is complete in the R-adic topology, and thus $\hat{M}$ is

complete in the R-adic topology. Furthermore we obtain that $\mu(M)$ is pure in $\hat{M}$. To prove this, suppose that $\hat{m}c = \mu(m)$, where $\hat{m} = (\cdots, m_i + MA_i, \cdots) \in \hat{M}$, $m \in M$ and c is a regular element of R. Since R is bounded, there is a non-zero ideal A_k such that $Rc \supseteq A_k$ and thus $Mc \supseteq MA_k$. Then $m_k c + MA_k = m + MA_k$ implies that $m_k c - m \in MA_k \subseteq Mc$ and so $\mu(m')c = \mu(m)$ for some $m' \in M$, as desired. Hence we have

Proposition 4. 6. *Let M be an R-module. Then $\hat{M}$ is the pure-injective envelope of $\mu(M)$.*

Theorem 4. 7. *The pure-injective envelope of an R-module M is isomorphic to the direct sum of the injective envelope of M^1 and the completion $\hat{M}$ of M.*

Proof. Let G be the pure-injective envelope of M. We write $G = D \oplus C$, where D is divisible and C is reduced. Then $M^1 = D \cap M$ by Lemma 4. 1. If $D \supsetneqq E(M^1)$, then we have $D = E(M^1) \oplus D_2$ with $D_2 \neq 0$. Clearly $D_2 \cap M = 0$ and $(D_2 \oplus M)/D_2$ is pure in G/D_2. Hence G is not a pure-essential extension of M, which is a contradiction and thus $D = E(M^1)$. Let $\bar{G} = G/E(M^1)$ and let $\bar{M} = M/M^1$ $(\simeq [M + E(M^1)]/E(M^1))$. Then $\bar{G}$ is a reduced pure-injective module and $\bar{M}$ is pure in $\bar{G}$. Hence we may assume that $\bar{M} \subset \hat{\bar{M}}(=\hat{M}) \subset \bar{G}$ by Proposition 6 of [14] and Proposition 4. 6. If $\hat{\bar{M}} \subsetneqq \bar{G}$, then there exists a proper submodule K of G such that $K \supseteq E(M^1)$, $K \supseteq M$ and $K/E(M^1) = \hat{\bar{M}}$. Hence $K \simeq E(M^1) \oplus \hat{\bar{M}}$. Thus K is a pure-injective module, which is a contradiction. Hence we have $\hat{M} = \hat{\bar{M}} = \bar{G}$, and thus $G \simeq E(M^1) \oplus \hat{M}$.

Following [6], an R-module M is a *P-adic module* if M is an $\hat{R}_P$-module.

Lemma 4. 8. *Let B_P, M_P be P-adic modules such that $B_P \subseteq M_P$. Let $B = \Sigma \oplus B_P$ and let $M = \Pi M_P$. Then*
  (i) *B_P is pure in M_P for every P if and only if B is pure in M.*
  (ii) *M_P/B_P is divisible for every P if and only if M/B is divisible.*

Proof. (i) This is immediate.
(ii) The "if" part is clear. The "only if" part: Let $m = (m_p)$ be any element of M and let c be any regular element of R. Since R is bounded, there exists a non-zero ideal A such that $Rc \supseteq A$, where $A = P_1^{a_1} \cdots P_k^{a_k}$. For any prime ideal P $(\neq P_1, \cdots, P_k)$, we have $R_P \supseteq R_P c \supseteq R_P A = R_P$, and hence c is a unit in R_P. Thus we obtain $m_p = m_p' c$ for some $m_p \in M_P$ $(P \neq P_1, \cdots, P_k)$. For any prime ideal P_i $(1 \leq i \leq k)$, $m_{p_i} = m_{p_i}' c + b_i$ $(m_{p_i}' \in M_{P_i}, b_i \in B_{P_i})$, because M_{P_i}/B_{P_i} is divisible. Now let $m' = (m_p')$, $b = \sum_{i=1}^k b_i$. Then we have $m = m'c + b$, as desired.

A submodule B of the R-module M is called a *basic submodule* if it is satisfies the following conditions:

(i) B is a direct sum of cyclic P-adic modules,

(ii) B is pure in M,

(iii) M/B is divisible.

Proposition 4. 9. *Let M be an algebraically compact R-module. Then*

(i) *M possesses a basic submodule.*

(ii) *Any two basic submodules of M are isormorphic.*

Proof. (i) Clearly we may restrict ourselves to the case when M is reduced. By Theorem 4. 2, we have $M = \Pi\, M_P$, where M_P is a P-adic module. By Theorem 3. 6, there exists a basic submodule B_P of M_P. Then $B = \Sigma \oplus B_P$ is a basic submodule of M by Lemma 4. 8.

(ii) Let B' be another basic submodule of M and let $B' = \Sigma \oplus B_P'$, where B_P' is the direct sum of all direct summands belonging to the same prime ideal P. Then, by Lemma 4. 8, $M = \hat{B}' \cong \Pi\, \hat{B}_P'$. By Theorem 4. 2, $\hat{B}_P' \cong M_P$ for every prime ideal P. It is clear that B_P' is a basic submodule of $\hat{B}_P'$ and so $B_P' \cong B_P$ by Theorem 3. 6. Hence $B' \cong B$ as an R-module.

Theorem 4. 10. *There is a one-to-one correspondence between the reduced algebraically compact modules M and the direct sums B of cyclic P-adic modules (P ranges over prime ideals of R): given M, we let its basic submodule B correspond to M: to a given B, there corresponds its R-adic completion.*

Proof. By Proposition 4. 9, the correspondence $M \to B$, where M is a reduced algebraically compact R-module and B is its basic submodule, is single-valued and $M = \hat{B}$. Conversely let B be a direct sum of cyclic P-adic modules. Then it is clear that $B^1 = 0$. Hence $\hat{B}$ is a pure-injective envelope of B by Proposition 4. 6 and so $\hat{B}$ is unique up to isomorphism. Clearly $\hat{B}$ is a reduced, algebraically compact module and B is a basic submodule of $\hat{B}$.

Corollary 4. 11. *There is a one-to-one correspondence between the P-adic, reduced, algebraically compact modules and the direct sums of cyclic P-adic modules.*

Let M be a reduced, algebraically compact R-module. M is called *adjusted* if it has no non-zero torsion-free direct summannds. Let B be a basic sub-module of a reduced, algbraically compact module M. We write $B = C \oplus D$, where C is the direct sum of all torsion-free, cyclic summands of B and D is the direct sum of all torsion, cyclic summsnds of B. Then $M = \hat{B} = \hat{C} \oplus \hat{D}$, $\hat{C}$ is a torsionfree, algebraically compact R-module and is an adjusted, algebraically compact R-module. Let T be the torsion submodule of M. Then clearly $\hat{D} \supseteq T \supseteq D$ and $\hat{D}/T$ is the maximal divisible submodule of M/T, because $\hat{D}/T$ is a homomorphic image of a divisible module $\hat{D}/D$ and $M/T \cong \hat{C} \oplus \hat{D}/T$. Hence we have

Proposition 4. 12.   *Let M be a reduced, algebraically compact R-module. Then there is a direct decomposition*

$$M = C \oplus D \, ,$$

where C is torsion-free, algebraically compact and D is an adjusted, algebraically compact module. D is a uniquely determined submodule of M.

By Theorem 4. 10 and the proof of Proposition 4. 12, we have

Corollary 4. 13.   *The mapping $B \to \hat{B}$ gives a one-to-one correspondence between the class of direct sums of torsion, cyclic modules and the class of adjusted, algebraically compact modules.*

5.   The Harrison duality theorem

In this section we shall study a structure of reduced, torsion-free, algebraically compact R-modules.   Let R be a bounded Dedekind prime ring and let Q be the quotient ring of R.   If P is a prime ideal of R, then $R_P = (D)_k$, where D is a local Dedekind prime domain with unique maximal ideal P_0.   Furthermore we have $P_0 = p_0 D = D p_0$, where $p_0 \in D$, $P' = p_0 R_P = R_P p_0$ and $\hat{P} = p_0 \hat{R}_P = \hat{R}_P p_0$ (see [11]).   The sequence $0 \to R_P/P'^n \overset{\theta_n}{\to} R_P/P'^{n+1}$ is exact, where $\theta_n(x + P'^n) = p_0 x + P'^{n+1}$ for every x in R_P.   The inductive limit $\varinjlim R_P/P'^n$ of the R-modules R_P/P'^n, under the homomorphism θ_n, is divisible, P-primary.   Since R_P/P'^n is isomorphic to $\hat{R}_P/\hat{P}^n$ in a natural correspondence, we have $\varinjlim R_P/P'^n \cong \varinjlim \hat{R}_P/\hat{P}^n$.   Hence if $\dim \hat{R}_P = d_P$, then $\dim (\varinjlim R_P/P'^n) = d_P$.   We define $Q_P = \{ q \in Q \mid q P^n \subseteq R$ for some non-negative integer $n\}$ for a fixed prime ideal P of R.   The module Q/R will be denoted by K and Q_P/R will be denoted by K_P.   Then we have

Lemma 5. 1.   (i) K_P *is isomorphic to Q/R_P as an (R, R)-bimodule.*

(ii) $K = \Sigma \oplus K_P$, *where P ranges over non-zero prime ideals of R. In particular, K_P is divisible, the P-primary part of K as a right and left R-module.*

(iii) K_P *is isomorphic to $\varinjlim R_P/P'^n$.*

(iv) $Hom_R(K_P, K_P) \cong \hat{R}_P$ *as an $\hat{R}_P$-module.*

Proof.   Since R is a bounded Asano order in Q, we can easily obtain that $Q_P + R_P = Q$, $Q_P \cap R_P = R$ and $(Q_{P_1} + \cdots + Q_{P_k}) \cap Q_{P_{k+1}} = R$, where P, P_i are prime ideals of R.   From these facts, (i) and (ii) are immediate.

(iii)   The map $\theta: \varinjlim R_P/P'^n \to Q/R_P$ ($\cong K_P$) defined by $q + P'^n \to p_0^{-n} q + R_P$, $q \in R_P$, is an R-monomorphism.   Let q be any element of Q.   Since $Q = R_P + Q_P$, we write $q = q_1 + q_2$, where $q_1 \in R_P$, $q_2 \in Q_P$.   Then $q_2 P^n \subseteq R$ for some n and so we have $q_2 = s p_0^{-n}$ ($s \in R_P$).   Since $p_0^{-n} R_P = R_P p_0^{-n}$, $q_2 = p_0^{-n} s_1$ ($s_1 \in R_P$).   Then $\theta(s_1 + P'^n) = p_0^{-n} s_1 + R_P = q + R_P$, and thus θ is an isomorphism.

(iv) By (i), we may prove that $\operatorname{Hom}_R(Q/R_P, \mathbf{Q}/R_P) \cong \hat{R}_P$. It is clear that $\operatorname{Hom}_R(Q/R_P, Q/R_P) = \operatorname{Hom}_{\hat{R}_P}(Q/R_P, Q/R_P)$. Let α be any element of $\operatorname{Hom}_R(Q/R_P, Q/R_P)$ and let $\alpha(p_0^{-n}+R_P) = r_n p_0^{-n}+R_P$, where $r_n \in R_P$. Then r_n is unique up to mod P'^n. Since $\alpha(p_0^{-n}+R_P) = \alpha(p_0^{-(n+1)}+R_P)p_0 = r_{n+1}p_0^{-n}+R_P$, we have $r_n - r_{n+1} \in P'^n$. Let $\hat{s} = (\cdots, s_n + P'^n, \cdots)$ be any element of $\hat{R}_P$. Then $\alpha\hat{s}(p_0^{-n}+R_P) = \alpha(s_n p_0^{-n}+R_P) = \alpha(p_0^{-n}s_n'+R_P) = r, s_n p_0^{-n}+R_P$, where $s_n' \in R_P$ with $s_n p_0^{-n} = p_0^{-n}s_n'$. Hence the map $\alpha \to \hat{r} = (\cdots, r_n + P'^n, \cdots)$ is an isomorphism as an $\hat{R}_P$-module.

Let M be an (R, R)-bimodule. We define $M_l[P^n] = \{x \in M \mid P^n x = 0\}$.

Lemma 5.2. *Let M be an (R, R)-bimodule such that M is torsion as a left R-module, let N be a right R-module and let $H = \operatorname{Hom}_R(M, N)$. Then*

(i) *H is a reduced, algebraically compact R-module.*

(ii) *If M is divisible as a left R-module, then H is torsion-free. Furthermore, if M is P-primary as a left R-module, then H is a P-adic module and $\alpha \in HP^n$ if and only if α annihilates $M_l[P^n]$.*

Proof. (i) follows from the same argument as in Theorem 46.1 of [6].

(ii) If $\alpha c = 0$, where $\alpha \in H$, c is a regular element in R, then $0 = \alpha c(M) = \alpha(cM) = \alpha(M)$ and thus $\alpha = 0$. Hence H is torsion-free. If M is P-primary, then clearly H is a P-adic module, because M is in a natural way a P-adic module by Lemma 3.14 of [11]. If $\alpha \in HP^n = Hp_0^n$, then clearly α annihilates $M_l[P^n]$. Conversely, assume that $\alpha(M_l[P^n]) = 0$. Let m be any element of M. Since $p_0^n M = M$, there exists an element $m' \in M$ such that $m = p_0^n m'$. Now define a map $\beta: M \to N$ by $\beta(m) = \alpha(m')$. Then β is well-defined, because m' is unique up to mod $M_l[P^n]$. Clearly β is additive. Let $r \in R$. Then $mr = p_0^n m' r$ and so $\beta(mr) = \alpha(m'r) = \beta(m)r$, i.e., $\beta \in H$. Since $p_0^n m = p_0^n m$, we have $(p_0^n m)' = m$. So $(\beta p_0^n)(m) = \beta(p_0^n m) = \alpha(m)$ and hence $\alpha = \beta p_0^n$, as desired.

Let M be an R-module. Then by Zorn's lemma there exist maximally independent sets of uniform submodules of M. By Theorem 1.10 of [12], the cardinal number of these sets is an invariant for M. We call it *the dimension of M*, and denote it by $\dim_R M$ ($=\dim M$). Let H be a torsion-free R-module. We define the *P-rank of H* to be the $\dim_R H/HP$ and denote it by P-rank H.

Lemma 5.3. *Let H, K be reduced, torsion-free, algebraically compact R-modules. Then*

(i) *P-rank $H = \dim_{\hat{R}_P}(B_P)$, where B_P is a basic submodule of P-adic component H_P of H (see Theorem 4.2).*

(ii) *$H \cong K$ if and only if P-rank $H = P$-rank K for every prime ideal P of R.*

Proof. (i) By Theorem 4.2, $H = \prod H_P$, where H_P is a P-adic component of H. Let B_P be a basic submodule of H_P and let $B = \Sigma \oplus B_P$. Then,

by Lemma 4. 8, B is a basic submodule of H. Hence we have $HP+B=H$ and $H/HP=(HP+B)/HP\cong B/(HP\cap B)=B/BP$. Because $BP=B_P P\oplus\Sigma_{P+P'} B_{P'}$, we have $H/HP\cong B_P/B_P P$ and thus P-rank $H=\dim B_P$.

(ii) Since $H_P=\hat{B}_P$, H_P is a pure-injective envelope of B_P. So (ii) follows immediately from (i).

Let M be a torsion R-module. We define the *P-dimension of M* as $\dim_R M[P]$, and denote it by P-dim M. Clearly P-dim $M=\dim M_P$, where M_P is the P-primary submodule of M.

Lemma 5. 4. *Let G_1, G_2 be torsion, divisible R-modules. Then $G_1\cong G_2$ if and only if P-dim $G_1=P$-dim G_2 for every prime ideal P of R.*

Proof. This is trivial.

For a convenience, we denote the R-module of type P^∞ by $R(P^\infty)$ and the cardinal number of a set S by $|S|$.

Lemma 5. 5. *Let M be a torsion, divisible R-module and let $H=\mathrm{Hom}_R(K, M)$. Then P-rank $H=P$-dim M for every prime ideal P of R.*

Proof. Let $K=\Sigma\oplus K_P$ and let $M=\Sigma\oplus M_P$ be the primary decomposition of M. Then we have

$$H\cong\Pi\,\mathrm{Hom}_R(K_P,M)\cong\Pi\,\mathrm{Hom}_R(K_P,\,M_P)\,,$$

as an R-module. By Lemma 5. 1, $\dim_{\hat{R}_P}[\mathrm{Hom}_P(K_P,\,K_P)]=d_P$, where $d_P=\dim\hat{R}_P$. Hence $\dim_{\hat{R}_P}[\mathrm{Hom}_R(K_P,R(P^\infty))]=1$ and $\mathrm{Hom}_R(K_P,\,R(P^\infty))$ is reduced by Lemma 5. 2. Thus we have $\mathrm{Hom}_R(K_P,\,R(P^\infty))\cong e\hat{R}_P$ by Lemma 3. 3, where e is a uniform idempotent in $\hat{R}_P$. Now we put $H_P=\mathrm{Hom}_R(K_P,\,M_P)$ and put $M_P=\Sigma_{i\in I}\oplus E_i$, where $|I|=\dim M_P$ and $E_i=R(P^\infty)$. Let $B_P=\Sigma_i\oplus\mathrm{Hom}_R(K_P,\,E_i)$. We shall prove that B_P is a basic submodule of H_P. It is clear that B_P is a direct sum of cyclic P-adic modules. If $\alpha c=\alpha_1\oplus\cdots\oplus\alpha_k$, where $\alpha\in H_P$, $\alpha_i\in\mathrm{Hom}_R(K_P,\,E_i)$ and c is a regular element of R. Then $\alpha(K_P)=\alpha(cK_P)=\alpha c(K_P)\subseteq E_1\oplus\cdots\oplus E_k$ and thus $\alpha\in B_P$. Hence B_P is pure in H_P. To prove that H_P/B_P is divisible, we let α be any element of H_P and let $S=S_1\oplus\cdots\oplus S_{d_P}$ be the socle of K_P. Then there are a finite number $i_1,\,\cdots,\,i_n\in I$ such that $\alpha(S)\subseteq E_{i_1}\oplus\cdots\oplus E_{i_n}$. Thus the restricted map $\alpha'=\alpha\,|\,S\colon S\to E_{i_1}\oplus\cdots\oplus E_{i_n}$ can be extended to a map $\beta\colon K_P\to E_{i_1}\oplus\cdots\oplus E_{i_n}$. Since $(\alpha-\beta)(S)=0$, we have $\alpha-\beta\in Hp_0$ by Lemma 5. 2 and $\beta\in B_P$. So $(H_P/B_P)p_0=H_P/B_P$. Let c be any regular element of $\hat{R}_P$. Then $\hat{R}_P c\supseteq\hat{P}^n$ for some n, because $\hat{R}_P$ is bounded. Hence we have $(H_P/B_P)c\supseteq(H_P/B_P)P^n=(H_P/B_P)p_0^n=H_P/B_P$, and thus B_P is a basic submodule of H_P. By Lemma 5. 3, we obtain that P-rank $H=\dim_{\hat{R}_P}B_{\hat{P}}=P$-dim M, as desired

Theorem 5. 6.   *The correspondence*

$$(*) \quad M \to H = \mathrm{Hom}_R(K, M)$$

is one-to-one between all divisible, torsion R-modules M and all reduced, torsion-free, algebraically compact R-modules H.   The inverse of (*) is given by the correspondence: $H \to H \otimes_R K$ *(cf. Proposition 2. 1 of [8]).*

Proof.   By Lemmas 5. 2, 5. 3, 5. 4 and 5. 5, the correspondence (*) is a monomorphism.   Let H be any reduced, torsion-free, algebraically compact R-module and let $H = \Pi \, H_P$ be as in Theorem 4. 2.   Furthermore let B_P be a basic submodule of H_P and let $B = \Sigma \oplus B_P$.   By Lemma 4. 8, the short exact sequence $0 \to B \to H \to H/B \to 0$ is pure-exact and thus we obtain that $0 \to B \otimes_R K \to H \otimes_R K \to (H/B) \otimes_R K \to 0$ is exact by Proposition 3 of [14].   Since H/B is divisible and K is torsion, $H/B \otimes_R K = 0$ and so $B \otimes_R K \cong H \otimes_R K$.   Since $B = \Sigma \oplus B_P$ and $K = \Sigma \oplus K_P$, we have $B \otimes_R K \cong \Sigma \oplus (B_P \otimes_R K_P)$.   Now we denote $B_P = \Sigma \oplus e\hat{R}_P$, where e is a uniform idempotent in $\hat{R}_P$.   Then $B_P \otimes_R K_P \cong \Sigma \oplus (e\hat{R}_P \otimes_R K_P)$.   Since $\hat{R}_P \otimes_R K_P \cong K_P$ and $\dim K_P = d_P$, where $d_P = \dim_{\hat{R}_P}(\hat{R}_P)$, we obtain that $e\hat{R}_P \otimes_R K_P$ is a P-primary, uniform, divisible R-module.   Hence $H \otimes_R K$ is a torsion, divisible R-module, and clearly $P\text{-dim}\,(H \otimes_R K) = \dim_{\hat{R}_P}(B_P) = P\text{-rank } H$ for every prime ideal P of R.   By Lemmas 5. 3, 5. 5, $H \cong \mathrm{Hom}_R(K, H \otimes_R K)$.   This completes the proof.

Corollary 5. 7.   *A torsion-free R-module H is a reduced, algebraically compact R-module if and only if it is isomorphic to a direct summand of a direct product of copies of the modules $e_P \hat{R}_P$, where e_P is a uniform idempotent in $\hat{R}_P$ and P ranges over non-zero prime ideals of R (cf. Propositions 2. 1 and 3. 6 of [8]).*

Proof.   The "if" part is clear.   The "only if" part: By Theorem 5. 6, $H = \mathrm{Hom}_R(K, M)$, where $M = \Sigma_P \oplus \Sigma \oplus R(P^\infty)$.   Since M is divisible, it is a direct summand of $\Pi_P \Pi \, R(P^\infty)$.   Hence H is a direct summand of $\mathrm{Hom}_R(K, \Pi_P \Pi \, R(P^\infty)) = \Pi_P \Pi \, \mathrm{Hom}_R(K, R(P^\infty))$, and $\mathrm{Hom}_R(K, R(P^\infty))$ is isomorphic to $e_P \hat{R}_P$, where e_P is a uniform idempotent in $\hat{R}_P$ (see the proof of Lemma 5. 5).

OSAKA UNIVERSITY

References

[1]   N. Bourbaki: Algèbre Commutative, (ch. 1 and 2) Hermann, Paris, 1961.
[2]   D. Eisenbud and J.C. Robson: *Modules over Dedekind prime rings,* J. Algebra **16** (1970), 86–104.
[3]   E.H. Feller and E.W. Swokowski: *Prime modules,* Canad. J. Math. **17** (1965), 1041–1052.

[4]   E.H. Feller and E.W. Swokowski: *Semi-prime modules,* Canad. J. Math. **18** (1966), 823–831.

[5]   L. Fuchs: Abelian Groups, Budapest, 1958.

[6]   L. Fuchs: Infinite Abelian Groups Vol. 1, Pure and Applied Math, Academic Press, New York, 1970.

[7]   A.W. Goldie: *Localization in non-commutative noetherian rings,* J. Algebra **5** (1967), 89–105.

[8]   D.K. Harrison: *Infinite abelian groups and homological methods,* Ann. of Math. **69** (1959), 336–391.

[9]   I. Kaplansky: *Modules over Dedekind rings and valuation rings,* Trans. Amer. Math. Soc. **72** (1952), 327–340.

[10] I. Kaplansky: Infinite Abelian Groups, Ann Arbor, 1969.

[11] H. Marubayashi: *Modules over bounded Dedekind prime rings,* Osaka J. Math. **9** (1972), 95–110.

[12] Y, Miyashita: *On quasi-injective modules,* J. Fac. Sci. Hokkaido Univ. **18** (1965), 158–187.

[13] T. Onodera: *Koenlich erzeugte Moduln und Kogeneratoren,* to appear.

[14] R.B. Warfield: *Purity and algebraic compactness for modules,* Pacific J. Math. **28** (1969), 699–719.