

TOPOLOGY OF CLASSICAL GROUPS^{*)}

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Introduction

T.T. Frankel [5] applied Morse theory to the classical groups using the trace function. The critical sets turn out to be Grassmann manifolds. Frankel's results show how the classical groups can be obtained by "attaching" plane-bundles over Grassmann manifolds. In this paper we apply Morse theory to the classical groups using another elementary function, namely "the length function". We think of the classical groups as imbedded in suitable euclidean space [6], and use methods similar to those of R. Bott [3]. Finally using some results on fixed point theory it is shown that the Morse inequalities are equalities. This method is due to Frankel [5]. The CW-decomposition and the Poincaré polynomials obtained in this paper, are, of course, well-known and have been obtained by several different ways.

Preliminaries

Let F be the field R of real numbers, the field C of complex or Q , the quaternions. Let $U(n; F)$ be the group of unitary matrices of degree n over F , i.e., $U(n; F) = \{A: A \bar{A}^t = I_n\}$, where 'bar' stands for complex or quaternionic conjugation as the case may be. Thus $U(n; R) = SO(n)$, (instead of $O(n)$ we take $SO(n)$) $U(n; C) = U(n)$, $U(n; Q) = Sp(n)$. Let $M(n; F)$ be the space of all matrices of degree n over F . Let $E = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$. Then the map of $U(n; F) \times U(n; F) \rightarrow M(2n; F)$ given by $\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} = A \rightarrow A A \bar{A}^t = \begin{pmatrix} 0 & X \bar{Y}^t \\ Y \bar{X}^t & 0 \end{pmatrix}$, where $X, Y \in U(n; F)$, induces an imbedding of $U(n; F) = \frac{U(n; F) \times U(n; F)}{\Delta}$ (here Δ is the diagonal) into $M(n; F)$. In this imbedding $X \in U(n; F)$ is sent into $X \in M(n; F)$. See Kobayashi [6]. This is an imbedding of $U(n; F)$ considered as a "symmetric space".

In Cartan's theory symmetric spaces arise as follows—Let G be a compact,

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connected Lie group with a left and right invariant Riemannian metric. Let $\sigma: G \rightarrow G$ be an automorphism of period 2 (i.e., an involution, $\sigma \neq \text{identity}$). Let K_e be the identity component of the full fixed set of σ . Then G/K_e is a symmetric space. If we take $G = U(n; F) \times U(n; F)$ and consider the involution $\sigma(x, y) = (y, x)$, the fixed set is $\Delta = \{(x, x) \mid x \in U(n; F)\}$. Thus $\frac{U(n; F) \times U(n; F)}{\Delta}$ is a symmetric space identified with $U(n; F)$ by the correspondence $(x, y) \rightarrow xy^{-1}$.

We use the identification $\frac{U(n; F) \times U(n; F)}{\Delta} \approx U(n; F)$ to think of $U(n; F)$ as an orbit imbedded in $M(n; F)$. Let L be a connected real non-compact Lie group and \mathfrak{l} its Lie algebra. Let K be a maximal connected compact subgroup of L and \mathfrak{k} its Lie algebra. We take $L = SO(n, n)$, $U(n, n)$ and $Sp(n, n)$. Then $K = U(n; F) \times U(n; F)$, $F = R, C$, or Q .

We recall that $SO(p, q)$ is the group of all matrices in $SL(p+q, R)$ which leave invariant the quadratic form $-x_1^2 - x_2^2 \dots - x_p^2 + x_{p+1}^2 + \dots + x_{p+q}^2$. The group $U(p, q)$ is the group of all matrices in $GL(p+q, C)$ which leave invariant the Hermitian form

$$-Z_1 \bar{Z}_1 - Z_2 \bar{Z}_2 - \dots - Z_p \bar{Z}_p + Z_{p+1} \bar{Z}_{p+1} + \dots + Z_{p+q} \bar{Z}_{p+q}$$

A similar definition can be given for $Sp(p, q)$. The Lie algebras are

$$\begin{aligned} \mathfrak{so}(p, q) &= \left\{ \begin{pmatrix} X_1 & X_2 \\ X_2^t & X_3 \end{pmatrix} \begin{array}{l} X_1, X_3 \text{ skew symmetric of orders } p \text{ and } q \text{ respectively} \\ X_2 \text{ arbitrary-all coefficients real} \end{array} \right\} \\ \mathfrak{u}(p, q) &= \left\{ \begin{pmatrix} Y_1 & Y_2 \\ \bar{Y}_2^t & Y_3 \end{pmatrix} \begin{array}{l} Y_1, Y_3 \text{ skew symmetric of orders } p \text{ and } q \text{ respectively} \\ Y_2 \text{ arbitrary-all coefficients complex} \end{array} \right\} \\ \mathfrak{sp}(p, q) &= \left\{ \begin{pmatrix} Z_1 & Z_2 \\ \bar{Z}_2^t & Z_3 \end{pmatrix} \begin{array}{l} Z_1, Z_3 \text{ matrices of orders } p \text{ and } q \text{ respectively} \\ Z_1 = -\bar{Z}_1^t, Z_3 = -\bar{Z}_3^t, \text{ quaternionic conjugation} \\ Z_2 \text{ arbitrary-all coefficients quaternions.} \end{array} \right\} \end{aligned}$$

The element $E = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \in \mathfrak{p}^*$ in each of the three cases and for the adjoint action of $K = U(n; F) \times U(n; F)$ on \mathfrak{p} , the orbit of E is precisely $\frac{U(n; F) \times U(n; F)}{\Delta} = U(n; F)$. Thus $U(n; F)$ is imbedded in $M(n; F)$. The Cartan-Killing form on \mathfrak{l} restricted to \mathfrak{p} induces the invariant metric $(X, Y) = \text{Re tr } X \bar{Y}^t$ up to a positive constant for $X, Y \in M(n; F)$, which agrees with the metric on $U(n; F)$. Thus the imbeddings are isometries. See Kobayashi [6].

The adjoint action of K on \mathfrak{p} becomes the action of $U(n; F) \times U(n; F)$ on $M(n; F)$ given by $(X, Y) \rightarrow X B Y^{-1}$, $B \in M(n; F)$.

*) the orthogonal complement of \mathfrak{k} in \mathfrak{l} with respect to an invariant metric.

Length function and its critical set

Let M^m be an m -dimensional manifold differentiably imbedded in a Euclidean space R^n . Let $p \in R^n - M^m$. Let $L_p(x)$ = square of the distance between $x \in M^m$ and the fixed point p . Then $q \in M^m$ is a critical point for $L_p(x)$ if and only if the straight line pq is perpendicular to the tangent space M_q of M at q . For details, see [3] or [7].

We are considering $U(n; F)$ as imbedded in $M(n; F)$. If we take the point $p = \begin{pmatrix} 3 & & & \\ & 6 & 0 & \\ & & \ddots & \\ & & & 3n \end{pmatrix}$ then the orbit of p is $\frac{U(n; F) \times U(n; F)}{\Lambda}$, $\Lambda = S^d \times \dots \times S^d$ (n copies) where $d=0$ if $F=R$, $d=1$ if $F=C$, and $d=3$ if $F=Q$. Also Λ is imbedded in $U(n; F) \times U(n; F)$ under the diagonal map. Here we are considering the action of $U(n; F) \times U(n; F)$ on $M(n; F)$.

It can be verified that the tangent space to the orbit of p at p is $Xp - pY$, all $X, Y \in \mathfrak{u}(n; F)$ the Lie algebra of $U(n; F)$. If we choose p as in previous paragraph, then the normal space to the orbit of p at p is D_R = all real diagonal matrices in $M(n; F)$. It is known that if a straight line is perpendicular to an orbit at a point, then it is perpendicular to all the orbits it meets [4, p. 967]. Hence all real diagonal matrices of $U(n; F)$ are in the critical set of L_p . On the other hand if $\sigma \in U(n; F)$ is a critical point for L_p , then the straight line $p\sigma$ must be in D_R since $p\sigma$ must be perpendicular to the orbit of p . Hence all the critical points of L_p are all the diagonal matrices in $U(n; F)$ i.e. diagonal matrices with $+1$ or -1 along the diagonal. If $F=R$ then there 2^{n-1} such matrices, because the determinant of such a matrix has to $+1$. If $F=C$ or Q then there are 2^n such critical points. These critical points are isolated.

Non-degeneracy and index of the critical points

We briefly state the procedure to find the index of the critical points. The full details and proofs can be found in [2], [3] or [7].

In general, let M^m be a manifold imbedded in a Euclidean space R^n . Let (x_1, x_2, \dots, x_n) be a system of co-ordinates for R^n and let $g(t)$ be a straight line given by $x_i = p_i + tq_i, i=1, \dots, n$. A variation of $g(t)$ will be a differentiable family of straight lines

$$V(\rho, t): x_i(\rho) = p_i(\rho) + tq_i(\rho), \quad -\infty < \rho < \infty$$

with $p_i(0) = p_i, q_i(0) = q_i, i=1, \dots, n$. The variational vector-field

$$\eta(t) = \left. \frac{dp_i}{d\rho}(\rho) \right|_{\rho=0} + t \left. \frac{dq_i(\rho)}{d\rho} \right|_{\rho=0}$$

induced by the variation $V(\rho, t)$ is called a *Jacobifield* (J-field) along g . The J-fields along g form a vector-space J_g of dimension $2n$. If M^m is a manifold imbedded in R^n , if g is orthogonal to M at g , then a *variation of g relative to M* is a variation $V(\rho, t)$ of g such that 1) $V(\rho, 0) \in M$ and 2) $V(\rho, t)$ are orthogonal to M at $V(\rho, 0)$. Let $J_g(M)$ be the J-fields along g which are induced by the variations of g relative to M . Then $J_g(M)$ is a sub-space of J_g and $\dim J_g(M) = n$. For every point $g(t_0)$ there is a restriction $\mathcal{X}_{t_0}: J_g(M) \rightarrow R_{g(t_0)}$, where $R_{g(t_0)}$ is the tangent space to R^n at $g(t_0)$. The segment $s = (g(0) g(t_0))$ is called a *focal segment* of M of multiplicity ν if $\dim \ker \mathcal{X}_{t_0} = \nu > 0$. This kernel will be denoted by $\Lambda_s(M)$ and is called the *focal kernel* of s relative to M . The following result shows how to compute the indices of the critical points.

Theorem [3, p. 31] *Let M be a proper differentiable submanifold of R^n . Let $a \in R^n$, $a \notin M$ and let $b \in M$ be a critical point of $L_a(x)$. Let $\nu(t)$ be the multiplicity of the focal segment $ta + (1-t)b$ if this segment is a focal segment and zero otherwise. Then the index of $b = \sum_{0 < t < 1} \nu(t)$.*

It is also known that along any segment S there can only be a finite number of focal points. Also, if p is not a focal point of (M, g) for any $g \in M$, the function L_p has non-degenerate critical points.

We combine the method of Bott [2, 3] and theory of Bott and Samelson [4] to find the indices of the critical points. Let π denote the adjoint action on \mathfrak{p} . [See section 2]. Then π becomes the action of $U(n; F) \times n; F$ on $M(n; F)$ given by $(X, Y) \rightarrow XBY^{-1}$, $X, Y \in U(n; F)$ and $B \in M(n; F)$.

In general let $\pi: K \times N \rightarrow N$ represent the action of K (a compact group of isometries of N) on N . Let M be an orbit of a point in N . A geodesic segment s will be called *transversal* if its initial direction is perpendicular to the orbit of its initial point. If p is the initial point of such a segment s , then $J_s(M)$ will be denoted by J_s^π . The focal kernel $\Lambda_s(M)$ be denoted as Λ_s^π . The Lie algebra \mathfrak{k} of K determines a representation $\dot{\pi}$ of \mathfrak{k} by vector-fields on N . These are called *infinitesimal K -motions* on N . By definition $\dot{\pi}(X)_p = \frac{\partial}{\partial t} \{ \pi(\exp tX)p \}_{t=0}$ ($X \in \mathfrak{k}, p \in N$). If s is a transversal geodesic segment in N , the variation $V_\alpha(t) = \pi(\exp \alpha X) s(t)$ is a variation relative to M . Hence $\dot{\pi}$ followed by restriction to s induces a map $\dot{\pi}_s: \mathfrak{k} \rightarrow J_s^\pi$. The action of K on N via π is called *variationally complete* if $\Lambda_s^\pi \subset \pi_s(\mathfrak{k})$ for any transversal geodesic segment s . Let $c(q)$, $q \in N$, be the subspace of \mathfrak{k} whose $\dot{\pi}$ -image vanishes at q . Let $c(s) \subset \mathfrak{k}$ be the kernel of $\dot{\pi}_s$. If s is transversal with end-point q , then variational completeness implies

$$0 \rightarrow c(s) \rightarrow c(q) \xrightarrow{\dot{\pi}_s} \Lambda_s^\pi \rightarrow 0$$

is exact. Hence

$$\dim \Lambda_s^\pi = \dim c(q) - \dim c(s).$$

In our present situation π is the adjoint action of K on \mathfrak{p} . Bott and Samelson [4] have proved that such action is variationally complete. If σ is a critical point and the point $p \in M(n; F)$ is chosen as already indicated, it can be verified that $\dim c(s) = 0$ if $F=R$, $=n$ if $F=C$, $=3n$ if $F=Q$. Also the focal points along $\sigma p = s(t)$ are those real diagonal matrices in $s(t)$ which have (for $0 < t < 1$) 1) Zero (s) on the diagonal or 2) two elements on the diagonal, with the same absolute value. These can be found easily and the $\dim c(q)$ can be computed in each case.

The relationship between critical points and the homology groups can be stated in terms of (weak) Morse inequalities, namely, $b_i(U(n; F)) \leq c_i$ where c_i is the number of critical points of index i , b_i is the i th Betti number of $U(n; F)$ with a field as coefficients. If $F=R$ then we use Z_2 as coefficients and in the other two cases we use any field as coefficients.

Applications of fixed point theory

We show that the (weak) Morse inequalities are equalities. For this purpose, we use Smith theory of fixed points of periodic transformations. The method is similar to those of Frankel [5]. We use the following two theorems (More general results and proofs can be found in [1].)

Theorem I. *If $\Gamma = \begin{pmatrix} \pm 1 & 0 \\ & \ddots \\ 0 & \pm 1 \end{pmatrix}$ i.e., $Z_2 \times \dots \times Z_2$ (n copies) acts on compact differentiable manifold M , if F is the fixed set, then $\sum_i b_i(F; Z_2) \leq \sum_i b_i(M; Z_2)$.*

Theorem II. *If a total group operates in a compact differentiable manifold M , and if F is the fixed set, then*

$$\sum_i b_i(F; K) \leq \sum_i b_i(M; K)$$

where $K=R$ or Z_p , p is prime.

Theorem I is applied to $SO(n)$. For the adjoint action of Γ on $SO(n)$, the fixed set consists of diagonal matrices with $+1$ or -1 on the diagonal. Since such matrices must have determinant $+1$, the fixed set consists of 2^{n-1} points. Since the total betti numbers (with Z_2 as coefficients) must be no more than 2^{n-1} , the Morse inequalities for $SO(n)$ have to become equalities.

Theorem II is applied to $U(n)$ and $Sp(n)$. The maximal torus T acts on $G(U(n)$ or $Sp(n))$ by adjoint action, i.e., $t \in T: g \rightarrow t g t^{-1}$. A point $g \in G$ is fixed under T if and only if $g \in T$. Hence by Theorem II,

$$2^n = \sum_i b_i(T; K) \leq \sum_i b_i(G; K)$$

where $n = \dim T$, and $K = R$ or Z_p , p prime. Again as in previous paragraph, the Morse inequalities for $U(n)$ and $Sp(n)$ become equalities. This also shows that $U(n)$ and $Sp(n)$ have no torsion.

By an inductive argument, the well-known Poincaré polynomials of $SO(n)$, $U(n)$ and $Sp(n)$ can be obtained.

An example

As a concrete example consider $U(2)$. The critical points for the functions $L_p(x)$, $p = \begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix}$ are $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}$, $\begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

Critical Point σ	Focal Points on σp	Multiplicity	Index of σ
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	none		0
$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 9/4 \end{pmatrix}$	1	1
$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 9/7 & 0 \\ 0 & 0 \end{pmatrix}$	1	3
	$\begin{pmatrix} 9/5 & 0 \\ 0 & 9/5 \end{pmatrix}$	2	
$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -3/7 & 0 \\ 0 & 0 \end{pmatrix}$	1	4
	$\begin{pmatrix} -3/11 & 0 \\ 0 & 3/11 \end{pmatrix}$	2	
	$\begin{pmatrix} 0 & 0 \\ 0 & 3/4 \end{pmatrix}$	1	

REMARK 1. The results obtained can be stated in terms of diagram, roots, etc. Also these methods can be applied to other orbits obtained by the action of $U(n; F) \times U(n; F)$ on $M(n; F)$.

REMARK 2. We have considered the pairs (L, K) , $L = SO(n, n)$, $U(n, n)$ and $Sp(n, n)$ and $K = U(n; F) \times U(n; F)$, $F = R, C$ or Q . Instead we could have looked at the pairs (G, K) , $G = SO(2n)$, $U(2n)$ or $Sp(2n)$ and K as above. The procedure will be exactly the same, except for a suitable modification of E .

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