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PARAMETRIZATION FOR A CLASS OF RAUZY FRACTALS

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Abstract
In this paper, we study a class of Rauzy fractals $\mathcal{R}_a$ given by the polynomial $x^3 - ax^2 + x - 1$ where $a \geq 2$ is an integer. In particular, we give explicitly an automaton that generates the boundary of $\mathcal{R}_a$ and using an unusual numeration system we prove that $\mathcal{R}_a$ is homeomorphic to a topological disk.

1. Introduction

The Rauzy fractal is a compact subset of $\mathbb{R}^n$, $n \geq 1$. It was studied by many mathematicians and is connected to many topics such as: numeration systems [8, 11, 22, 19], geometrical representation of symbolic dynamical systems [6, 20], multidimensional continued fractions and simultaneous approximations [5, 14], auto-similar tilings [6, 22], substitutions and tilings [12] and Markov partitions of Hyperbolic automorphisms of Torus [20, 22].

Let $\beta > 1$ be a fixed real number. Any positive real number x can be expanded as

$$x = \sum_{i=N_0}^{\infty} a_{-i}\beta{-i} = a_{-N_0}\beta^{-N_0} + a_{-N_0-1}\beta^{-N_0-1} + ...$$

with $a_i \in \mathbb{Z} \cap [0, \beta)$ and we are assuming the greedy condition

$$\left| x - \sum_{i=N_0}^{N} a_{-i}\beta^{-i} \right| < \beta^{-N},$$

for all $N \geq N_0$. We call this expansion a beta expansion of x in base β. A Pisot number is an algebraic integer whose conjugates other than itself have modulus less than one. Let $Fin(\beta)$ be a set consisting of all finite beta expansions and consider the condition

$$(F) \qquad Fin(\beta) = \mathbb{Z}[\beta^{-1}]_{\geq 0}.$$

Consider the beta expansion of the positive number

$$0 < 1 - [\beta]\beta^{-1} = c_{-2}\beta^{-2} + c_{-3}\beta^{-3} + ... = .0c_{-2}c_{-3}....$$

If we put $c_{-1} = [\beta]$, we can write

$$1 = .c_{-1}c_{-2}c_{-3}....$$

This expansion $.c_{-1}c_{-2}c_{-3}...$ is called the expansion of 1 and denoted by $d(1, \beta)$. We can identify this expression with the word $c_{-1}c_{-2}c_{-3}...$ generated by $\mathbb{A} = \mathbb{Z} \cap [0, \beta)$. Every

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finite word generated by $\mathbb{A}$ represents a beta expansion in base β if and only if the word is lexicographically less than $d(1,\beta)$ at any starting point. This fact can be generalized to infinite words apart from certain exceptions (see [21]).

In [16] they proved that if $\beta > 1$ is an integer then (F) holds and, conversely, the condition (F) implies that β is a Pisot number. A Pisot number β is called a Pisot unit if it is also a unit of the integer ring of $\mathbb{Q}[\beta]$. In [2] we have the following results :

Theorem 1.1. *Let β ba a cubic Pisot number. Then $\beta > 1$ has property (F) if and only if β is a root of the following polynomial with integer coefficients:*

$$x^3 - ax^2 - bx - 1, a \geq 0, \text{ and } -1 \leq b \leq a + 1.$$

Lemma 1.2. *Let $\beta > 1$ be a cubic Pisot number with $Irr(\beta) = x^3 - ax^2 - bx - 1$. Then the expansion of 1 in base β is given by:*

i) $d(1,\beta) = .(a-1)(a+b-1)(\widetilde{a+b})$, *if* $-a+1 \leq b \leq -2$;
ii) $d(1,\beta) = .ab1$, *if* $0 \leq b \leq a$;
iii) $d(1,\beta) = .(a-1)(a-1)01$, *if* $b = -1$;
iv) $d(1,\beta) = .(a+1)00a1$, *if* $b = a+1$.

Here $\widetilde{w}$ is the periodic expansion wwww....

Theorem 1.3. *A cubic Pisot unit β has property (F) if and only if $d(1,\beta)$ is finite.*

To each cubic Pisot unit satisfying (F), we can associate a Rauzy fractal. In the case where β is a Pisot number satisfying condition (ii) of Lemma 1.2, the Rauzy fractal was studied in [17] and [18]. In [17] the authors proved that if $2b > a - 3$, then the boundary of the Rauzy fractal is not homeomorfic to a circle. If β is a cubic Pisot unit satisfiyng (iii) of Lemma 1.2 and $\alpha, \overline{\alpha}$ its Galois conjugates the fractal associated is given by

$$\mathcal{R}_a = \left\{ \sum_{i=2}^{\infty} a_i \alpha^i,\ a_i a_{i-1} a_{i-2} a_{i-3} <_{lex} (a-1)(a-1)01,\ \forall i \geq 5 \right\},$$

where $<_{lex}$ is the lexicographic order on finite words. In [9] the authors proved the topological and arithmetical properties of $\mathcal{R}_a$. In particular, they proved there exists an explicit finite state automaton $\mathcal{A}$ such that the boundary of $\mathcal{R}_a$ is recognized by $\mathcal{A}$. With this automaton they proved that for $a = 2$, the boundary of $\mathcal{R}_2$ is homeomorfic to a circle. Their proof cannot be extended to the case $a \geq 3$. The parametrization of the boundary of $\mathcal{R}_a$, $a \geq 3$ is different from the case $a = 2$. It uses an unusual numeration system.

In this paper we will study the fractal associated to a number β satisfying the condition (iii) of Lemma 1.2 with $a \geq 3, b = -1$. In this case the polinomial $p(x) = x^3 - ax^2 + x - 1 = (x-\beta)(x-\alpha)(x-\overline{\alpha})$,where $\beta > 1$ and $\alpha, \overline{\alpha} \in \mathbb{C}\backslash\mathbb{R}$.

The purpose of this work is to present a complete description of the boundary of $\mathcal{R}_a$, $a \geq 3$. Our main result is the following.

Theorem 1.4. *$\partial\mathcal{R}_a$ is homeomorfic to S^1.*

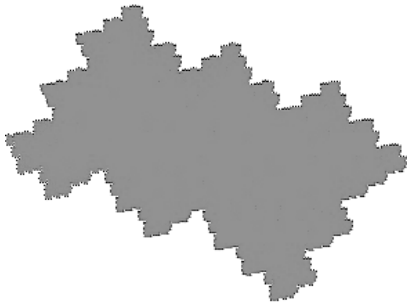

Fig. 1. $\mathcal{R}_3$

2. Background, notations and definitions

In this section we will give more informations about β-numeration, Rauzy fractal, automaton and we will present some notations that will be used in the next sections.

Assume that β is a Pisot number of degree $d \geq 3$. We denote by $\beta_2, \beta_3, ..., \beta_r$ the real Galois conjugate of β and by $\beta_{r+1}, ..., \beta_{r+s}, \beta_{r+s+1} = \overline{\beta_{r+1}}, ..., \beta_{r+2s} = \overline{\beta_{r+s}}$ its complex Galois conjugates. Let

$$\psi = (\beta_2, ..., \beta_r, \beta_{r+1}, ..., \beta_{r+s}) \in \mathbb{R}^{r-1} \times \mathbb{C}^s$$

and put $\psi^i = (\beta_2^i, ..., \beta_r^i, \beta_{r+1}^i, ..., \beta_{r+s}^i)$, $\forall i \in \mathbb{Z}$. The Rauzy fractal is by definition the set

$$\mathcal{R} = \left\{ \sum_{i=0}^{\infty} a_i \psi^i, \ (a_i)_{i \geq 0} \in E_\beta \right\}$$

where $E_\beta = \{(x_i)_{i \geq k}, k \in \mathbb{Z} | \forall n \geq k, (x_i)_{k \leq i \leq n}$ is a finite β expansion$\}$.

An important class of Pisot numbers are those such that the associated Rauzy fractal has 0 as an interior point. This numbers where characterized by Akiyama ([3]), and they are exactly the Pisot numbers satisfying condition (F).

In this paper we will work with sequences $(a_n)_{n \in \mathbb{Z}}$ belonging to $\{0, 1, ..., a-1\}^{\mathbb{Z}}$ and the following set

$\mathcal{N} = \{(a_n)_{n \in \mathbb{Z}}, \ \exists k \in \mathbb{Z}, \ a_k > 0, \ a_i = 0$ for all $i < k, \ a_i a_{i-1} a_{i-2} a_{i-3} <_{lex} (a-1)(a-1)01, \ \forall i \geq k\}$.

If $(a_n) \in \mathcal{N}$ we will call it an admissible sequence.

Take $(a_n, b_n)_{n \in \mathbb{Z}}$ an infinite path on the automaton $\mathcal{A}$ starting in the initial state. If $(a_n), (b_n) \in \mathcal{N}$ we will call it an admissible path.

In [9] the following results were proved.

(1) Let $z = \sum_{i=2}^{\infty} a_i \alpha^i \in \mathcal{R}_a$. Then $z \in \partial \mathcal{R}_a$ if and only if there exists $(b_i)_{i \geq l} \in \mathcal{N}$, $l < 1$, $b_l \neq 0$ such that $\sum_{i=2}^{\infty} a_i \alpha^i = \sum_{i=l}^{\infty} b_i \alpha^i$.

(2) There exists an explicit finite state automaton $\mathcal{A}$ (see figure 2 below) such that $\sum_{i=l}^{\infty} \epsilon_i \alpha^i = \sum_{i=l}^{\infty} \epsilon'_i \alpha^i$, $(\epsilon_i), (\epsilon'_i) \in \mathcal{N}$ if and only if $(\epsilon_i, \epsilon'_i)_{i \geq l}$ is an admissible path.

Let us explain the behavior of this automaton. Let $\varepsilon = (\varepsilon_i)_{i \geq l}$ and $\varepsilon' = (\varepsilon'_i)_{i \geq l}$ belonging to $\mathcal{N}$, $x = \sum_{i=l}^{\infty} \varepsilon_i \alpha^i$ and $y = \sum_{i=l}^{\infty} \varepsilon'_i \alpha^i$. Suppose $x = y$. For all $k \geq l$ we put

$$A_k(\varepsilon, \varepsilon') = \alpha^{-k+2} \sum_{i=l}^{k} (\varepsilon_i - \varepsilon'_i)\alpha^i. \tag{2.1}$$

In [9] the authors proved that $A_k(\varepsilon, \varepsilon') \in S = \{0, \pm\alpha, \pm\alpha^2, \pm(\alpha-\alpha^2), \pm(1+(a-1)\alpha^2), \pm(1+(a-2)\alpha^2), \pm(1-\alpha+(a-1)\alpha^2), \pm(1-2\alpha+a\alpha^2)\}$. We can see that for all $k \geq l$,

$$A_{k+1}(\varepsilon, \varepsilon') = \frac{A_k(\varepsilon, \varepsilon')}{\alpha} + (\varepsilon_{k+1} - \varepsilon'_{k+1})\alpha^2. \tag{2.2}$$

Let s be the smallest integer such that $\varepsilon_s \neq \varepsilon'_s$. Hence $A_i(\varepsilon, \varepsilon') = 0$ for $i \in \{l, \dots, s-1\}$. Suppose $\varepsilon_s > \varepsilon'_s$. Then, $A_s = (\varepsilon_s - \varepsilon'_s)\alpha^2 = \alpha^2$. From (2.2) we deduce $A_{s+1}(\varepsilon, \varepsilon') = \alpha + (\varepsilon_{s+1} - \varepsilon'_{s+1})\alpha^2$ which should belong to S. Hence $A_{s+1}(\varepsilon, \varepsilon') = \alpha$ if $\varepsilon_{s+1} = \varepsilon'_{s+1}$ or $A_{s+1}(\varepsilon, \varepsilon') = \alpha - \alpha^2$ if $(\varepsilon_{s+1}, \varepsilon'_{s+1}) = (t_1, t_1 + 1)$, where $0 \leq t_1 \leq a - 2$. Continuing by the same way and using the fact that the set of states S is finite, we obtain the following finite state automaton shown in Figure 2.

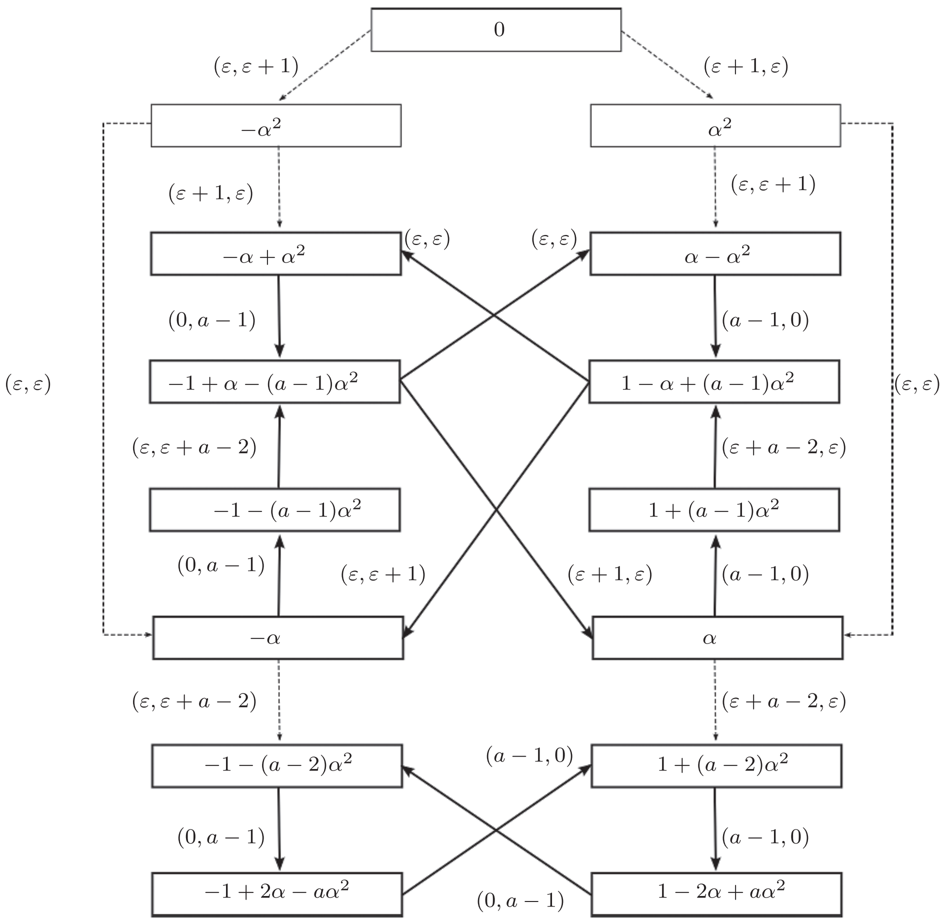

Fig. 2. Automaton $\mathcal{A}$

Another result proved in [9] is the following.

Proposition 2.1. *$\mathcal{R}_a$ induces a periodic tiling of the plane $\mathbb{C}$ modulo $\mathbb{Z}u + \mathbb{Z}\alpha u$ where $u = \alpha - 1$. Moreover $\partial\mathcal{R}_a = \bigcup_{v\in B} \mathcal{R}_a \cap (\mathcal{R}_a + v)$, where $B = \{\pm u, \pm\alpha u, \pm(1+\alpha)u, \pm(\alpha-1)u\}$ and $\mathcal{R}_a \cap (\mathcal{R}_a + (1+\alpha)u)) = \{-1\}$, $\mathcal{R}_a \cap (\mathcal{R}_a + (\alpha-1)u) = \{-\alpha\}$.*

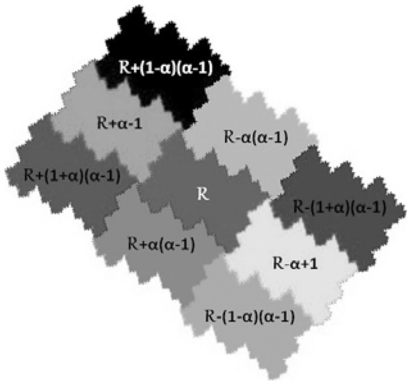

Fig.3. Tiling induced by $\mathcal{R}_3$

REMARK 2.2. In this paper we will use the following relations:

$$\alpha^n - a\alpha^{n-1} + \alpha^{n-2} - \alpha^{n-3} = 0,\ \alpha^n = (a-1)\alpha^{n-1} + (a-1)\alpha^{n-2} + \alpha^{n-4}, \forall\, n \in \mathbb{Z}. \quad (2.3)$$

Lemma 2.3. *Let $z \in \mathcal{B}_{\alpha-1}$. Then $z = (\alpha - 1) + \sum_{i=2}^{\infty} a_i\alpha^i$ and $z = \sum_{i=2}^{\infty} b_i\alpha^i$ with $a_2 = 0$, $b_2 = a - 1$ and $b_4 = 0$.*

Proof. Take $z \in \mathcal{B}_{\alpha-1}$. Using relation (2.3) we have $z = \alpha^{-3} + (a-1)\alpha^{-1} + (a-2) + \sum_{i=2}^{\infty} a_i\alpha^i$ and $z = \sum_{i=2}^{\infty} b_i\alpha^i$. Then the admissible path, starting from 0, in the automaton associated to z is

$$(1,0)(0,0)(a-1,0)(a-2,0)(0,0)(a_2,b_2)(a_3,b_3)....$$

Using the automaton we have that $a_2 = 0$, $b_2 = a - 1$ and then we can write $z = \alpha^{-3} + (a-1)\alpha^{-1} + (a-2) + \sum_{i=3}^{\infty} a_i\alpha^i = \alpha - 1 + \sum_{i=3}^{\infty} a_i\alpha^i$ and $z = (a-1)\alpha^2 + \sum_{i=3}^{\infty} b_i\alpha^i$. We also have that $b_4 = 0$. □

3. Parametrization of ∂R_a, $a \geq 3$

In this section we give a complete description of $\partial\mathcal{R}_a$, $a \geq 3$. By Proposition (2.1) we have that $\partial\mathcal{R}_a = \bigcup_{v\in B} \mathcal{R}_a \cap (\mathcal{R}_a + v)$ where $B = \{\pm(\alpha^{-3} + \alpha^{-1}) = \pm(\alpha - 1),\ \pm(\alpha^2 - \alpha),\ \pm(\alpha^2 - 1),\ \pm(\alpha - 1)^2\}$. Since $\mathcal{R}_a \cap (\mathcal{R}_a \pm v)$ is a point if $v = \alpha^2 - 1$ or $v = (\alpha - 1)^2$, we will study the others four regions $\mathcal{B}_v = \mathcal{R}_a \cap (\mathcal{R}_a + v)$ where $v \in \{\pm(\alpha - 1), \pm(\alpha^2 - \alpha)\}$. For this we will use the set $\mathcal{B}_{\alpha-1} = \mathcal{R}_a \cap (\mathcal{R}_a + \alpha - 1)$ described by

$$\mathcal{B}_{\alpha-1} = \left\{ z \in \mathbb{C}, z = \alpha - 1 + \sum_{i=3}^{\infty} a_i\alpha^i = (a-1)\alpha^2 + \sum_{i=3}^{\infty} b_i\alpha^i \right\}.$$

In particular, we will prove the following results.

Proposition 3.1. *Let f_i, $i = 1,2,3$, be the functions defined by $f_1(z) = \alpha^{-1} - 1 + \alpha^{-1}z$, $f_2(z) = -(a-1)\alpha + \alpha^{-1}z$ and $f_3(z) = 1 - \alpha + z$. Then we have the following properties:*

(1) $\mathcal{B}_{\alpha^2-\alpha} = f_1(\mathcal{B}_{\alpha-1})$,
(2) $\mathcal{B}_{\alpha-\alpha^2} = f_2(\mathcal{B}_{\alpha-1})$,
(3) $\mathcal{B}_{1-\alpha} = f_3(\mathcal{B}_{\alpha-1})$.
(4) $\mathcal{B}_{\alpha-1} \cap f_1(\mathcal{B}_{\alpha-1}) = \mathcal{B}_{\alpha-1} \cap \mathcal{B}_{\alpha^2-\alpha} = \{-1\}$.
(5) $f_1(\mathcal{B}_{\alpha-1}) \cap f_3(\mathcal{B}_{\alpha-1}) = \mathcal{B}_{\alpha^2-\alpha} \cap \mathcal{B}_{1-\alpha} = \{-\alpha\}$.
(6) $f_2(\mathcal{B}_{\alpha-1}) \cap f_3(\mathcal{B}_{\alpha-1}) = \mathcal{B}_{\alpha-\alpha^2} \cap \mathcal{B}_{1-\alpha} = \{-\alpha^2\}$.
(7) $\mathcal{B}_{\alpha-1} \cap f_2(\mathcal{B}_{\alpha-1}) = \mathcal{B}_{\alpha-1} \cap \mathcal{B}_{\alpha-\alpha^2} = \{-(a-1)\alpha - \alpha^{-1}\}$.
(8) $\mathcal{B}_{\alpha-1} \cap f_3(\mathcal{B}_{\alpha-1}) = \mathcal{B}_{\alpha-1} \cap \mathcal{B}_{1-\alpha} = \emptyset$.
(9) $f_1(\mathcal{B}_{\alpha-1}) \cap f_2(\mathcal{B}_{\alpha-1}) = \mathcal{B}_{\alpha^2-\alpha} \cap \mathcal{B}_{\alpha-\alpha^2} = \emptyset$.

Proposition 3.2. *Let g_i, $i = 0,1,...,2(a-1)$, be the functions defined by $g_{2k+1}(z) = -1 - k\alpha^3 + \alpha^3 z$ for $k = 0,...,a-2$, and $g_{2k}(z) = \alpha - 1 + (a-1-k)\alpha^3 + \alpha^2 z$ for $k = 0,...,a-1$. Then*

$$\mathcal{B}_{\alpha-1} = \bigcup_{i=0}^{2(a-1)} g_i(X_i),$$

where $X_i = \mathcal{B}_{\alpha-1}$ if i is an odd number or $i = 2(a-1)$ and $X_i = \mathcal{B}'_{\alpha-1} = \{z \in \mathcal{B}_{\alpha-1}; a_3 \neq a-1\}$ if i is an even number.

Remark 3.3. Using Proposition 3.2 we will construct an explicit continuous and bijective application from $[0,1]$ to $\mathcal{R}_{\alpha-1}$. Using this fact and Proposition 3.1 we obtain an explicit homeomorphism between the circle and the boundary of $\mathcal{R}_a$.

Proof of Proposition 3.1. According to (2.3) we have $\alpha^2 - \alpha = \alpha^{-2} + (a-1) + (a-2)\alpha$.

(1) Take $z \in \mathcal{B}_{\alpha-1}$. According to Lemma 2.3 $z = \alpha - 1 + \sum_{i=3}^{\infty} a_i\alpha^i = (a-1)\alpha^2 + \sum_{i=3}^{\infty} b_i\alpha^i$.
Then
$$f_1(z) = \alpha^{-1} - 1 + \alpha^{-1}((a-1)\alpha^2 + \sum_{i=3}^{\infty} b_i\alpha^i) = \alpha^{-2} + (a-1) + (a-2)\alpha + \sum_{i=3}^{\infty} b_i\alpha^{i-1}$$
$\in \mathcal{R}_a + \alpha^2 - \alpha$.
We also have
$$f_1(z) = \alpha^{-1} - 1 + \alpha^{-1}(\alpha - 1 + \sum_{i=3}^{\infty} a_i\alpha^i) = \sum_{i=3}^{\infty} a_i\alpha^{i-1} \in \mathcal{R}_a.$$
Therefore $f_1(\mathcal{B}_{\alpha-1}) \subseteq \mathcal{B}_{\alpha^2-\alpha}$.
Take $z \in \mathcal{B}_{\alpha^2-\alpha}$, $z = \alpha^{-2} + (a-1) + (a-2)\alpha + \sum_{i=2}^{\infty} a_i\alpha^i = \sum_{i=2}^{\infty} b_i\alpha^i$. Then
$$f_1^{-1}(z) = \alpha - 1 + \alpha(\alpha^{-2} + (a-1) + (a-2)\alpha + \sum_{i=2}^{\infty} a_i\alpha^i) = (a-1)\alpha^2 + \sum_{i=2}^{\infty} a_i\alpha^{i+1} \in \mathcal{R}_a.$$
We also have

$$f_1^{-1}(z) = \alpha - 1 + \alpha(\sum_{i=2}^{\infty} b_i\alpha^i) = \alpha - 1 + \sum_{i=2}^{\infty} b_i\alpha^{i+1} \in \mathcal{R}_a + \alpha - 1.$$

Therefore $f_1^{-1}(\mathcal{B}_{\alpha^2-\alpha}) \subseteq \mathcal{B}_{\alpha-1}$ and then

$$f_1(\mathcal{B}_{\alpha-1}) = \mathcal{B}_{\alpha^2-\alpha}.$$

(2) Take $z \in \mathcal{B}_{\alpha-\alpha^2}$. Then $z + \alpha^2 - \alpha$ belongs to $\mathcal{B}_{\alpha^2-\alpha}$ and according to what was done before there exists $w \in \mathcal{B}_{\alpha-1}$ such that $z + \alpha^2 - \alpha = g_1(w)$. Then $z+\alpha^2-\alpha = \alpha^{-1}-1+\alpha^{-1}(w) \Rightarrow z = \alpha^{-1}-1+\alpha-\alpha^2+\alpha^{-1}(w) = -(a-1)\alpha+\alpha^{-1}(w)$. Therefore

$$f_2(\mathcal{B}_{\alpha-1}) = \mathcal{B}_{\alpha-\alpha^2}.$$

We also know that $f_1^{-1}(z+\alpha^2-\alpha) = \alpha-1+\alpha(z+\alpha^2-\alpha) = (a-1)\alpha^2+\alpha z = f_2^{-1}(z) \in \mathcal{B}_{\alpha-1}$ and then $f_2^{-1}(\mathcal{B}_{\alpha-\alpha^2}) \subseteq \mathcal{B}_{\alpha-1}$. Therefore $f_2(\mathcal{B}_{\alpha-1}) = \mathcal{B}_{\alpha-\alpha^2}$.

(3) This item can be done by the same manner of item (2).

(4) Take $z \in \mathcal{B}_{\alpha-1} \cap \mathcal{B}_{\alpha^2-\alpha} = \mathcal{R} \cap (\mathcal{R} + \alpha - 1) \cap (\mathcal{R} + \alpha^2 - \alpha)$. Then $z - \alpha + 1 \in \mathcal{R} \cap (\mathcal{R} - \alpha + 1) \cap (\mathcal{R} + (\alpha-1)^2) \subseteq \mathcal{R} \cap (\mathcal{R} + (\alpha-1)^2) = \{-1\}$. Therefore, $z - \alpha + 1 = -\alpha$, and $z = -1$.

(8) Take $z \in \mathcal{B}_{\alpha-1} \cap f_3(\mathcal{B}_{\alpha-1})$. Then there is $z_1 \in \mathcal{B}_{\alpha-1}$ such that $z = 1 - \alpha + z_1$. Then

$$\alpha + z = 1 + z_1.$$

Since $z,\ z_1 \in \mathcal{B}_{\alpha-1}$ we know that $z = (a-1)\alpha^2 + \sum_{i=3}^{\infty} a_i\alpha^i$ and $z_1 = (a-1)\alpha^2 + \sum_{i=3}^{\infty} b_i\alpha^i$.

Then the equality above becomes

$$1 + (a-1)\alpha^2 + \sum_{i=3}^{\infty} a_i\alpha^i = \alpha + (a-1)\alpha^2 + \sum_{i=3}^{\infty} b_i\alpha^i.$$

So, we conclude that $(1,0)(0,1)(a-1,a-1)\ldots$ is an admissible path on the automaton $\mathcal{A}$ starting from 0. But there is no such path on the automaton and then

$$\mathcal{B}_{\alpha-1} \cap f_3(\mathcal{B}_{\alpha-1}) = \mathcal{B}_{\alpha-1} \cap \mathcal{B}_{1-\alpha} = \emptyset.$$

Following the ideas of items (4) and (8) we can prove (5), (6), (7) and (9). For more details see [10] □

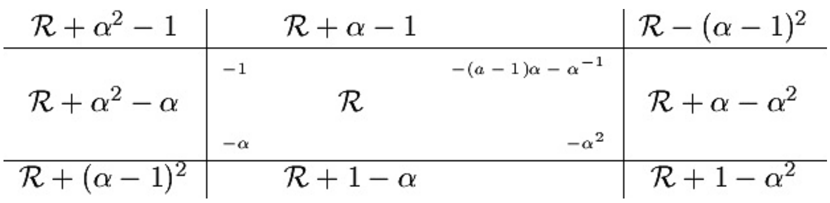

Fig.4. Boundary of $\mathcal{R}_a$

Proof of Proposition 3.2. Let z be an element of $\mathcal{B}_{\alpha-1}$. Using the automaton $\mathcal{A}$ we can write $z = \alpha - 1 + \sum_{i=3}^{\infty} a_i\alpha^i = (a-1)\alpha^2 + \sum_{i=3}^{\infty} b_i\alpha^i$ where $(a_3, b_3) = (t,t),\ t = 0, 1, ..., a-1$ or

$(a_3, b_3) = (t, t-1),\ t = 1, ..., a-1$. Let $\mathcal{B}^{1,t}_{\alpha-1},\ \mathcal{B}^{2,t}_{\alpha-1}$ be the following sets:
$\mathcal{B}^{1,t}_{\alpha-1} = \{z \in \mathcal{B}_{\alpha-1}; (a_3, b_3) = (t,t),\ t = 0, 1..., a-1\}$,

$\mathcal{B}^{2,t}_{\alpha-1} = \{z \in \mathcal{B}_{\alpha-1}; (a_3, b_3) = (t, t-1),\ t = 1, 2..., a-1\}$. Since

$$\mathcal{B}_{\alpha-1} = \left[\bigcup_{t=0}^{a-1} \mathcal{B}^{1,t}_{\alpha-1}\right] \bigcup \left[\bigcup_{t=1}^{a-1} \mathcal{B}^{2,t}_{\alpha-1}\right],$$

in order to prove this theorem we need to show that $g_{2k+1}(\mathcal{B}_{\alpha-1}) = \mathcal{B}^{2,a-1-k}_{\alpha-1}, k = 0, ..., a-2$, $g_{2(a-1)}(\mathcal{B}_{\alpha-1}) = \mathcal{B}^{1,0}_{\alpha-1}$ and $g_{2k}(\mathcal{B}'_{\alpha-1}) = \mathcal{B}^{1,a-1-k}_{\alpha-1}, k = 0, ..., a-2$.
1)- Indeed since $z \in \mathcal{B}_{\alpha-1}$ then:
$g_{2k+1}(z) = -1 - k\alpha^3 + \alpha^3(\alpha - 1 + \sum_{i=3}^{\infty} a_i\alpha^i) = (a-1)\alpha^2 + (a-2-k)\alpha^3 + \sum_{i=3}^{\infty} a_i\alpha^{i+3}$ and

$g_{2k+1}(z) = -1 - k\alpha^3 + \alpha^3((a-1)\alpha^2 + \sum_{i=3}^{\infty} b_i\alpha^i) = \alpha - 1 + (a-1-k)\alpha^3 + (a-1)\alpha^4 + (a-2)\alpha^5 + \sum_{i=3}^{\infty} b_i\alpha^{i+3}$, that is $g_{2k+1}(\mathcal{B}_{\alpha-1}) \subseteq \mathcal{B}^{2,a-1-k}_{\alpha-1}$. On the other hand if we take $w \in \mathcal{B}^{2,a-1-k}_{\alpha-1}$,

$w = \alpha-1+(a-1-k)\alpha^3+(a-1)\alpha^4+(a-2)\alpha^5+\sum_{i=6}^{\infty} u_i\alpha^i = (a-1)\alpha^2+(a-2-k)\alpha^3+\sum_{i=6}^{\infty} v_i\alpha^i$ then

$z = \alpha - 1 + \sum_{i=6}^{\infty} v_i\alpha^{i-3} = (a-1)\alpha^2 + \sum_{i=6}^{\infty} u_i\alpha^{i-3}$ is an element of $\mathcal{B}_{\alpha-1}$ such that $g_{2k+1}(z) = w$.
Therefore we conclude that

$$g_{2k+1}(\mathcal{B}_{\alpha-1}) = \mathcal{B}^{2,a-1-k}_{\alpha-1}, k = 0, ..., a-2.$$

$g_{2(a-1)}(z) = \alpha - 1 + \alpha^2(\alpha - 1 + \sum_{i=3}^{\infty} a_i\alpha^i) = (a-1)\alpha^2 + \sum_{i=3}^{\infty} a_i\alpha^{i+2}$ and

$g_{2(a-1)}(z) = \alpha - 1 + \alpha^2((a-1)\alpha^2 + \sum_{i=3}^{\infty} b_i\alpha^i) = \alpha - 1 + (a-1)\alpha^4 + \sum_{i=3}^{\infty} b_i\alpha^{i+2}$, that is

$g_{2(a-1)}(\mathcal{B}_{\alpha-1}) \subseteq \mathcal{B}^{1,0}_{\alpha-1}$. On the other hand if we take $w \in R^{1,0}_{\alpha-1}$, $w = \alpha-1+(a-1)\alpha^4+\sum_{i=5}^{\infty} u_i\alpha^i =$

$(a-1)\alpha^2 + \sum_{i=5}^{\infty} v_i\alpha^i$ then $z = \alpha - 1 + \sum_{i=5}^{\infty} v_i\alpha^{i-2} = (a-1)\alpha^2 + \sum_{i=5}^{\infty} u_i\alpha^{i-2}$ is an element of $\mathcal{B}_{\alpha-1}$
such that $g_{2(a-1)}(z) = w$. Therefore

$$g_{2(a-1)}(\mathcal{B}_{\alpha-1}) = \mathcal{B}^{1,0}_{\alpha-1}.$$

2)- Let $z \in \mathcal{B}'_{\alpha-1}$ given by $z = \alpha - 1 + \sum_{i=3}^{\infty} a_i\alpha^i = (a-1)\alpha^2 + \sum_{i=3}^{\infty} b_i\alpha^i$. Since $a_3 \neq a-1$ then
using the automaton we have $b_3 \neq a-1$. Then
$g_{2k}(z) = \alpha - 1 + (a-1-k)\alpha^3 + \alpha^2(\alpha - 1 + \sum_{i=3}^{\infty} a_i\alpha^i) = (a-1)\alpha^2 + (a-1-k)\alpha^3 + \sum_{i=3}^{\infty} a_i\alpha^{i+2}$ and

$g_{2k}(z) = \alpha - 1 + (a-1-k)\alpha^3 + \alpha^2((a-1)\alpha^2 + \sum_{i=3}^{\infty} b_i\alpha^i) = \alpha - 1 + (a-1-k)\alpha^3 + (a-1)\alpha^4 +$

$\sum_{i=3}^{\infty} b_i\alpha^{i+2}$. So we have $g_{2k}(\mathcal{B}'_{\alpha-1}) \subseteq \mathcal{B}^{1,a-1-k}_{\alpha-1}$. On the other hand if we take $w \in \mathcal{B}^{1,a-1-k}_{\alpha-1}$, $w = \alpha - 1 + (a-1-k)\alpha^3 + (a-1)\alpha^4 + \sum_{i=5}^{\infty} u_i\alpha^i = (a-1)\alpha^2 + (a-1-k)\alpha^3 + \sum_{i=5}^{\infty} v_i\alpha^i$ then we have $u_5, v_5 \neq a-1$ (again use the automaton) and $z = \alpha-1+\sum_{i=5}^{\infty} v_i\alpha^{i-2} = (a-1)\alpha^2+\sum_{i=5}^{\infty} u_i\alpha^{i-2}$ is an element of $\mathcal{B}'_{\alpha-1}$ such that $g_{2k}(z) = w$. Therefore

$$g_{2k}(\mathcal{B}'_{\alpha-1}) = \mathcal{B}^{1,a-1-k}_{\alpha-1}, k = 0, ..., a-2.$$

□

Using the previous notation and taking $u = -1, v = -(a-1)\alpha - \alpha^{-1}, w = -1 - \alpha^3$, we have the following lemmas.

Lemma 3.4.

(1) *Take* $k_1 \leq k_2$. *Then*

$$g_{2k_1}(\mathcal{B}'_{\alpha-1}) \bigcap g_{2k_2+1}(\mathcal{B}_{\alpha-1}) = \begin{cases} \emptyset, \ \textit{if}\ k_2 > k_1 \\ -1 - \alpha^2 - k\alpha^3 - (a-1)\alpha^4, \ \textit{if}\ k_2 = k_1. \end{cases}$$

Therefore $-1 - \alpha^2 - k\alpha^3 - (a-1)\alpha^4 = g_{2k}(w) = g_{2k+1}(v)$.

(2) *Take* $k_1 < k_2$. *Then*

$$g_{2k_1+1}(\mathcal{B}_{\alpha-1}) \bigcap g_{2k_2}(\mathcal{B}'_{\alpha-1}) = \begin{cases} \emptyset, \ \textit{if}\ k_2 > k_1 + 1 \\ -1 - (k_1+1)\alpha^3, \ \textit{if}\ k_2 = k_1 + 1. \end{cases}$$

Therefore $-1 - (k_1+1)\alpha^3 = g_{2k+1}(u) = g_{2(k+1)}(v)$.

(3) $g_{2k_1}(\mathcal{B}'_{\alpha-1}) \bigcap g_{2k_2}(\mathcal{B}'_{\alpha-1}) = \emptyset$, *if* $k_1 \neq k_2$.

(4) $g_{2k_1+1}(\mathcal{B}_{\alpha-1}) \bigcap g_{2k_2+1}(\mathcal{B}_{\alpha-1}) = \emptyset$, *if* $k_1 \neq k_2$.

(5) $\lim_{n\longrightarrow\infty} (g_0 \circ g_{2(a-1)})^n(z) = u = -1, \ \forall z \in \mathcal{B}_{\alpha-1}$.

(6) $\lim_{n\longrightarrow\infty} (g_{2(a-1)} \circ g_0)^n(z) = v = -(a-1)\alpha - \alpha^{-1}, \ \forall z \in \mathcal{B}'_{\alpha-1}$.

Proof. 1)– Take $z \in g_{2k_1}(\mathcal{B}'_{\alpha-1}) \bigcap g_{2k_2+1}(\mathcal{B}_{\alpha-1}), k_1 \leq k_2$.
Then $z = g_{2k_1}(z_1) = g_{2k_2+1}(z_2), z_1 \in \mathcal{B}'_{\alpha-1}, z_2 \in \mathcal{B}_{\alpha-1}$ and

$$\alpha - 1 + (a-1-k_1)\alpha^3 + \alpha^2 z_1 = -1 - k_2\alpha^3 + \alpha^3 z_2.$$

If we suppose $k_2 = k_1 + k$ we have

$$\alpha + (a-1)\alpha^3 + \alpha^2 z_1 = -k\alpha^3 + \alpha^3 z_2,$$

and multiplying by α^{-3}

$$\alpha^{-2} + (a-1) + \alpha^{-1} z_1 = -k + z_2.$$

Since $z_1 \in \mathcal{B}'_{\alpha-1}, z_2 \in \mathcal{B}_{\alpha-1}$ we know that $z_1 = \alpha - 1 + \sum_{i=3}^{\infty} a_i\alpha^i, a_3 \neq a-1$ and $z_2 = \alpha - 1 + \sum_{i=3}^{\infty} b_i\alpha^i$. Then the equality above becomes

$$\alpha^{-2}+a-\alpha^{-1}+\sum_{i=3}^{\infty}a_i\alpha^{i-1}=-k+\alpha-1+\sum_{i=3}^{\infty}b_i\alpha^{i},$$

and since $\alpha^{-2}+a-1=\alpha$ then

$$(k+1)+\sum_{i=3}^{\infty}a_i\alpha^{i-1}=\sum_{i=3}^{\infty}b_i\alpha^{i}.$$

So we conclude that $(k+1,0)(0,0)(a_3,0)(a_4,b_3)(a_5,b_4)\ldots$ is an admissible path on the automaton $\mathcal{A}$ starting from 0. Using the automaton, since $a_3\neq a-1$, we see that the only possibility is

$$(1,0)(0,0)(a-2,0)(a-1,0)(0,a-1)(0,a-1)(a-1,0)(a-1,0)\ldots$$

Then $k=0$,

$$z_1=\alpha-1+(a-2)\alpha^3+(a-1)\alpha^4+\sum_{i=2}^{\infty}[(a-1)\alpha^{4i-1}+(a-1)\alpha^{4i}]=-1-\alpha^3,$$

and

$$z_2=\alpha-1+\sum_{i=1}^{\infty}[(a-1)\alpha^{4i}+(a-1)\alpha^{4i+1}]=-(a-1)\alpha-\alpha^{-1}.$$

2)– Take $z\in g_{2k_1+1}(\mathcal{B}_{\alpha-1})\bigcap g_{2k_2}(\mathcal{B}'_{\alpha-1})$, $k_1<k_2$.
Then $z=g_{2k_1+1}(z_1)=g_{2k_2}(z_2)$, $z_1\in\mathcal{B}_{\alpha-1}, z_2\in\mathcal{B}'_{\alpha-1}$ and

$$-1-k_1\alpha^3+\alpha^3z_1=\alpha-1+(a-1-k_2)\alpha^3+\alpha^2z_2.$$

If we suppose $k_2=k_1+k$ we have

$$\alpha^3z_1=\alpha+(a-1-k)\alpha^3+\alpha^2z_2,$$

and multiplying by α^{-3}

$$z_1=\alpha^{-2}+(a-1-k)+\alpha^{-1}z_2.$$

Since $z_1\in\mathcal{B}_{\alpha-1}, z_2\in\mathcal{B}'_{\alpha-1}$ we know that $z_1=(a-1)\alpha^2+\sum_{i=3}^{\infty}a_i\alpha^i$ and $z_2=(a-1)\alpha^2+\sum_{i=3}^{\infty}b_i\alpha^i$. Then the equality above becomes

$$(a-1)\alpha^2+\sum_{i=3}^{\infty}a_i\alpha^{i}=\alpha^{-2}+(a-1-k)+(a-1)\alpha+\sum_{i=3}^{\infty}b_i\alpha^{i-1}.$$

So we conclude that $(1,0)(0,0)(a-1-k,0)(a-1,0)(b_3,a-1)(b_4,a_3)(b_5,a_4)\ldots$ is an admissible path on the automaton $\mathcal{A}$ starting from 0. Using the automaton, we see that the only possibility is

$$(1,0)(0,0)(a-2,0)(a-1,0)(0,a-1)(0,a-1)(a-1,0)(a-1,0)\ldots$$

Then $k=1$,

$$z_2=(a-1)\alpha^2+\sum_{i=1}^{\infty}[(a-1)\alpha^{4i+1}+(a-1)\alpha^{4i+2}]=-(a-1)\alpha-\alpha^{-1},$$

and

$$z_1 = \sum_{i=1}^{\infty}[(a-1)\alpha^{4i-2} + (a-1)\alpha^{4i-1}] = -1.$$

3)−Take $z \in g_{2k_1}(\mathcal{B}'_{\alpha-1}) \bigcap g_{2k_2}(\mathcal{B}'_{\alpha-1})$, $k_1 < k_2$.
Then $z = g_{2k_1}(z_1) = g_{2k_2}(z_2)$, $z_1 \in \mathcal{B}'_{\alpha-1}, z_2 \in \mathcal{B}'_{\alpha-1}$ and

$$\alpha - 1 + (a-1-k_1)\alpha^3 + \alpha^2 z_1 = \alpha - 1 + (a-1-k_2)\alpha^3 + \alpha^2 z_2.$$

If we suppose $k_2 = k_1 + k$ we have

$$k\alpha^3 + \alpha^2 z_1 = \alpha^2 z_2,$$

and multiplying by α^{-2}

$$k\alpha + z_1 = z_2.$$

Since $z_1 \in \mathcal{B}'_{\alpha-1}, z_2 \in \mathcal{B}'_{\alpha-1}$ we know that $z_1 = (a-1)\alpha^2 + \sum_{i=3}^{\infty} a_i\alpha^i$, $z_2 = (a-1)\alpha^2 + \sum_{i=3}^{\infty} b_i\alpha^i$, and, by Example 2.3, $a_4 = b_4 = 0$. Then the equality above becomes

$$k\alpha + (a-1)\alpha^2 + \sum_{i=3}^{\infty} a_i\alpha^i = (a-1)\alpha^2 + \sum_{i=3}^{\infty} b_i\alpha^i.$$

So we conclude that if the intersection is not empty, $(k,0)(a-1,a-1)(a_3,b_3)(0,0)(a_5,b_5)...$ is an admissible path on the automaton $\mathcal{A}$ starting from 0. But there is no such path on the automaton and then

$$g_{2k_1}(\mathcal{B}'_{\alpha-1}) \bigcap g_{2k_2}(\mathcal{B}'_{\alpha-1}) = \emptyset.$$

Using the same ideas we can prove that $g_{2k_1+1}(\mathcal{B}_{\alpha-1}) \bigcap g_{2k_2+1}(\mathcal{B}_{\alpha-1}) = \emptyset$.
5)- Using induction we can prove that

$$(g_0 \circ g_{r-1})^n(z) = \sum_{i=1}^{n} \left[(a-1)\alpha^{4i-2} + (a-1)\alpha^{4i-1}\right] + \alpha^{4n}z.$$

Then $\lim_{n\longrightarrow\infty} (g_0 \circ g_{r-1)})^n(z) = \sum_{i=1}^{\infty} \left[(a-1)\alpha^{4i-2} + (a-1)\alpha^{4i-1}\right] = -1.$
Indeed for $n = 1$ and $z \in \mathcal{B}_{\alpha-1}$ by proposition 3.2 we have $g_{r-1}(z) = \alpha - 1 + \alpha^2 z \in \mathcal{B}^{1,0}_{\alpha-1}$. By definition we have

$$\begin{aligned} g_0(g_{r-1}(z)) &= \alpha - 1 + (a-1)\alpha^3 + \alpha^2(\alpha - 1 + \alpha^2 z) = \alpha - 1 + (a-1)\alpha^3 + \alpha^3 - \alpha^2 + \alpha^4 z \\ &= (a-1)\alpha^2 + (a-1)\alpha^3 + \alpha^4 z. \end{aligned}$$

Suppose the formula is true for $k \geq 1$, that is

$$(g_0 \circ g_{r-1})^k(z) = \sum_{i=1}^{k} \left[(a-1)\alpha^{4i-2} + (a-1)\alpha^{4i-1}\right] + \alpha^{4k}z. \quad (*)$$

We have to prove the formula for $n = k+1$. Since $(g_0 \circ g_{r-1})^{k+1}(z) = (g_0 \circ g_{r-1}) \circ (g_0 \circ g_{r-1})^k(z)$ using $(*)$ we have to prove that

$$g_0\Big(g_{r-1}\Big(\sum_{i=1}^{k}\Big[(a-1)\alpha^{4i-2}+(a-1)\alpha^{4i-1}\Big]+\alpha^{4k}z\Big)\Big)=\sum_{i=1}^{k+1}\Big[(a-1)\alpha^{4i-2}+(a-1)\alpha^{4i-1}\Big]+\alpha^{4(k+1)}z.$$

Indeed

$$\begin{aligned}
&g_{r-1}\Big(\sum_{i=1}^{k}\Big[(a-1)\alpha^{4i-2}+(a-1)\alpha^{4i-1}\Big]+\alpha^{4k}z\Big)\\
&=\alpha-1+\alpha^2\Big(\sum_{i=1}^{k}\Big[(a-1)\alpha^{4i-2}+(a-1)\alpha^{4i-1}\Big]+\alpha^{4k}z\Big)\\
&=\alpha-1+\sum_{i=1}^{k}\Big[(a-1)\alpha^{4i}+(a-1)\alpha^{4i+1}\Big]+\alpha^{4k+2}z,
\end{aligned}$$

and

$$\begin{aligned}
&g_0\Big(g_{r-1}\Big(\sum_{i=1}^{k}\Big[(a-1)\alpha^{4i-2}+(a-1)\alpha^{4i-1}\Big]+\alpha^{4k}z\Big)\Big)\\
&=g_0\Big(\alpha-1+\sum_{i=1}^{k}\Big[(a-1)\alpha^{4i}+(a-1)\alpha^{4i+1}\Big]+\alpha^{4k+2}z\Big)\\
&=\alpha-1+(a-1)\alpha^3+\alpha^2\Big(\alpha-1+\sum_{i=1}^{k}\Big[(a-1)\alpha^{4i}+(a-1)\alpha^{4i+1}\Big]+\alpha^{4k+2}z\Big)\\
&=(a-1)\alpha^2+(a-1)\alpha^3+\sum_{i=1}^{k}\Big[(a-1)\alpha^{4i+2}+(a-1)\alpha^{4i+3}\Big]+\alpha^{4(k+1)}z\\
&=\sum_{i=1}^{k+1}\Big[(a-1)\alpha^{4i-2}+(a-1)\alpha^{4i-1}\Big]+\alpha^{4(k+1)}z.
\end{aligned}$$

6)− Using induction we can prove that

$$(g_{r-1}\circ g_0)^n(z)=(a-1)\alpha^2+\sum_{i=1}^{n-1}\Big[(a-1)\alpha^{4i+1}+(a-1)\alpha^{4i+2}\Big]+(a-1)\alpha^{4n+1}+\alpha^{4n}z,$$

and then $\lim_{n\longrightarrow\infty}(g_{r-1}\circ g_0)^n(z)=(a-1)\alpha^2+\sum_{i=1}^{\infty}\Big[(a-1)\alpha^{4i+1}+(a-1)\alpha^{4i+2}\Big]=-(a-1)\alpha-\alpha^{-1}.$

□

Proposition 3.5. *Let $t\in[0,1], a\geq 3, r=2a-1$. Then there exists an unusual expansion $(a_i)_{i\geq1}\in\{0,1,...,r-1\}^{\mathbb{N}}$, $n_i,m_i\in\mathbb{N}$ such that we can write*

$$t=\frac{a_1}{r}+\sum_{\substack{i=2\\ m_i+n_i=k}}^{\infty}\frac{a_i}{r^{n_i}(r-2)^{m_i}},$$

where the digits a_i and the numbers n_i,m_i satisfy the following properties:

(1) *if $a_1\in\{1,3,5,...,r-2,r-1\}$ then $a_2\in\{0,1,...,r-1\}$, $n_2=2$, $m_2=0$;*
(2) *if $a_1\in\{0,2,4,...,r-3\}$ then $a_2\in\{0,1,...,r-3\}$, $n_2=1$, $m_2=1$;*
and for $i\geq3$ we have:

(3) $a_i \in \{0, 1, 2..., r-1\}$, $m_i = m_{i-1}$, $n_i = n_{i-1} + 1$ *if one of the following conditions are satisfied*
  (a) $a_{i-1} = 0$ *and* $i - 1$ *even;*
  (b) $a_{i-1} = r - 1$ *and* $i - 1$ *odd;*
  (c) $a_{i-1} = 2n - 1$, $n = 1, ..., a - 1$;
  (d) $a_{i-1} = r - 3$, $i - 1$ *odd,* $n_{i-1} = n_{i-2}$ *and* $m_{i-1} = m_{i-2} + 1$

(4) $a_i \in \{0, 1, 2..., r-3\}$, $m_i = m_{i-1} + 1$, $n_i = n_{i-1}$ *if one of the following conditions are satisfied*
  (a) $a_{i-1} = 0$ *and* $i - 1$ *odd;*
  (b) $a_{i-1} = r - 1$ *and* $i - 1$ *even;*
  (c) $a_{i-1} = 2n$, $n = 1, ..., a - 3$ *or* $a_{i-1} = r - 3$, $i - 1$ *even or odd,* $n_{i-1} = n_{i-2} + 1$ *and* $m_{i-1} = m_{i-2}$;
  (d) $a_{i-1} = r - 3$, $i - 1$ *even,* $n_{i-1} = n_{i-2}$ *and* $m_{i-1} = m_{i-2} + 1$

Proof. We can write

$$1 = \frac{r-1}{r} + \sum_{i=0}^{\infty} \frac{r-1}{r^{2+i}(r-2)^i} + \frac{r-3}{r^{2+i}(r-2)^{1+i}}.$$

Given $t \in [0, 1)$, we can prove by induction that for each $k \geq 1$ we can write

$$t = \frac{a_1}{r} + \sum_{\substack{i=2 \\ m_i+n_i=i}}^{k} \frac{a_k}{r^{n_i}(r-2)^{m_i}} + c_k,\ 0 \leq c_k < \frac{1}{r^{n_k}(r-2)^{m_k}}.$$

Indeed if $t \in [0, 1)$, then there exist $a_1 \in \{0, 1, ..., r-1\}$ such that $rt = a_1 + t_1$, with $0 \leq t_1 < 1$. Then $t = \frac{a_1}{r} + \frac{t_1}{r} = \frac{a_1}{r} + c_1$ with $c_1 < \frac{1}{r}$.
If $a_1 \in \{1, 3, ..., r-2, r-1\}$ then there exist $a_2 \in \{0, 1, ..., r-1\}$ such that $rc_1 = \frac{a_2}{r} + t_2$, with $0 \leq t_2 < \frac{1}{r}$ and then

$$t = \frac{a_1}{r} + c_1 = \frac{a_1}{r} + \frac{a_2}{r^2} + \frac{t_2}{r} = \frac{a_1}{r} + \frac{a_2}{r^2} + c_2,\ 0 < c_2 < \frac{1}{r^2}.$$

If $a_1 \in \{0, 2, ..., r-3\}$ then there exist $a_2 \in \{0, 1, ..., r-3\}$ such that $rc_1 = \frac{a_2}{r-2} + t_2$, with $0 \leq t_2 < \frac{1}{r-2}$ and then

$$t = \frac{a_1}{r} + c_1 = \frac{a_1}{r} + \frac{a_2}{r(r-2)} + \frac{t_2}{r} = \frac{a_1}{r} + \frac{a_2}{r(r-2)} + c_2,\ 0 < c_2 < \frac{1}{r(r-2)}.$$

Then the result is true for $k = 1, 2$. Suppose that it is true for $2 < k$, that is

$$t = \frac{a_1}{r} + \sum_{\substack{i=2 \\ m_i+n_i=i}}^{k} \frac{a_k}{r^{n_i}(r-2)^{m_i}} + c_k,$$

where $0 \leq c_k < \frac{1}{r^{n_k}(r-2)^{m_k}}$.
If a_k and k satisfy condition $(3a)$ or $(3b)$ or $(3c)$ or $(3d)$ then there exist $a_{k+1} \in \{0, 1, ..., r-1\}$ such that
$r^{n_k}(r-2)^{m_k} c_k = \frac{a_{k+1}}{r} + t_{k+1}$, $0 \leq t_{k+1} < \frac{1}{r}$ and then

$$t = \frac{a_1}{r} + \sum_{\substack{i=2 \\ m_i+n_i=i}}^{k} \frac{a_k}{r^{n_i}(r-2)^{m_i}} + c_k = \frac{a_1}{r} + \sum_{\substack{i=2 \\ m_i+n_i=i}}^{k} \frac{a_k}{r^{n_i}(r-2)^{m_i}} + \frac{a_{k+1}}{r^{n_k+1}(r-2)^{m_k}} + c_{k+1},$$

where $0 \leq c_{k+1} = \dfrac{t_{k+1}}{r^{n_k}(r-2)^{m_k}} < \dfrac{1}{r^{n_k+1}(r-2)^{m_k}}$. Then the result is true for $k+1$.
If a_k and k satisfy condition (4a) or (4b) or (4c) or (4d) then there exist $a_{k+1} \in \{0, 1, ..., r-3\}$ such that
$r^{n_k}(r-2)^{m_k} c_k = \frac{a_{k+1}}{r-2} + t_{k+1},\ 0 \leq t_{k+1} < \frac{1}{r-2}$ and then

$$t = \frac{a_1}{r} + \sum_{\substack{i=2 \\ m_i+n_i=i}}^{k} \frac{a_k}{r^{n_i}(r-2)^{m_i}} + c_k = \frac{a_1}{r} + \sum_{\substack{i=2 \\ m_i+n_i=i}}^{k} \frac{a_k}{r^{n_i}(r-2)^{m_i}} + \frac{a_{k+1}}{r^{n_k}(r-2)^{m_k+1}} + c_{k+1},$$

where $0 \leq c_{k+1} = \dfrac{t_{k+1}}{r^{n_k}(r-2)^{m_k}} < \dfrac{1}{r^{n_k}(r-2)^{m_k+1}}$. Then the result is true for $k+1$. Therefore the result is true for every $k \geq 1$. □

REMARK 3.6. Let $t \in [0, 1]$ written as $t = \dfrac{a_1}{r} + \sum_{\substack{i=2 \\ m_i+n_i=i}}^{k} \dfrac{a_k}{r^{n_i}(r-2)^{m_i}}$. In order to simplify the demonstration of some of the results in this paper, a simpler notation will be used, i.e., t will be represented as

$$t = \frac{a_1}{r} + \sum_{\substack{i=2 \\ m_i+n_i=i}}^{\infty} \frac{a_i}{(n_i, m_i)}.$$

Proposition 3.7. *Let t, t' be elements in* $[0, 1]$, $t = \frac{a_1}{r} + \sum_{i=2}^{\infty} \frac{a_i}{(n_i,m_i)}$, $t' = \frac{a'_1}{r} + \sum_{i=2}^{\infty} \frac{a'_i}{(n'_i,m'_i)}$, *$a_i$ and a'_i as in* Proposition (3.4). *Suppose that $a_i = a'_i, i = 1, 2, ..., k-1$ and $a_k < a'_k$. If $|t' - t| < r^{-N}$ with $k < N$ then $t = T_1 + T_2 + T_3,\ t' = T_1 + T'_2 + T'_3$ where*
$T_1 = \dfrac{a_1}{r} + \sum_{\substack{i=2 \\ m_i+n_i=i}}^{k-1} \dfrac{a_i}{(n_i, m_i)},\ T_3 = \sum_{\substack{i \geq N+1 \\ m_i+n_i=i}}^{\infty} \dfrac{a_i}{(n_i, m_i)},\ T'_3 = \sum_{\substack{i \geq N+1 \\ m_i+n_i=i}}^{\infty} \dfrac{a'_i}{(n'_i, m'_i)}$ *and*

1) *if k is even and a_k satisfies items* (3a) *or* (3b) *or* (3c) *of* Proposition 3.5 *then*
$T_2 = \frac{a_k}{(n_k,m_k)} + \frac{r-1}{(n_k+1,m_k)} + \frac{r-1}{(n_k+2,m_k)} + \frac{r-3}{(n_k+2,m_k+1)} + \frac{r-1}{(n_k+3,m_k+1)} + \frac{r-3}{(n_k+3,m_k+2)} + ... + \frac{a_N}{(n_N,m_N)}$,

$T'_2 = \frac{a_k+1}{(n_k,m_k)} + \frac{0}{(n'_{k+1},m'_{k+1})} + \frac{0}{(n'_{k+2},m'_{k+2})} + \frac{0}{(n'_{k+3},m'_{k+3})} + \frac{0}{(n'_{k+4},m'_{k+4})} + \frac{0}{(n'_{k+5},m'_{k+5})} + ... + \frac{0}{(n'_N,m'_N)}$,
Moreover we have $t = t'$ if and only if
$t = T_1 + \frac{a_k}{(n_k,m_k)} + \frac{r-1}{(n_k+1,m_k)} + \sum_{i=0}^{\infty} \left(\frac{r-1}{(n_k+2+i,m_k+i)} + \frac{(r-3)}{(n_k+2+i,m_k+1+i)} \right)$ *and*

$t' = T_1 + \frac{a_k+1}{(n_k,m_k)} + \sum_{i>k+1}^{\infty} \frac{0}{(n'_i,m'_i)}$.
2) *if k is odd and and a_k satisfies item* (3c) *of* Proposition 3.5 *then*
$T_2 = \frac{a_k}{(n_k,m_k)} + \frac{r-1}{(n_k+1,m_k)} + \frac{r-3}{(n_k+1,m_k+1)} + \frac{r-1}{(n_k+2,m_k+1)} + \frac{r-3}{(n_k+2,m_k+2)} + ... + \frac{a_N}{(n_N,m_N)}$,

$T'_2 = \frac{a_k+1}{(n_k,m_k)} + \frac{0}{(n'_{k+1},m'_{k+1})} + \frac{0}{(n'_{k+2},m'_{k+2})} + \frac{0}{(n'_{k+3},m'_{k+3})} + \frac{0}{(n'_{k+4},m'_{k+4})} + ... + \frac{0}{(n'_N,m'_N)}$,
Moreover we have $t = t'$ if and only if

$t = T_1 + \frac{a_k}{(n_k,m_k)} + \sum_{i=0}^{\infty}\left(\frac{r-1}{(n_k+1+i,m_k+i)} + \frac{(r-3)}{(n_k+1+i,m_k+1+i)}\right)$ *and*

$t' = T_1 + \frac{a_k+1}{(n_k,m_k)} + \sum_{i>k+1}^{\infty}\frac{0}{(n'_i,m'_i)}$.

3) *if k is odd and a_k satisfies items* (4*a*) *or* (4*b*) *of* Proposition 3.5 *then*

$$T_2 = \frac{a_k}{(n_k,m_k)} + \frac{r-3}{(n_k,m_k+1)} + \frac{r-3}{(n_k,m_k+2)} + \frac{r-1}{(n_k+1,m_k+2)} + \frac{r-3}{(n_k+1,m_k+3)} + \frac{r-1}{(n_k+2,m_k+3)} + \dots + \frac{a_N}{(n_N,m_N)},$$

$$T'_2 = \frac{a_k+1}{(n_k,m_k)} + \frac{0}{(n'_{k+1},m'_{k+1})} + \frac{0}{(n'_{k+2},m'_{k+2})} + \frac{0}{(n'_{k+3},m'_{k+3})} + \frac{0}{(n'_{k+4},m'_{k+4})} + \frac{0}{(n'_{k+5},m'_{k+5})} + \dots + \frac{0}{(n'_N,m'_N)},$$

Moreover we have $t = t'$ if and only if

$t = T_1 + \frac{a_k}{(n_k,m_k)} + \frac{r-3}{(n_k,m_k+1)} + \sum_{i=0}^{\infty}\left(\frac{r-3}{(n_k+i,m_k+2+i)} + \frac{(r-1)}{(n_k+1+i,m_k+2+i)}\right)$ *and*

$t' = T_1 + \frac{a_k+1}{(n_k,m_k)} + \sum_{i>k+1}^{\infty}\frac{0}{(n'_i,m'_i)}$.

4) *if k is even and a_k satisfies item* (4*c*) *of* Proposition 3.5 *then*

$$T_2 = \frac{a_k}{(n_k,m_k)} + \frac{r-3}{(n_k,m_k+1)} + \frac{r-1}{(n_k+1,m_k+1)} + \frac{r-3}{(n_k+1,m_k+2)} + \frac{r-1}{(n_k+2,m_k+2)} + \dots + \frac{a_N}{(n_N,m_N)},$$

$$T'_2 = \frac{a_k+1}{(n_k,m_k)} + \frac{0}{(n'_{k+1},m'_{k+1})} + \frac{0}{(n'_{k+2},m'_{k+2})} + \frac{0}{(n'_{k+3},m'_{k+3})} + \frac{0}{(n'_{k+4},m'_{k+4})} + \dots + \frac{0}{(n'_N,m'_N)},$$

Moreover we have $t = t'$ if and only if

$t = T_1 + \frac{a_k}{(n_k,m_k)} + \sum_{i=0}^{\infty}\left(\frac{r-3}{(n_k+i,m_k+1+i)} + \frac{(r-1)}{(n_k+1+i,m_k+1+i)}\right)$ *and*

$t' = T_1 + \frac{a_k+1}{(n_k,m_k)} + \sum_{i>k+1}^{\infty}\frac{0}{(n'_i,m'_i)}$.

Proof. Take $t,t' \in [0,1]$ such that $|t'-t| < r^{-N}$, $t = \frac{a_1}{r} + \sum_{\substack{k=2\\ m_k+n_k=k}}^{\infty}\frac{a_k}{(n_k,m_k)}$, $t' = \frac{a'_1}{r} + \sum_{\substack{k=2\\ m'_k+n'_k=k}}^{\infty}\frac{a'_k}{(n'_k,m'_k)}$, $a_i = a'_i$, $\forall\, i = 1,2,\dots,k-1$ and $a_k < a'_k$. Then

$$\begin{aligned} t'-t &= \frac{(a'_k-a_k)}{(n_k,m_k)} + \sum_{i>k}^{\infty}\left[\frac{a'_i}{(n'_i,m'_i)} - \frac{a_i}{(n_i,m_i)}\right] \\ &= \frac{(a'_k-a_k-1)}{(n_k,m_k)} + \frac{1}{(n_k,m_k)} + \sum_{i>k}^{\infty}\left[\frac{a'_i}{(n'_i,m'_i)} - \frac{a_i}{(n_i,m_i)}\right], \end{aligned}$$

and since $m_k + n_k = k$, $|t'-t| < r^{-N}$ then $a'_k - a_k - 1 = 0$, that is, $a'_k = a_k + 1$.

(1) Let k be an even number and $a_k = 0$ or $2n-1$, $n = 1,\dots,a-1$. Then $a_{k+1} \in \{0,1,\dots,r-1\}$ and we can write

$$\frac{1}{(n_k,m_k)} = \frac{r-1}{(n_k+1,m_k)} + \sum_{i=0}^{\infty}\left(\frac{r-1}{(n_k+2+i,m_k+i)} + \frac{(r-3)}{(n_k+2+i,m_k+1+i)}\right).$$

Therefore

$$\begin{aligned} t'-t &= \frac{a'_{k+1}}{(n'_{k+1},m'_{k+1})} - \frac{a_{k+1}}{(n_k+1,m_k)} + \frac{(r-1)}{(n_k+1,m_k)} + \dots \\ &= \frac{a'_{k+1}}{(n'_{k+1},m'_{k+1})} + \frac{r-1-a_{k+1}}{(n_k+1,m_k)} + \dots, \end{aligned}$$

where $m'_{k+1} + n'_{k+1} = m_k + n_k + 1 = k + 1$. As $|t' - t| < r^{-N} < (r-2)^{-m} r^{-n}, m + n = N \geq k + 1$ then $\frac{a'_{k+1}}{(n'_{k+1}, m'_{k+1})} + \frac{r-1-a_{k+1}}{(n_k+1, m_k)} = 0$ and it is possible only with $a'_{k+1} = 0$ and $a_{k+1} = r - 1$.

As $a_{k+1} = r - 1$ and $k + 1$ is an odd number, then $a_{k+2} \in \{0, 1, ..., r - 1\}$ and we have

$$\begin{aligned} t' - t &= \frac{a'_{k+2}}{(n'_{k+2}, m'_{k+2})} - \frac{a_{k+2}}{(n_k + 2, m_k)} + \frac{r-1}{(n_k + 2, m_k)} + ... \\ &= \frac{a'_{k+2}}{(n'_{k+2}, m'_{k+2})} + \frac{r - 1 - a_{k+2}}{(n_k + 2, m_k)} + ... \end{aligned}$$

with $m'_{k+2} + n'_{k+2} = m_k + n_k + 2 = k + 2$. Again we have $a'_{k+2} = 0$ and $a_{k+2} = r - 1$. Now $a_{k+2} = r - 1$ and $k + 2$ is an even number. Then $a_{k+3} \in \{0, 1, ..., (r - 3)\}$ and

$$\begin{aligned} t' - t &= \frac{a'_{k+3}}{(n'_{k+3}, m'_{k+3})} - \frac{a_{k+3}}{(n_k + 2, m_k + 1)} + \frac{(r-3)}{(n_k + 2, m_k + 1)} + ... \\ &= \frac{a'_{k+3}}{(n'_{k+3}, m'_{k+3})} + \frac{(r - 3) - a_{k+3}}{(n_k + 2, m_k + 1)} + ... \end{aligned}$$

with $m'_{k+3} + n'_{k+3} = m_k + n_k + 3 = k + 3$. Therefore $a'_{k+3} = 0$ and $a_{k+3} = (r - 3)$. Following this idea we have the result.

(2) To prove this part we use the same ideas of (1) and the equality

$$\frac{1}{(n_k, m_k)} = \sum_{i=0}^{\infty} \left(\frac{(r-1)}{(n_k + 1 + i, m_k + i)} + \frac{(r-3)}{(n_k + 1 + i, m_k + 1 + i)} \right).$$

(3) To prove this part we use the same ideas of (1) and the equality

$$\frac{1}{(n_k, m_k)} = \frac{(r-3)}{(n_k, m_k + 1)} + \sum_{i=0}^{\infty} \left(\frac{(r-3)}{(n_k + i, m_k + 2 + i)} + \frac{r-1}{(n_k + 1 + i, m_k + 2 + i)} \right).$$

(4) To prove this part we use the same ideas of (1) and the equality

$$\frac{1}{(n_k, m_k)} = \sum_{i=0}^{\infty} \left(\frac{(r-3)}{(n_k + i, m_k + 1 + i)} + \frac{r-1}{(n_k + 1 + i, m_k + 1 + i)} \right).$$

□

Now we will give an explicit parametrization of $\mathcal{B}_{\alpha-1}$ and hence for the boundary $\partial\mathcal{R}_a$. Let z be an element of $\mathcal{B}_{\alpha-1}$. Using Proposition 3.2, there exists a sequence $(z_n)_{n\geq 1}$ in $\mathcal{B}_{\alpha-1}$, such that

$$z = g_{a_1} \circ g_{a_2} \circ \ldots \circ g_{a_n}(z_n), \forall\, n \geq 1.$$

If x is an element of $\mathcal{B}'_{\alpha-1}$, the sequence $y_n = g_{a_1} \circ g_{a_2} \circ \ldots \circ g_{a_n}(x)$ converges to z because the functions $g_i, i = 0, 1, ..., r - 1$ are contractions.

Let $A = \{0, 1, ..., r - 1\}$ be a subset of $\mathbb{N}$ and consider the function

$$\begin{aligned} \psi :\ & A^{\mathbb{N}} \longrightarrow A^{\mathbb{N}} \\ & (a_i) \longmapsto \psi((a_i)) = (b_i) \end{aligned}$$

given by:
$b_1 = a_1$;
$b_{2k} = r-1-a_{2k}$;
$b_{2k+1} = a_{2k+1}$ if $a_{2k} \in \{0\} \cup \{2n-1 : n = 1..., a-1\}$;
$b_{2k+1} = a_{2k+1} + 2$ if $a_{2k} \in \{2n : n = 1, ..., a-1\}$.
Take $x_0 \in \mathcal{B}'_{\alpha-1}$ and consider $f : [0,1] \longrightarrow \mathcal{B}_{\alpha-1}$ defined as follows:
if $t = \frac{a_1}{r} + \sum_{\substack{k=2 \\ m_k+n_k=k}}^{\infty} \frac{a_k}{r^{n_k}(r-2)^{m_k}}, (a_i) \in A^{\mathbb{N}}$, then $f(t) = \lim_{n\to\infty} g_{b_1} \circ g_{b_2} \circ ... \circ g_{b_n}(x_0)$ where $\psi((a_i)) = (b_i)$.

Theorem 3.8. *f is a continuous, bijective function satisfying $f(0) = u = -1$ and $f(1) = v$.*

Proof. (1)− f is a well defined function.
We are going to use the following notation:

$$g_{b_1} \circ g_{b_2} \circ ... \circ g_{b_{k-1}} \circ g_{b_k}(x_0) = g_{b_1...b_k}(x_0).$$

According Lemma 3.4 we have

$$u = -1 = g_{0(r-1)0(r-1)...}(x_0) = g_{\overline{0(r-1)}}(x_0).$$
$$v = -(a-1)\alpha - \alpha^{-1} = g_{(r-1)0(r-1)0...}(x_0) = g_{\overline{(r-1)0}}(x_0).$$
$$w = -1 - \alpha^3 = g_{2(r-1)0(r-1)0...}(x_0) = g_{2\overline{(r-1)0}}(x_0).$$

Taking $t, t' \in [0,1]$ such that $t = t'$. We have to prove that $f(t) = f(t')$ and for this we use Proposition 3.7 and the definition of ψ. We have to consider some cases.
- k be an even number and $a_k = 0$ or $2n-1,\ n = 1, ..., a-1$.
Then $a_k + 1 = 1$ or $2n$ and by Proposition 3.7 we have
$t = \frac{a_1}{r} + ... + \frac{a_{k-1}}{(n_{k-1}, m_{k-1})} + \frac{a_k}{(n_k, m_k)} + \frac{r-1}{(n_k+1, m_k)} + \sum_{i=0}^{\infty} \left(\frac{r-1}{(n_k+2+i, m_k+i)} + \frac{r-3}{(n_k+2+i, m_k+1+i)} \right)$
and
$t' = \frac{a_1}{r} + ... + \frac{a_{k-1}}{(n_{k-1}, m_{k-1})} + \frac{a_k+1}{(n_k, m_k)} + \sum_{i=k+1}^{\infty} \frac{0}{(n'_i, m'_i)}$. Using the definition of ψ we have

$f(t) = g_{b_1...b_{k-1}(r-1)\overline{(r-1)0}}(x_0) = g_{b_1...b_{k-1}(r-1)}(v)$,
$f(t') = g_{b_1...b_{k-1}(r-2)\overline{0(r-1)}}(x_0) = g_{b_1...b_{k-1}(r-2)}(u)$,
if $a_k = 0$ or
$f(t) = g_{b_1...b_{k-1}(r-2n)\overline{(r-1)0}}(x_0) = g_{b_1...b_{k-1}(r-2n)}(v)$
$f(t') = g_{b_1...b_{k-1}(r-2n-1)2\overline{(r-1)0}}(x_0) = g_{b_1...b_{k-1}(r-2n)}(w)$ if $a_k = 2n-1,\ n = 1, ..., a-1$.
By Lemma 3.4 $f(t) = f(t')$.
Using the same ideas we can prove the following cases (see [10]).
- k be an odd number and $a_k = 2n-1,\ n = 1, ..., a-1$,
- k be an odd number and $a_k = 0$ or $2n,\ n = 1, ..., a-2$,
- k be an even number and $a_k = 2n,\ n = 1, ..., a-2$.
(2)− f is injective.
Suppose that $f(t) = f(t')$. According to Lemma 3.4 we have two possibilities:

- $f(t) = g_{b_1...b_{k-1}b_k}(u), b_k \in \{1, 3, 5, ..., r-2\}$ and $f(t') = g_{b_1...b_{k-1}(b_k+1)}(v)$.
  Using the above notations we have
  $f(t) = g_{b_1...b_{k-1}b_k\overline{0(r-1)}}(x_0)$ and $f(t') = g_{b_1...b_{k-1}(b_k+1)\overline{(r-1)0}}(x_0)$. We need to consider the following cases:

- k is an even number, $b_k \neq r-2$. In this case $b_k = r-1-a_k$ and then $a_k = r-1-b_k$ is an odd number. By the definition of ψ we have:
  - $a_{k+1} = 0$ because $b_{k+1} = 0,\ a_k$ odd number,
  - $a_{k+2} = 0$ because $b_{k+2} = r-1,\ k+2$ even number.
  Following this idea is easy to see that $a_i = 0,\ \forall\, i \geq k+1$.
  Therefore $t = \frac{a_1}{r} + ... + \frac{a_{k-1}}{(n_{k-1},m_{k-1})} + \frac{r-1-b_k}{(n_k,m_k)} + \sum_{i=k+1}^{\infty} \frac{0}{(n_i,m_i)}$.
  We also have $b'_k = b_k + 1 = r-1-a'_k$ and then $a'_k = r-2-b_k \neq 0$ is an even number. By the definition of ψ and Proposition 3.5 we have:
  -$a'_{k+1} = r-3,\ n'_{k+1} = n'_k,\ m'_{k+1} = m'_k + 1$ because $b'_{k+1} = r-1,\ a'_k$ even,
  -$a'_{k+2} = r-1$ because $b'_{k+2} = 0,\ k+2$ even ,
  -$a_{k+3} = r-3,\ n'_{k+3} = n'_{k+2},\ m'_{k+3} = m'_{k+2} + 1$.
  Following this idea we have
  $t' = \frac{a_1}{r} + ... + \frac{a_{k-1}}{(n_{k-1},m_{k-1})} + \frac{r-2-b_k}{(n_k,m_k)} + \sum_{i=0}^{\infty}\left(\frac{r-3}{(n'_k+i,m'_k+1+i)} + \frac{r-1}{(n'_k+1+i,m'_k+1+i)}\right)$ and then $t = t'$.

Using the same ideas we can prove the following cases (see [10]).

- k is an even number, $b_k = r-2$,
- k is an odd number.

– $f(t) = g_{b_1...b_{k-1}b_k}(-1-\alpha^3), b_k \in \{0, 2, ..., r-3\}$ and $f(t') = g_{b_1..b_{k-1}(b_k+1)}(v)$. Using the above notations we have $f(t) = g_{b_1...b_{k-1}b_k2\overline{(r-1)0}}(x_0)$ and $f(t') = g_{b_1...b_{k-1}(b_k+1)\overline{(r-1)0}}(x_0)$. We have to consider the following cases:

- k is an even number. In this case $b_k = r-1-a_k$ and $a_k = r-1-b_k$ is an even number too and we can prove that

$$t = \frac{a_1}{r} + ... + \frac{a_{k-1}}{(n_{k-1},m_{k-1})} + \frac{r-1-b_k}{(n_k,m_k)} + \sum_{i=k+1}^{\infty} \frac{0}{(n_i,m_i)}$$

  and

$$t' = \frac{a_1}{r} + ... + \frac{a_{k-1}}{(n_{k-1},m_{k-1})} + \frac{r-2-b_k}{(n_k,m_k)} + \frac{r-1}{(n'_k+1,m'_k)} + \sum_{i=0}^{\infty}\left(\frac{r-1}{(n'_k+2+i,m'_k+i)} + \frac{r-3}{(n'_k+2+i,m'_k+1+i)}\right).$$

  Then $t = t'$.
- k is an odd number. In this case $b_k = a_k$ or $b_k = a_k + 2$ and then $a_k = b_k$ or $a_k = b_k - 2$. We can prove that

$$t = \frac{a_1}{r} + ... + \frac{a_{k-1}}{(n_{k-1},m_{k-1})} + \frac{a_k}{(n_k,m_k)} + \frac{r-3}{(n_k,m_k+1)} + \sum_{i=0}^{\infty}\left(\frac{r-3}{(n_k+i,m_k+2+i)} + \frac{r-1}{(n_k+1+i,m_k+2+i)}\right)$$

  and

$$t' = \frac{a_1}{r} + ... + \frac{a_{k-1}}{(n_{k-1},m_{k-1})} + \frac{a_k+1}{(n_k,m_k)} + \sum_{i=k+1}^{\infty} \frac{0}{(n'_i,m'_i)}.$$

  Then $t = t'$.

(3)– f is a continuous function.

Let us consider $t, t' \in [0,1]$, $|t' - t| < r^{-N}$ as in Proposition 3.7. We have to consider the following cases:

– t and t' satisfying (1) of Proposition 3.7.
Here we have:
(1) $f(t) = g_{b_1...b_{k-1}(r-1)(r-1)0(r-1)0...b_{N+1}...}(x_0)$ and
$f(t') = g_{b_1...b_{k-1}(r-2)0(r-1)0(r-1)...b'_{N+1}...}(x_0)$ if $a_k = 0$. Then

$$|f(t) - f(t')| = |g_{b_1b_2...b_{k-1}r-1}(z_1) - g_{b_1b_2...b_{k-1}r-2}(z_2)|$$
$$\leq |\alpha|^{2(k-1)}|g_{r-1}(z_1) - g_{r-2}(z_2)|.$$

As $g_{r-2}(u) = g_{r-1}(v)$ then

$$|f(t) - f(t')| \leq |\alpha|^{2(k-1)} (|g_{r-1}(z_1) - g_{r-1}(v)| + |g_{r-2}(z_2) - g_{r-2}(u)|)$$
$$\leq |\alpha|^{2(k-1)}(|\alpha|^2 + |\alpha|^3)diam(\mathcal{B}_{\alpha-1}) = |\alpha|^{2k}(1 + |\alpha|)diam(\mathcal{B}_{\alpha-1}),$$

where $diam(\mathcal{B}_{\alpha-1})$ is the diameter of $\mathcal{B}_{\alpha-1}$.

(2) $f(t) = g_{b_1...b_{k-1}(r-2n)(r-1)0(r-1)0...b_{N+1}...}(x_0)$ and
$f(t') = g_{b_1...b_{k-1}(r-2n-1)2(r-1)0(r-1)0...b'_{N+1}...}(x_0)$ if $a_k = 2n - 1$. Then

$$|f(t) - f(t')| = |g_{b_1b_2...b_{k-1}(r-2n)}(z_1) - g_{b_1b_2...b_{k-1}(r-2n-1)}(z_2)|$$
$$\leq |\alpha|^{2(k-1)}|g_{r-2n}(z_1) - g_{r-2n-1}(z_2)|.$$

As $g_{r-2n-1}(w) = g_{r-2n}(v)$ then

$$|f(t) - f(t')| \leq |\alpha|^{2(k-1)} (|g_{r-2n}(z_1) - g_{r-2n}(v)| + |g_{r-2n-1}(z_2) - g_{r-2n-1}(w)|)$$
$$\leq |\alpha|^{2(k-1)}(|\alpha|^2 + |\alpha|^3)diam(\mathcal{B}_{\alpha-1}) = |\alpha|^{2k}(1 + |\alpha|)diam(\mathcal{B}_{\alpha-1}).$$

Using the same ideas we can prove the next cases (see [10]).

- t and t' satisfying item (2) of Proposition 3.7.
- t and t' satisfying item (3) of Proposition 3.7.
- t and t' satisfying item (4) of Proposition 3.7.

In all that cases we conclude that f is a continuous function. □

Now we can finally prove Theorem 1.4.
Proof. Let $\mathcal{Q} \subseteq \mathbb{R}^2$ be the set

$$\mathcal{Q} = \{(0, y), 0 \leq y \leq 1\} \cup \{(x, 1), 0 \leq x \leq 1\} \cup \{(1, y), 0 \leq y \leq 1\} \cup \{(x, 0), 0 \leq y \leq 1\}.$$

Using Proposition 3.1, Theorem 3.8 and Figure 4 we can prove that $F : \mathcal{Q} \longrightarrow \partial\mathcal{R}_a$ given by

$$F(x,y) = \begin{cases} f(y), \text{if } (x, y) = (0, y), 0 \leq y \leq 1 \\ (f_2 \circ f)(x), \text{if } (x, y) = (x, 1), 0 \leq x \leq 1 \\ (f_3 \circ f)(y), \text{if } (x, y) = (1, y), 0 \leq y \leq 1 \\ (f_1 \circ f)(x), \text{if } (x, y) = (x, 0), 0 \leq x \leq 1 \end{cases}$$

is an homeomorphism . □

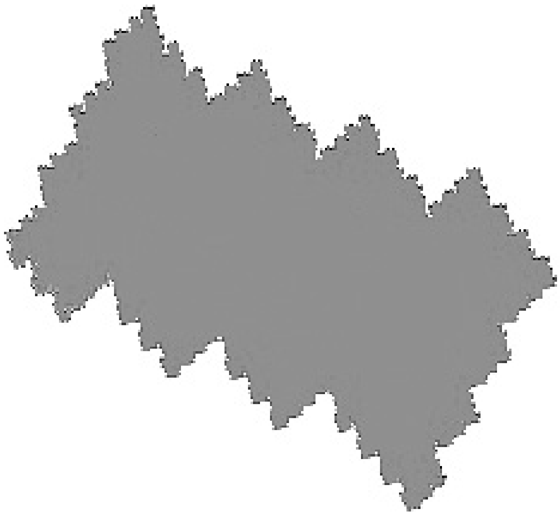

Fig. 5. $\mathcal{R}_4$

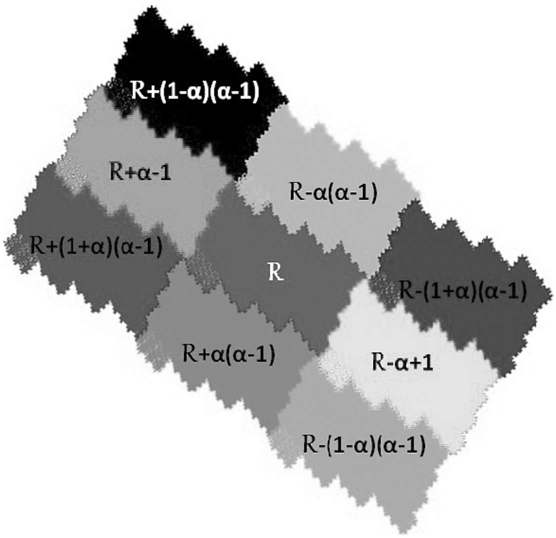

Fig. 6. Tiling induced by $\mathcal{R}_4$

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