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FOLDINGS AND TWO-SIDED TILTING COMPLEXES FOR BRAUER TREE ALGEBRAS

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Abstract

In this note, for a Brauer tree algebra A and a star-shaped Brauer tree algebra B which is derived equivalent to A, we give operations on the two-sided tilting complex D_T of $A \otimes B^{op}$-modules constructed in [3] which is isomorphic to the Rickard tree-to-star complex T constructed in [5] in $D^b(A)$, and we show that the operations on D_T correspond to operations called *foldings* on the Rickard tree-to-star complex T given in [7].

1. Introduction

Throughout this paper, we denote by A a Brauer tree algebra with e edges numbered as $0, 1, \cdots, e-1$ and by B a star-shaped Brauer tree algebra with the same number of edges as A and with the same multiplicity as A.

Brauer tree algebras with the same number of edges and the same multiplicity are derived equivalent. This fact is shown by Rickard in [5, Theorem4.2] by constructing a one-sided tilting complex of A-modules with endomorphism ring B. We denote the one-sided tilting complex constructed by Rickard by T, and call T a Rickard tree-to-star complex. Also in [7, Section 3] Rickard and Schaps constructed various one-sided tilting complexes of A-modules with endomorphism ring B by using pointings on the Brauer tree (see Section 2.4). We call each of the one-sided tilting complexes a Rickard-Schaps tree-to-star complex. In particular the Rickard tree-to-star complex is one of the Rickard-Schaps tree-to-star complexes. For any Rickard-Schaps tree-to-star complex T', by applying operations, called *foldings*, several times to the Rickard tree-to-star complex T we can get it (the definition of the folding can be seen in Remark 2.19).

$$T \xrightarrow{\text{folding}} T_1 \xrightarrow{\text{folding}} T_2 \longrightarrow \cdots \longrightarrow T_{n-1} \xrightarrow{\text{folding}} T_n = T'$$

On the other hand, in [6] and [2] it is shown that for a one-sided tilting complex over an algebra Γ there exists a two-sided tilting complex isomorphic to the one-sided tilting complex in the bounded derived category of Γ-modules in theory. The existence of the two-sided tilting complexes are abstract, but in [3], by using concrete bimodules we constructed a two-sided tilting complex of A-B-bimodules isomorphic to the Rickard tree-to-star complex T in the bounded derived category of A-modules for any Brauer tree algebra A. We denote this two-sided tilting complex by D_T. This two-sided tilting complex D_T has an indecomposable

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A-B-bimodule M in degree 0 inducing a stable equivalence of Morita type induced by T, and each non-zero terms except for degree 0 is of the form $\bigoplus P(S_i) \otimes_k P(V_j)^*$ for some simple A-modules S_i and some simple B-modules V_j (see Section 2.2). Hence all of the terms of the two-sided tilting complex D_T are projective on both sides, so we have an equivalence $D_T \otimes_B^L - \cong D_T \otimes_B -$ between $D^b(A)$ and $D^b(B)$ with the reverse equivalence $D_T^* \otimes_A -$.

Since any Rickard-Schaps tree-to-star complex T' is obtained by applying foldings several times to the Rickard tree-to-star complex T, there should be operations corresponding to foldings which enable us to construct a two-sided tilting complex $D_{T'}$ with $D_{T'} \cong T'$ in $D^b(A)$. We call these operations on two-sided tilting complexes *two-sided folings*, and we get the following commutative diagram, where vertical arrows mean the restrictions to A from $A \otimes_k B^{op}$ and where $D_{T_i} \cong T_i$ in $D^b(A)$ for each $1 \leq i \leq n$.

$$\begin{tikzcd}
D_T \arrow[r, "\text{two-sided folding}"] \arrow[d] & D_{T_1} \arrow[r, "\text{two-sided folding}"] \arrow[d] & D_{T_2} \arrow[r] \arrow[d] & \cdots \arrow[r] & D_{T_{n-1}} \arrow[r, "\text{two-sided folding}"] \arrow[d] & D_{T_n} \arrow[d] \\
T \arrow[r, "\text{folding}"] & T_1 \arrow[r, "\text{folding}"] & T_2 \arrow[r] & \cdots \arrow[r] & T_{n-1} \arrow[r, "\text{folding}"] & T_n
\end{tikzcd}$$

The aim of this paper is giving such operations on the two-sided tilting complexes.

The first result of this paper is the following.

Theorem 1. *Let D_T be a two-sided tilting complex of $A \otimes_k B^{op}$-modules isomorphic to the Rickard tree-to-star complex T of A-modules. Let C be a complex of $A \otimes_k B^{op}$-modules obtained by applying the following operations several times to D_T:*

(i) *Deleting the direct summand of the form $P(U) \otimes_k X$ from the leftmost non-zero term and the one of the form $P(U) \otimes_k X'$ from the second leftmost term for a simple module U not adjacent to the exceptional vertex.*

(ii) *Deleting the direct summand of the form $\bigoplus_{i=1}^{n} P(U_i) \otimes_k X_i$ from the leftmost non-zero term and the one of the form $\bigoplus_{i=1}^{n} P(U_i) \otimes_k X_i'$ from the second leftmost term for the all simple modules $U_1, \cdots, U_n$ adjacent to the exceptional vertex.*

Then the resulting complex C is a two-sided tilting complex of $A \otimes_k B^{op}$-modules.

We remark that we can apply the operations in Theorem 1 to D_T any number of times by using an isomorphism

$$\begin{aligned}
C &= (\cdots \to 0 \to P_n \to \cdots \to P_1 \to M) \\
&\cong (\cdots \to 0 \to P_n \to \cdots \to P_1 \xrightarrow{\varphi} P(\Omega^{-1}M) \to P(\Omega^{-2}M) \to \cdots \to P(\Omega^{-l}M) \to \Omega^{-l}M)
\end{aligned}$$

where φ is the composition $P_1 \to M \hookrightarrow I(M) \cong P(\Omega^{-1}M)$ and by the fact that $P(\Omega^{-n'}M)$ is of the form $\bigoplus_{i=0}^{e-1} P(S_i) \otimes_k P(V_{j,-n'})^*$ for each n' by [8, Lemma 2].

The second result of this paper is that operations in Theorem 1 are two-sided foldings. Hence for any Rickard-Schaps tree-to-star complex T' we can construct a two-sided tilting complex corresponding to T' by using concrete bimodules.

Theorem 2. *Let D_T be a two-sided tilting complex of $A \otimes_k B^{op}$-modules isomorphic to the Rickard tree-to-star complex T in $D^b(A)$ and T' be a Rickard-Schaps tree-to-star complex. Then we can construct a two-sided tilting complex $D_{T'}$ corresponding to T' by applying the operations in* Theorem 1 *to D_T several times.*

Since the resulting two-sided tilting complexes are projective on both sides, the derived equivalences induced by them are given by taking the ordinary tensor, which are easy to understand in comparison with the ones induced by the one-sided tilting complexes. Hence we can understand how the equivalence induced by the Rickard tree-to-star complex is different from the one by each Rickard-Schaps tree-to-star complex obtained by applying foldings to T several times. We give the detailed construction of the two-sided tilting complex $D_{T'}$ in Section 4.

Throughout this paper, let k be an algebraically closed field. Algebras mean finite dimensional symmetric k-algebras, and modules mean finitely generated left modules unless otherwise stated. We denote the k-dual $\mathrm{Hom}_k(-,k)$ by $(-)^*$ and denote $\otimes_k$ by $\otimes$ to simplify these. For k-algebras Γ and Λ, we denote the opposite algebra of Γ by Γ^{op} and we identify Γ-Λ-bimodules with $\Gamma\otimes\Lambda^{op}$-modules. For a Γ-module U, we denote its projective cover by $P(U)$, and denote $U/\mathrm{rad}\,U$ by $\mathrm{top}\,U$.

Our 'complexes' are chain complexes, so the differentials will have degree -1. If

$$X : \cdots \xrightarrow{d^X_{d+2}} X_{i+1} \xrightarrow{d^X_{i+1}} X_i \xrightarrow{d^X_i} X_{i-1} \xrightarrow{d^X_{i-1}} \cdots$$

is a complex, then $X[1]$ will be a complex with i-th term $X[1]_i = X_{i-1}$ and with i-th differential $d_i^{X[1]} = (-1)d^X_{i-1}$, in the other words, $X[1]$ will be the following complex

$$X[1] : \cdots \xrightarrow{-d^X_{d+1}} X_i \xrightarrow{-d^X_i} X_{i-1} \xrightarrow{-d^X_{i-1}} X_{i-2} \xrightarrow{-d^X_{i-2}} \cdots$$

where X_{i-1} is in degree i. Moreover we define $X[m] := X[m-1][1]$ inductively, and define $X[-m]$ as a complex satisfying that $X[-m][m] = X$.

For an algebra Γ, $K^b(\Gamma)$ means the homotopy category consisting of bounded complexes of Γ-modules and $D^b(\Gamma)$ means the derived category consisting of bounded complexes of Γ-modules. For a morphism f of complexes, $C(f)$ means the mapping cone of f.

2. Preliminaries

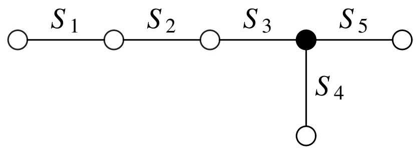
2.1. Rickard's result. In [5, Theorem 4.2], Rickard showed that two Brauer tree algebras with the same number of edges and with the same multiplicity are derived equivalent. The proof of this statement is done by constructing a tree-to-star complex of A-modules with endomorphism ring B. The construction is as follows:

For each edge i of the Brauer tree of A, there is a unique path in the Brauer tree from the exceptional vertex to the farther vertex of the two adjacent to the edge i. This path defines a sequence $i_0, i_1, \cdots, i_r = i$ of the edges. For this sequence, we define a complex corresponding to i as follows:

$$T(i) : \cdots \to 0 \to P(i_0) \to P(i_1) \to \cdots \to P(i_r) \to 0 \to \cdots$$

where $P(i_0)$ is in the degree 0, and where all the maps are non-zero. Finally, put $T' = \bigoplus_{i=0}^{e-1} T(i)$. Then T' is a tree-to-star complex. In this paper, we put $T = T'[m-1]$, where m is the maximal distance of the all edges of A from the exceptional vertex, so that the rightmost non-zero term of T' is in the degree 0. We call this tilting complex T a Rickard tree-to-star complex.

Example 2.1. Let A be a Brauer tree algebra associated to the following Brauer tree.



Then T is as follows,

$$\begin{array}{ccccc} P(S_3) & \to & P(S_2) & \to & P(S_1) \\ & & \oplus & & \\ P(S_3) & \to & P(S_2) & & \\ & & \oplus & & \\ P(S_3) & & & & \\ & & \oplus & & \\ P(S_4) & & & & \\ & & \oplus & & \\ P(S_5) & & & & \end{array}$$

where $P(S_1)$ is in the degree 0.

2.2. Two-sided tilting complexes corresponding to T. We fix some notation on the Brauer tree algebra A. Let M be an indecomposable $A \otimes B^{op}$-module inducing a stable equivalence of Morita type induced by the Rickard tree-to-star complex T. For a simple A-module S_i we define its distance to be the number of edges between the exceptional vertex and the farther vertex on the edge S_i, and denote it by $d(S_i)$ and the maximal distance of all simple A-modules by m. We fix a simple A-module with distance m, and denote it by S. Then $M^* \otimes_A S$ is a simple B-module (see [3, Proposition 3.8]), and we denote this simple module by V.

In [3], we constructed a two-sided tilting complex of $A \otimes B^{op}$-modules which is isomorphic to T in $D^b(A)$. The construction is as follows:

As will be explained in Remark 2.6 below, since B is the Brauer star algebra, $\{\Omega^{2i} V | 0 \leq i \leq e-1\}$ is the set of all simple B-modules. By standard properties of the functor Ω, since M is projective on both sides, so is $\Omega^n M$ for each n. By [8, Lemma 2], we have

$$P(\Omega^n M) \cong \bigoplus_{i=0}^{e-1} P(\Omega^n M \otimes_B \Omega^{2i} V) \otimes P(\Omega^{2i} V)^* \cong \bigoplus_{i=0}^{e-1} P(\Omega^{n+2i} S) \otimes P(\Omega^{2i} V)^*.$$

We then get a minimal projective resolution

$$\cdots \to P(\Omega^{n-1} M) \to \cdots \to P(M) \to M \to 0$$

of M as an $A \otimes B^{op}$-module, where M is in the degree 0 and $P(\Omega^{n-1} M)$ is in the degree n. Then from this projective resolution we define $D_T = (D_n, d_n)$ as follows:

$$D_n = \begin{cases} M & n = 0, \\ \displaystyle\bigoplus_{d(\operatorname{top} \Omega^{n-1+2i} S) \leq m-n} P(\Omega^{n-1+2i} S) \otimes P(\Omega^{2i} V)^* & 1 \leq n \leq m-1, \\ 0 & \text{otherwise}, \end{cases}$$

and $d_n : D_n \to D_{n-1}$ is the restriction of the differential of the projective resolution to each left term for each n. Then this complex D_T is a two-sided tilting complex and isomorphic to T in $D^b(A)$.

Example 2.2. We consider the case where A is the Brauer tree algebra in Example 2.1. Let M be the indecomposable $A \otimes B^{op}$-module inducing a stable equivalence of Morita type between A and B induced by T, and let V_i be the simple B-module such that $\operatorname{top}(M \otimes_B V_i) \cong S_i$ for $1 \leq i \leq 5$. We then have a minimal projective resolution of M as an $A \otimes B^{op}$-module as follows.

$$\cdots \to \begin{array}{c} P(S_2) \otimes P(V_1)^* \\ \oplus \\ P(S_1) \otimes P(V_2)^* \\ \oplus \\ P(S_4) \otimes P(V_3)^* \\ \oplus \\ P(S_5) \otimes P(V_4)^* \\ \oplus \\ P(S_3) \otimes P(V_5)^* \end{array} \to \begin{array}{c} P(S_1) \otimes P(V_1)^* \\ \oplus \\ P(S_2) \otimes P(V_2)^* \\ \oplus \\ P(S_3) \otimes P(V_3)^* \\ \oplus \\ P(S_4) \otimes P(V_4)^* \\ \oplus \\ P(S_5) \otimes P(V_5)^* \end{array} \to {}_AM_B$$

By deleting some summands from each term, we have a two-sided tilting complex D_T of $A \otimes B^{op}$-modules isomorphic to the Rickard tree-to-star complex described in Example 2.1 in $D^b(A)$ as follows.

$$\begin{array}{c} \\ \\ P(S_4) \otimes P(V_3)^* \\ \oplus \\ P(S_5) \otimes P(V_4)^* \\ \oplus \\ P(S_3) \otimes P(V_5)^* \end{array} \to \begin{array}{c} P(S_2) \otimes P(V_2)^* \\ \oplus \\ P(S_3) \otimes P(V_3)^* \\ \oplus \\ P(S_4) \otimes P(V_4)^* \\ \oplus \\ P(S_5) \otimes P(V_5)^* \end{array} \to {}_AM_B$$

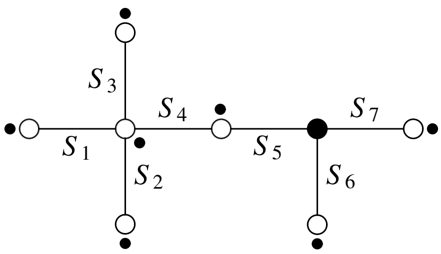
2.3. Pointed Brauer tree. In this section we explain the pointed Brauer tree.

Definition 2.3. Let $\mathcal{T}$ be a Brauer tree. A pointing on $\mathcal{T}$ is a choice of a sector from an edge adjacent to v to the previous one in the cyclic ordering for each non-exceptional vertex v, and we indicate each sector by placing a point there. We denote a sector from an edge i to the previous edge j in the cyclic ordering by (i, j), where if there is only one edge i at the vertex v we denote the pointing by (i, i). A Brauer tree $\mathcal{T}$ with this additional structure is called a pointed Brauer tree.

We define some notation on pointed Brauer trees.

Notation 2.4. Given a pointed Brauer tree $\mathcal{T}$, we define a Brauer tree algebra Γ associated to $\mathcal{T}$ as usual. For each edge of $\mathcal{T}$, we associate the edge to the farther point of the two on vertices adjacent to the edge. Hence we have one-to-one correspondences among all the points, all the edges and all the non-exceptional vertices. For a point $r = (i, j)$ of a pointed Brauer tree, let $U(r) = U(i, j)$ be a uniserial module whose composition factors are given by $S_i, \cdots, S_j$ in the cyclic ordering around the vertex v adjacent to the edges i and j if $i \neq j$. If $i = j$, then $U(r) = S_i$. Any pointed Brauer tree does not have a point on the exceptional vertex, but, for convenience, we define a uniserial module corresponding to a point on the exceptional vertex to be a uniserial module given by the turning around the exceptional vertex from the point μ times, where μ is a multiplicity.

EXAMPLE 2.5. We consider the following pointed Brauer tree.



We assign each point r_i to the number of the closest edge from the exceptional vertex adjacent to the vertex on which the point is. Then each point r_i can be denoted by as follows: $r_1 = (S_1, S_1), r_2 = (S_2, S_2), r_3 = (S_3, S_3), r_4 = (S_4, S_2), r_5 = (S_4, S_5), r_6 = (S_6, S_6), r_7 = (S_7, S_7)$. Then each uniserial module $U(r_i)$ has the following structure.

$U(r_1)$	$U(r_2)$	$U(r_3)$	$U(r_4)$	$U(r_5)$	$U(r_6)$	$U(r_7)$
S_1	S_2	S_3	$\begin{bmatrix} S_4 \\ S_3 \\ S_1 \\ S_2 \end{bmatrix}$	$\begin{bmatrix} S_4 \\ S_5 \end{bmatrix}$	S_6	S_7

REMARK 2.6. It can easily be seen that for the simple A-module S, $\Omega^n S$ can be written as $U(r)$ for some point r on the Brauer tree. Letting r^0 be the point corresponding to the terminal vertex on the edge S and letting $r^0, r^1, \cdots, r^n, \cdots$ be the sequence of points in unpointed Brauer tree encountered by moving along a Green's walk [1] from S, we then have $\Omega^n S \cong U(r^n)$.

By this remark, we have that $\{\Omega^{2i} V | 0 \leq i \leq e - 1\}$ is the set of all simple B-modules.

REMARK 2.7. For an integer n and a uniserial module $\Omega^n S$, if $\operatorname{top} \Omega^n S$ and $\operatorname{soc} \Omega^n S$ are adjacent to the exceptional vertex, then the vertex corresponding to $\Omega^n S$ is the exceptional vertex.

On these uniserial modules, we easily see the following result.

Lemma 2.8. *Let A be a Brauer tree algebra associated to a Brauer tree $\mathcal{T}$. For points (i, j) adjacent to the vertex v and (i', j') adjacent to the vertex v' of a Brauer tree $\mathcal{T}$,* $\operatorname{Hom}_A(U(i, j), U(i', j')) \neq 0$ *if and only if $v = v'$ or $i = j'$. Moreover if $v = v'$ and it is a non-exceptional vertex or if $i = j'$ and $v \neq v'$, then* $\dim \operatorname{Hom}_A(U(i, j), U(i', j')) = 1$ *and* *if $v = v'$ and it is the exceptional vertex, then* $\dim \mathrm{Hom}_A(U(i, j), U(i', j'))$ *is equal to the exceptional multiplicity.*

We associate the two-sided tilting complex D_T to a pointed Brauer tree. We now first bring the following lemma.

Lemma 2.9. [3] *Let D_T be the two-sided tilting complex defined in* [3] *isomorphic to the Rickard tree-to-star complex T in $D^b(A)$. This two-sided tilting complex D_T has the following properties.*

(1) *For the above simple A-module S and for any simple B-module $\Omega^{2i}V$ there exists an integer l_i such that $D_T \otimes_B \Omega^{2i}V \cong \Omega^{2i+l_i}S[l_i]$ in $D^b(A)$.*
(2) *$\Omega^{2i+l_i}S$ is a uniserial module with multiplicity of each composition factor one and a pair of simple modules* $(\mathrm{soc}\,\Omega^{2i+l_i}S, \mathrm{top}\,\Omega^{2i+l_i}S)$ *are in the cyclic ordering at the vertex adjacent to both of* $\mathrm{soc}\,\Omega^{2i+l_i}S$ *and* $\mathrm{top}\,\Omega^{2i+l_i}S$. *Thus* $(\mathrm{top}\,\Omega^{2i+l_i}S, \mathrm{soc}\,\Omega^{2i+l_i}S)$ *is a point of the Brauer tree and* $\Omega^{2i+l_i}S \cong U(\mathrm{top}\,\Omega^{2i+l_i}S, \mathrm{soc}\,\Omega^{2i+l_i}S)$.
(3) *For the uniserial module $\Omega^{2i+l_i}S$, the edge corresponding to* $\mathrm{soc}\,\Omega^{2i+l_i}S$ *is closer to the exceptional vertex than the edge corresponding to any composition factor of* $\Omega^{2i+l_i}S/\mathrm{soc}\,\Omega^{2i+l_i}S$.

Since D_T is a two-sided tilting complex of $A \otimes B^{op}$-modules, for non-isomorphic simple B-modules $\Omega^{2i}V$ and $\Omega^{2j}V$ it holds that $\mathrm{Hom}_{D^b(A)}(D_T \otimes_B \Omega^{2i}V, D_T \otimes_B \Omega^{2j}V[-n]) = 0$ for all $n \geq 0$. Combining this fact for $n = |l_i - l_j|$ with Lemma 2.9 gives

$$\mathrm{Hom}_A(\Omega^{2i+l_i}S, \Omega^{2j+l_j}S) = 0 \quad \text{or} \quad \mathrm{Hom}_A(\Omega^{2j+l_j}S, \Omega^{2i+l_i}S) = 0.$$

Hence by Lemma 2.8 we have the following result.

Proposition 2.10. *There is a one-to-one correspondence between all of the non-exceptional vertices of $\mathcal{T}$ and $\{\Omega^{2i+l_i}S \mid 0 \leq i \leq e-1\}$.*

Notation 2.11. By this proposition and Lemma 2.9 (2), by giving a point $(\mathrm{top}\,\Omega^{2i+l_i}S, \mathrm{soc}\,\Omega^{2i+l_i}S)$ on each vertex of $\mathcal{T}$ we have a pointed Brauer tree. We denote this pointed Brauer tree by $\mathcal{T}_{D_T}$. By Lemma 2.9 (3), $\mathcal{T}_{D_T}$ is a pointed Brauer tree with each point in the sector which the Green's walk from the exceptional vertex meets first.

Example 2.12. We consider the two-sided tilting complex D_T in Example 2.2. We can rewrite this two-sided tilting complex as follows, where $V = V_1$:

$$\begin{array}{ccccc}
 & & P(S_2) \otimes P(\Omega^8 V)^* & & \\
 & & \oplus & & \\
P(S_4) \otimes P(\Omega^2 V)^* & \to & P(S_3) \otimes P(\Omega^2 V)^* & \to & {}_A M_B \\
\oplus & & \oplus & & \\
P(S_5) \otimes P(\Omega^4 V)^* & & P(S_4) \otimes P(\Omega^4 V)^* & & \\
\oplus & & \oplus & & \\
P(S_3) \otimes P(\Omega^6 V)^* & & P(S_5) \otimes P(\Omega^6 V)^* & &
\end{array}$$

For the projective resolution $P_\bullet(M)$ of the $A \otimes B^{op}$-module M, $P_\bullet(M) \otimes_B \Omega^{2i}V$ is a projective resolution $P_\bullet(M \otimes_B \Omega^{2i}V)$ of an A-module $M \otimes_B \Omega^{2i}V \cong \Omega^{2i}(M \otimes_B V) \cong \Omega^{2i}S$. Also,

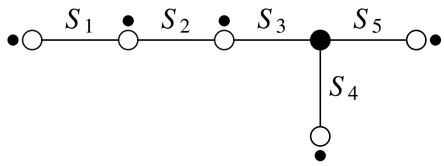
$P(\Omega^{2i}V)^*\otimes_B\Omega^{2j}V \cong \mathrm{Hom}(P(\Omega^{2i}V), \Omega^{2j}V) \cong \delta_{ij}k$ for $0 \le i \le 4$. Hence we have $D_T\otimes_B\Omega^{2i}V \cong \sigma_{\ge l_i}P_\bullet(\Omega^{2i}S_1)$ for some l_i where $\sigma_{\ge l_i}$ means stupid truncation (see [9, Remark 3.5.21]), and we have $\sigma_{\ge l_i}P_\bullet(\Omega^{2i}S_1) \cong \Omega^{2i+l_i}S_1[l_i]$ in $D^b(A)$. Hence we have the following results (we refer the reader to [3, Proposition 5.3] for the details).

i	0	1	2	3	4
$D_T \otimes_B \Omega^{2i}V$	S_1	$\Omega^{2+2}S_1[2]$	$\Omega^{4+2}S_1[2]$	$\Omega^{6+2}S_1[2]$	$\Omega^{8+1}S_1[1]$

Since we have

$$\Omega^4 S_1 \cong S_4,\ \Omega^6 S_1 \cong S_5,\ \Omega^8 S_1 \cong \begin{bmatrix} S_2 \\ S_3 \end{bmatrix} \text{ and } \Omega^9 S_1 \cong \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$$

by easy calculations, we have the following pointed Brauer tree $\mathcal{T}_{D_T}$.



2.4. Rickard-Schaps tree-to-star complexes. In [5] for any Brauer tree algebra A, Rickard constructed a one-sided tilting complex of A-modules with endomorphism ring B. By the complex, we get that Brauer tree algebras with the same number of edges and the same multiplicity are derived equivalent. In that sense, one-sided tilting complexes over Brauer tree algebras with endomorphism rings star-shaped Brauer tree algebras play important roles.

For a Brauer tree algebra such a complex is not uniquely determined in general. Rickard constructed one particular tree-to-star complex for each Brauer tree algebra in [5]. In [7] Rickard and Schaps constructed some tree-to-star complexes by using pointings. In particular for a Brauer tree algebra A, the Rickard tree-to-star complex in [5] is obtained from a pointing on the Brauer tree of A (see Remark 2.14). Moreover for any pointing, a Rickard-Schaps tree-to-star complex of A-modules associated to the pointing is obtained by applying the operation called folding to the Rickard tree-to-star complex several times.

For a pointing on a Brauer tree, the tree-to-star complex is obtained by the following process.

ALGORITHM 2.13. [7, Section 3]

(1) Pick an arbitrary edge at the exceptional vertex as a starting point, and let the exceptional vertex be numbered as 0.
(2) Number all non-exceptional vertices by taking the Green's walk around the tree, assigning a number to each vertex whenever the corresponding point is reached.
(3) Give each edge the same number as the vertex farthest from the exceptional vertex.
(4) We build the complex $T' = \bigoplus T_s$ by recursion on the distance of s from the exceptional vertex. For any s, we let P_s be the projective module corresponding to the edge numbered by s in (2) and let d be the distance of s from the exceptional vertex.
    (a) If the edge numbered s is adjacent to the exceptional vertex, then T_s is the stalk complex with P_s in degree 0.
    (b) If $n(1), \cdots, n(d) = s$ are the numbers assigned to the edges in a minimal path from

the exceptional vertex to the edge s, assuming we know by recursion that $T_{n(d-1)}$ contains one copy of $P_{n(d-1)}$, then we distinguish two cases:

(b.1) $n(d-1) > n(d)$: Let j_{d-1} be the integer such that $P_{n(d-1)}$ is on the j_{d-1}-th degree of $T_{n(d-1)}$. Then we can take a non-zero morphism

$$f_s : P_s[j_{d-1}] \to T_{n(d-1)}$$

uniquely up to scalar multiplication, and we define $T_s = C(f_s)$. We then have $j_d = j_{d-1} + 1$, since taking the cone of f_s will shift the degree of P_s by 1.

(b.2) $n(d-1) < n(d)$: Let j_{d-1} be the integer such that $P_{n(d-1)}$ is on the j_{d-1}-th degree of $T_{n(d-1)}$. Then we can take a non-zero morphism

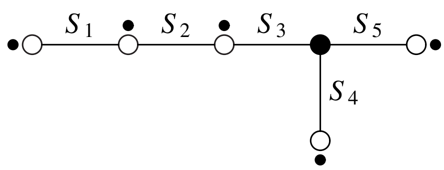
$$g_s : T_{n(d-1)} \to P_s[j_{d-1}]$$

uniquely up to scalar multiplication, and we define $T_r = C(g_s)[-1]$. We then have $j_d = j_{d-1} - 1$, since taking the cone of g_s will not shift the degree of P_s.

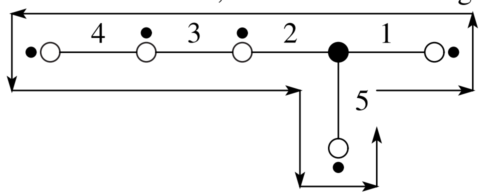
REMARK 2.14. It can easily be seen that the Rickard tree-to-star complex T in [5] is obtained from the pointing such that the sequence in (b) satisfies the condition that $n(j) < n(j+1)$ for any $1 \le j \le d-1$ and any d, that is, $\mathcal{T}_{D_T}$ in Notation 2.11 (see Example 2.16).

REMARK 2.15. We define the Rickard-Schaps tree-to-star complexes in the opposite direction from the original ones in [7], because the cyclic ordering of Brauer tree is defined moving by clockwise in [7], but we define it to be counter-clockwise.

EXAMPLE 2.16. We consider a Brauer tree algebra associated to the following pointed Brauer tree.



By taking the Green's walk around the tree, we number each edge 1 to 5.

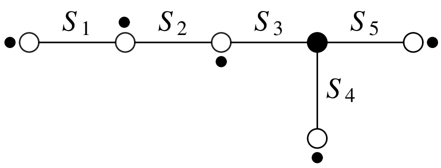


From this numbering, we have the following tree-to-star complex. (Note that each projective module now has two names, either $P(S_i)$ accoding to the numbering of the unpointed Brauer tree, or P_s according to its numbering in the pointed Brauer tree.)

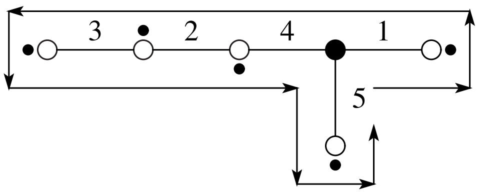
$$
\begin{array}{ccccc}
P(S_3) & \rightarrow & P(S_2) & \rightarrow & P(S_1) \\
 & & \oplus & & \\
P(S_3) & \rightarrow & P(S_2) & & \\
 & & \oplus & & \\
P(S_3) & & & & \\
 & & \oplus & & \\
P(S_4) & & & & \\
 & & \oplus & & \\
P(S_5) & & & &
\end{array}
$$

This complex coincides with the Rickard tree-to-star complex.

EXAMPLE 2.17. We consider a Brauer tree algebra associated to the following pointed Brauer tree.

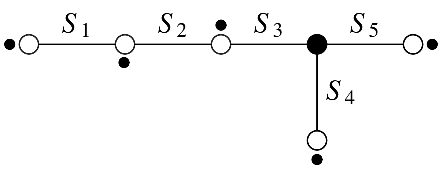


Similarly as Example 2.16, we have the following numbering and tree-to-star complex.

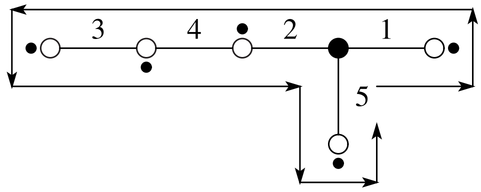


$$
\begin{array}{ccc}
P(S_2) & \rightarrow & P(S_1) \oplus P(S_3) \\
 & \oplus & \\
P(S_2) & \rightarrow & P(S_3) \\
 & \oplus & \\
 & & P(S_3) \\
 & \oplus & \\
 & & P(S_4) \\
 & \oplus & \\
 & & P(S_5)
\end{array}
$$

EXAMPLE 2.18. We consider a Brauer tree algebra associated to the following pointed Brauer tree.



Similarly as Example 2.16, we have the following numbering and tree-to-star complex.

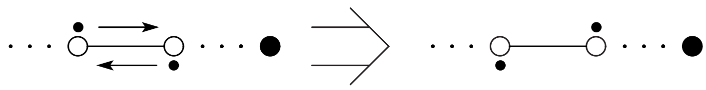


$$\begin{array}{ccc}
P(S_1) \oplus P(S_3) & \to & P(S_2) \\
\oplus & & \\
P(S_3) & \to & P(S_2) \\
\oplus & & \\
P(S_3) & & \\
\oplus & & \\
P(S_4) & & \\
\oplus & & \\
P(S_5) & &
\end{array}$$

REMARK 2.19. We consider how Rickard-Schaps tree-to-star complexes change when given points are moved. We consider two pointed Brauer trees $\mathcal{T}_1$ and $\mathcal{T}_2$.

(i) Suppose $\mathcal{T}_2$ is given by moving two points on an edge U not adjacent to the exceptional vertex of $\mathcal{T}_1$ along the reverse Green's walk as follows.

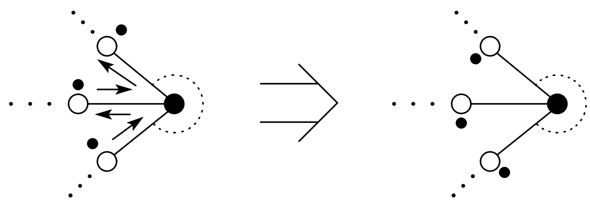
$\mathcal{T}_1$ $\mathcal{T}_2$



Then a Rickard-Schaps tree-to-star complex associated to $\mathcal{T}_2$ is given by a -2 shift of $P(U)$ in a Rickard-Schaps tree-to-star complex associated to $\mathcal{T}_1$.

(ii) Suppose that $\mathcal{T}_1$ is such that for all non-exceptional vertices on the edges adjacent to the exceptional vertex, the points on the vertices are in the first sector in a Green's walk from the exceptional vertex. Moreover suppose $\mathcal{T}_2$ is given by moving all the points on the edges adjacent to the exceptional vertex of $\mathcal{T}_1$ along the reverse Green's walk twice via the exceptional vertex as follows.

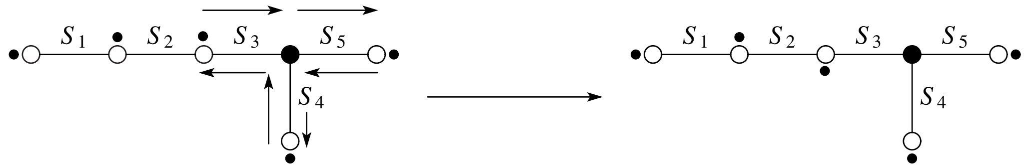
$\mathcal{T}_1$ $\mathcal{T}_2$



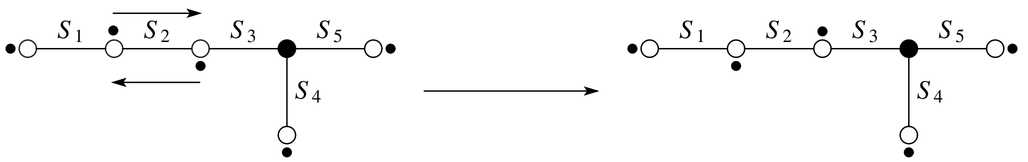
Then a Rickard-Schaps tree-to-star complex associated to $\mathcal{T}_2$ is given by a -2 shift of $\bigoplus_{i=1}^n P(U_i)$ in a Rickard-Schaps tree-to-star complex associated to $\mathcal{T}_1$, where $U_1, \cdots,$ U_n are all edges adjacent to the exceptional vertex.

We call these two operations to the Rickard-Schaps tree-to-star complexes foldings.

EXAMPLE 2.20. The pointing in Example 2.17 is obtained from the one in Example 2.16 by moving all the points on the edges adjacent to the exceptional vertex along the reverse Green's walk.



In this case we can regard the tree-to-star complex in Example 2.17 as a complex obtained by folding all the projective modules associated to the simple modules adjacent to the exceptional vertex in the one in Example 2.16. Moreover the pointing in Example 2.18 is obtained from the one in Example 2.17 by moving the points on the edges S_2 along the reverse Green's walk.



In this case we can regard the tree-to-star complex in Example 2.18 as a complex obtained by folding the projective modules associated to the simple modules S_2 in the one in Example 2.17.

REMARK 2.21. Let $\mathcal{T}$ be a Brauer tree, and $\mathcal{T}_i$ a pointed Brauer tree of $\mathcal{T}$ for $i = 1, 2$. Then we can get $\mathcal{T}_2$ by applying the "moving points operations" in Remark 2.19 to $\mathcal{T}_1$ several times. In particular, we have any Rickard-Schaps tree-to-star complex is obtained by applying the foldings several times to the Rickard tree-to-star complex T.

This remark can be explained as follows:

If $\mathcal{T}$ has 2 edges, then the statement is clear. Suppose the statement holds for pointed Brauer trees of any Brauer tree with $e-1$ edges for $e \geq 3$. Let $\mathcal{T}$ be a Brauer tree with e edges and let an edge numbered by e be on an end of $\mathcal{T}$ and the terminal vertex of the edge e is not the exceptional vertex and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be pointed Brauer trees of $\mathcal{T}$. For the Brauer tree $\mathcal{T}$, let $\mathcal{T}'$ be a Brauer tree obtained by removing the edge e from $\mathcal{T}$, and denote the vertex which $\mathcal{T}'$ and the edge e have in common by v and the cyclic ordering of v in $\mathcal{T}$ by $\cdots, e-1, e, e-2, \cdots$ (if there are only two edges around v, let $e-1$ be equal to $e-2$). Moreover let $\mathcal{T}_i'$ be a pointed Brauer tree of $\mathcal{T}'$ obtained by removing the edge e from the pointed Brauer tree $\mathcal{T}_i$ for $i = 1, 2$. Then by the assumption, we can get $\mathcal{T}_2'$ from $\mathcal{T}_1'$ by applying the operations for points in Remark 2.19 (i) and (ii) several times:

$$\mathcal{T}_1' = \mathcal{T}'^{(1)} \xrightarrow{a_1'} \cdots \xrightarrow{a_s'} \mathcal{T}'^{(s+1)} = \mathcal{T}_2'$$

where a_j' is any of the following for each j: operation in Remark 2.19 (i), operation in Remark 2.19 (ii), not move.

First suppose that v is not the exceptional vertex and the other vertex on the edge $e-2$ is not the exceptional vertex. If a_1' is not the operation in Remark 2.19 (i) with respect to $e-2$, then we can consider a_1' as an operation to $\mathcal{T}_1$, and put $a_1 = a_1'$. If a_1' is the operation in Remark 2.19 (i) with respect to $e-2$ and if there exists no point the sector from $e-1$ to e in $\mathcal{T}_1$, then we can consider a_1' as an operation to $\mathcal{T}_1$ too, and put $a_1 = a_1'$. If a_1' is the operation in Remark 2.19 (i) with respect to $e-2$ and if there exists a point in the sector from $e-1$ to

e in $\mathcal{T}_1$, we can consider a'_1 as an operation to $\mathcal{T}$ after applying the operation in Remark 2.19 (i) with respect to e to $\mathcal{T}$, and denote this by a_1. Similarly we determine $a_2, \cdots, a_s$, and if necessary, applying the operation in Remark 2.19 (i) with respect to e after a_s, we have $\mathcal{T}_2$:

$$\mathcal{T}_1 = \mathcal{T}^{(1)} \xrightarrow{a_1} \cdots \xrightarrow{a_s} \mathcal{T}^{(s+1)} = \mathcal{T}_2.$$

Next suppose that v is not the exceptional vertex and the other vertex on the edge $e-2$ is the exceptional vertex. In this case, we can demonstrate the statement by replacing "the operation in Remark 2.19 (i) with respect to $e-2$" in the above demonstration with "the operation in Remark 2.19 (ii)".

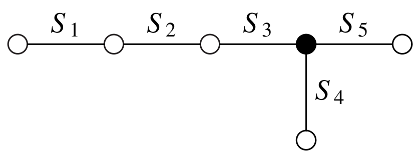
Finally suppose that the vertex v is the exceptional vertex. In this case, we set a_j by the operation in Remark 2.19 (ii) to $\mathcal{T}^{(j)}$ if a'_j the operation in Remark 2.19 (ii) to $\mathcal{T}'^{(j)}$. Otherwise we consider a'_j as an operation to $\mathcal{T}^{(j)}$, and put $a_j = a'_j$. Then we have $\mathcal{T}_2$:

$$\mathcal{T}_1 = \mathcal{T}^{(1)} \xrightarrow{a_1} \cdots \xrightarrow{a_s} \mathcal{T}^{(s+1)} = \mathcal{T}_2.$$

3. Examples and Proof of Theorem 1

In this section we first give examples of Theorem 1 and then give its proof. The statement of Theorem 1 was given in the introduction.

EXAMPLE 3.1. We consider the case where A is a Brauer tree algebra in Example 2.1, that is, A is associated to the following Brauer tree.



By Example 2.2, we have the two-sided tilting complex D_T isomorphic to the Rickard tree-to-star complex T in $D^b(A)$.

$$\begin{array}{ccccccc}
 & & & & P(S_2)\otimes P(V_2)^* & & \\
 & & & & \oplus & & \\
D_T: & & P(S_4)\otimes P(V_3)^* & \to & P(S_3)\otimes P(V_3)^* & \to & {}_AM_B \\
 & & \oplus & & \oplus & & \\
 & & P(S_5)\otimes P(V_4)^* & & P(S_4)\otimes P(V_4)^* & & \\
 & & \oplus & & \oplus & & \\
 & & P(S_3)\otimes P(V_5)^* & & P(S_5)\otimes P(V_5)^* & &
\end{array}$$

Applying the operation in Theorem 1 (ii) to this two-sided tilting complex D_T, we have the following two-sided tilting complex of $A\otimes B^{op}$-modules.

$$C_1: \quad P(S_2)\otimes P(V_2)^* \quad \to \quad {}_AM_B$$

This two-sided tilting complex coincides with the one constructed by Rouquier in [8]. This complex is isomorphic to the following complex in $D^b(A\otimes B^{op})$, where the middle term is the injective hull of M.

$$
\begin{array}{ccccc}
 & & P(S_1)\otimes P(V_1)^* & & \\
 & & \oplus & & \\
P(S_2)\otimes P(V_2)^* & & P(S_3)\otimes P(V_2)^* & & \\
 & & \oplus & & \\
 & \to & P(S_2)\otimes P(V_3)^* & \to & {}_A\Omega^{-1}M_B \\
 & & \oplus & & \\
 & & P(S_4)\otimes P(V_4)^* & & \\
 & & \oplus & & \\
 & & P(S_5)\otimes P(V_5)^* & &
\end{array}
$$

Applying the operation in Theorem 1 (i) to this two-sided tilting complex, we have the following two-sided tilting complex of $A\otimes B^{op}$-modules.

$$
\begin{array}{cccc}
 & P(S_1)\otimes P(V_1)^* & & \\
 & \oplus & & \\
C_2 : & P(S_3)\otimes P(V_2)^* & \to & {}_A\Omega^{-1}M_B \\
 & \oplus & & \\
 & P(S_4)\otimes P(V_4)^* & & \\
 & \oplus & & \\
 & P(S_5)\otimes P(V_5)^* & &
\end{array}
$$

To prove Theorem 1, we prepare the following lemma.

Lemma 3.2. *Let M be an $A\otimes B^{op}$-module inducing a stable equivalence of Morita type between A and B. If a complex obtained in* Theorem 1

$$C = (\cdots \to 0 \to R_{l-1} \to \cdots \to R_1 \to M \to 0 \to \cdots)$$

of $A\otimes B^{op}$-modules satisfies the following conditions for $0 \le i, j \le e-1$ and $n \ge 0$, then C is a two-sided tilting complex inducing equivalences between $D^b(A)$ and $D^b(B)$ and between $K^b(A)$ and $K^b(B)$:

$$\mathrm{Hom}_{D^b(A)}(C\otimes_B \Omega^{2j}V, C\otimes_B \Omega^{2i}V[-n]) \cong \begin{cases} k & i=j \text{ and } n=0,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. Any simple B-module can be denoted by $\Omega^{2i}V$ for $0\le i\le e-1$, and $\Omega^{2i}V \not\cong \Omega^{2j}V$ for $0 \le i \ne j \le e-1$. Hence, to show that C is a two-sided tilting complex of $A\otimes B^{op}$-modules, it suffices to show the complex C satisfies the following three conditions of Proposition 4.4 in [3] for $0\le i,j\le e-1$:

(1) For each $1\le i\le l-1$, R_i is a projective $A\otimes B^{op}$-modules, and M is projective on both sides,
(2) $\mathrm{Hom}_{D^b(B\otimes B^{op})}(C^*\otimes_A C, \Omega^{2i}V\otimes\Omega^{2j}V^*) \cong \delta_{ij}k$, where δ_{ij} is the Kronecker delta,
(3) $\mathrm{Hom}_{D^b(B\otimes B^{op})}(C^*\otimes_A C, \Omega^{2i}V\otimes\Omega^{2j}V^*[-n]) = 0$ for any positive integer n.

Condition 1 is clear. Also, since there is an isomorphism

$$\mathrm{Hom}_{D^b(B\otimes B^{op})}(C^* \otimes_A C, \Omega^{2i}V \otimes \Omega^{2j}V^*[-n]) \cong \mathrm{Hom}_{D^b(A)}(C \otimes_B \Omega^{2j}V^*, C \otimes_B \Omega^{2i}V[-n])$$

for any positive integer $n \geq 0$, we have that C satisfies Condition 2 and Condition 3 by the assumptions. Therefore C is a two-sided tilting complex. □

Remark 3.3. If M is an $A \otimes B^{op}$-module inducing a stable equivalence of Morita type between A and B, then $\Omega^{-n}M$ induces a stable equivalence of Morita type between A and B too by [4, Proposition 2.9]. Hence even if the rightmost non-zero term of C is $\Omega^{-n}M$ for some n, we can use Lemma 3.2 to show that C is a two-sided tilting complex.

We will demonstrate first Theorem 1 (i) and then Theorem 1 (ii). Let D be a complex of $A \otimes B^{op}$-modules given by applying the operations in Theorem 1 to D_T several times, and suppose D is a two-sided tilting complex. First we will show that a complex C given by applying the operation in Theorem 1 (i) to D once is a two-sided tilting complex in Proposition 3.4. Next we will show that a complex C given by applying the operation in Theorem 1 (ii) to D once is a two-sided tilting complex in Proposition 3.6.

Proposition 3.4. *Let D be a complex of $A\otimes B^{op}$-modules given by applying the operations in* Theorem 1 *to D_T several times, and suppose D is a two-sided tilting complex. Then a complex C given by applying the operation in* Theorem 1 (i) *to D once is a two-sided tilting complex*

Proof. Let D be a complex obtained by applying the operations in Theorem 1 several times to D_T, and suppose D is a two-sided tilting complex. Let C be a complex obtained by applying the operation in Theorem 1 (i) to D once, and we show that C is a two-sided tilting complex.

By Lemma 3.2 we shall show that for $0 \leq i, j \leq e-1$ and $n \geq 0$ the complex C satisfies the following conditions:

$$\mathrm{Hom}_{D^b(A)}(C \otimes_B \Omega^{2i}V, C \otimes_B \Omega^{2j}V[-n]) = \begin{cases} k & \text{if } i = j \text{ and } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

By the construction of C, for each $0 \leq i \leq e-1$ we have $C \otimes_B \Omega^{2i}V \cong \Omega^{2i+l_i}S[l_i]$ in $D^b(A)$ for some integer l_i. Using this notation, we have the following isomorphisms:

$$\begin{aligned} \mathrm{Hom}_{D^b(A)}(C \otimes_B \Omega^{2i}V, C \otimes_B \Omega^{2j}V[-n]) &\cong \mathrm{Hom}_{D^b(A)}(\Omega^{2i+l_i}S[l_i], \Omega^{2j+l_j}S[l_j - n]) \\ &\cong \mathrm{Hom}_{D^b(A)}(\Omega^{2i+l_i}S, \Omega^{2j+l_j}S[l_j - l_i - n]). \end{aligned}$$

Hence we show

$$\mathrm{Hom}_{D^b(A)}(\Omega^{2i+l_i}S, \Omega^{2j+l_j}S[l_j - l_i - n]) = \begin{cases} k & \text{if } i = j \text{ and } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

We divide into 3 cases: $l_j - l_i - n$ is positive, negative and zero.

Case 1. $l_j - l_i - n < 0$.

If $i = j$, then we have $n \neq 0$ by the assumption. Hence it suffices to show

$$\mathrm{Hom}_{D^b(A)}(\Omega^{2i+l_i}S, \Omega^{2j+l_j}S[l_j - l_i - n]) = 0.$$

However since $l_j - l_i - n < 0$, we have

$$\mathrm{Hom}_{D^b(A)}(\Omega^{2i+l_i}S, \Omega^{2j+l_j}S[l_j - l_i - n]) \cong \mathrm{Ext}_A^{l_j-l_i-n}(\Omega^{2i+l_i}S, \Omega^{2j+l_j}S) = 0.$$

CASE 2. $l_j - l_i - n = 0$.
Under the assumption $l_j - l_i - n = 0$, we have

$$\mathrm{Hom}_{D^b(A)}(\Omega^{2i+l_i}S, \Omega^{2j+l_j}S[l_j - l_i - n]) \cong \mathrm{Hom}_A(\Omega^{2i+l_i}S, \Omega^{2j+l_j}S).$$

We shall show under the assumption $l_j - l_i - n = 0$,

$$\mathrm{Hom}_A(\Omega^{2i+l_i}S, \Omega^{2j+l_j}S) = \begin{cases} k & \text{if } i = j \text{ and } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Now we remark that if $C\otimes_B\Omega^{2i}V \not\cong D\otimes_B\Omega^{2i}V$, then we have $D\otimes_B\Omega^{2i}V \cong \Omega^{2i+l_i+1}S[l_i+1]$ or $D\otimes_B\Omega^{2i}V \cong \Omega^{2i+l_i+2}S[l_i+2]$ by the construction of C. If $D\otimes_B\Omega^{2i}V \cong \Omega^{2i+l_i+1}S[l_i+1]$, then there is another integer $0 \le i' \le e-1$ different from i such that $D\otimes_B\Omega^{2i'}V \cong \Omega^{2i'+l_{i'}+1}S[l_{i'}+1]$ uniquely. If $D \otimes_B \Omega^{2i}V \cong \Omega^{2i+l_i+2}S[l_i + 2]$, then for any $0 \le j \le e - 1$ different from i we have $D \otimes_B \Omega^{2j}V \cong C \otimes_B \Omega^{2j}V$.

Suppose $D \otimes_B \Omega^{2i}V \cong \Omega^{2i+l_i+2}S[l_i + 2]$. Then by [3, Proposition 5.3] we know that $D \otimes_B \Omega^{2i}V$ is of the form

$$\cdots \to 0 \to P(U) \to P(U) \to \cdots \to \Omega^{2i}S \to 0 \to \cdots,$$

where the leftmost nonzero term $P(U)$ is in the degree l_i+2, that is, $D\otimes_B\Omega^{2i}V$ is isomorphic to the stupidly truncated complex $\sigma_{\ge l_i+2}P_\bullet(\Omega^{2i}S)$ of $P_\bullet(\Omega^{2i}S)$ at $l_i + 2$. Moreover we know that $D \otimes_B \Omega^{2j}V \cong C \otimes_B \Omega^{2j}V$ for any $j \in \{0, 1, \cdots, e-1\} - \{i\}$. Since $\mathrm{top}\,\Omega^{2i+l_i+1}S \cong U \cong \mathrm{soc}\,\Omega^{2i+l_i+1}S$ and U is not adjacent to the exceptional vertex, the vertex corresponding to $\Omega^{2i+l_i+1}S$ is on an end of the Brauer tree. Hence the vertex corresponding to $\Omega^{2i+l_i}S$ coincides with the one to $\Omega^{2i+l_i+2}S$ by Remark 2.6. Since $P(U)$ is a projective cover of $\Omega^{2i+l_i}S$, we have $\mathrm{top}\,\Omega^{2i+l_i}S \cong U$. Now suppose $\mathrm{Hom}_A(\Omega^{2i+l_i}S, \Omega^{2j+l_j}S) \neq 0$ for some $j \neq i$. By the construction of C, we have $\mathrm{soc}\,\Omega^{2j+l_j}S \not\cong U \cong \mathrm{top}\,\Omega^{2i+l_i}S$ because injective hulls of U in the degree $l_i + 2$ and in $l_i + 1$ are the deleted terms and $D \otimes_B \Omega^{2j}V \cong C \otimes_B \Omega^{2j}V$. Hence by Lemma 2.8 the vertex corresponding to $\Omega^{2i+l_i}S$ coincides with the one to $\Omega^{2j+l_j}S$. Therefore we have the vertex corresponding to $\Omega^{2i+l_i+2}S$, isomorphic to $D \otimes_B \Omega^{2i}V[-l_i - 2]$, coincides with the one to $\Omega^{2j+l_j}S$, isomorphic to $D \otimes_B \Omega^{2j}V[-l_j]$, which implies that

$$\mathrm{Hom}_{D^b(A)}(D \otimes_B \Omega^{2i}V, D \otimes_B \Omega^{2j}V[-n']) \neq 0 \;\text{ or }\; \mathrm{Hom}_{D^b(A)}(D \otimes_B \Omega^{2j}V, D \otimes_B \Omega^{2i}V[-n']) \neq 0$$

for $n' = |l_j - l_i - 2|$. Since $i \neq j$, this contradicts to the fact that D is a two-sided tilting complex. Therefore we have $\mathrm{Hom}_A(\Omega^{2i+l_i}S, \Omega^{2j+l_j}S) = 0$. Moreover we have $\mathrm{Hom}_A(\Omega^{2j+l_j}S, \Omega^{2i+l_i}S) = 0$ similarly. Also since the vertex corresponding to $\Omega^{2i+l_i}S$ coincides with the one to $\Omega^{2i+l_i+2}S$ which is non-exceptional vertex, by Lemma 2.6 we have $\mathrm{Hom}_A(\Omega^{2i+l_i}S, \Omega^{2i+l_i}S) = k$.

Next we assume that if $C \otimes_B \Omega^{2i}V \not\cong D \otimes_B \Omega^{2i}V$ then $D \otimes_B \Omega^{2i}V \cong \Omega^{2i+l_i+1}S[l_i + 1]$. Then there is another $j \neq i$ such that if $C\otimes_B\Omega^{2j}V \not\cong D\otimes_B\Omega^{2j}V$ then $D\otimes_B\Omega^{2j}V \cong \Omega^{2j+l_j+1}S[l_j+1]$. We divide into 4 cases:

Case 2.1. $C \otimes_B \Omega^{2i} V \cong D \otimes_B \Omega^{2i} V$ and $C \otimes_B \Omega^{2j} V \cong D \otimes_B \Omega^{2j} V$.

In this case, since D is a tilting complex, it is clear that the complex C has the required properties.

Case 2.2. $C \otimes_B \Omega^{2i} V \ncong D \otimes_B \Omega^{2i} V$ and $C \otimes_B \Omega^{2j} V \ncong D \otimes_B \Omega^{2j} V$.

We remark that the assumption implies that $D \otimes_B \Omega^{2i} V \cong \Omega^{2i+l_i+1} S\,[l_i + 1]$, $D \otimes_B \Omega^{2j} V \cong \Omega^{2j+l_j+1} S\,[l_j + 1]$ and that $\operatorname{soc} \Omega^{2i+l_i+1} S \cong U \cong \operatorname{soc} \Omega^{2j+l_j+1} S$.

First, suppose $i = j$. To show $\mathrm{Hom}_A(\Omega^{2i+l_i} S, \Omega^{2i+l_i} S) = k$, we only need to show that a point corresponding to the uniserial module $\Omega^{2i+l_i} S$ is not on the exceptional vertex by Lemma 2.8. By the assumption the socle of the uniserial module $\Omega^{2i+l_i+1} S$ isomorphic to U is not adjacent to the exceptional vertex. Since there is an isomorphism

$$\operatorname{soc} \Omega^{2i+l_i+1} S \cong \operatorname{top} \Omega^{2i+l_i} S,$$

the top of the uniserial module $\Omega^{2i+l_i} S$ is not adjacent to the exceptional vertex. In particular the point corresponding to the uniserial module $\Omega^{2i+l_i} S$ is not on the exceptional vertex.

Next suppose $i \neq j$. By the construction of C, we have $l_i = l_j - 1$ or $l_j = l_i - 1$, but by the assumption that $l_j - l_i - n = 0$ and that $n \geq 0$, the case it may occur is only the case $l_i = l_j - 1$ and $n = 1$. Now by the construction of C, we have

$$\operatorname{top} \Omega^{2i+l_i} S \cong \operatorname{soc} \Omega^{2i+l_i+1} S \cong U \cong \operatorname{soc} \Omega^{2j+l_j+1} S \cong \operatorname{top} \Omega^{2j+l_j} S.$$

Hence we can denote the points corresponding to $\Omega^{2i+l_i} S$ and $\Omega^{2j+l_j} S$ by (U, U_i) and (U, U_j) respectively. If the vertices on which these points are coincide, then these points coincide. Hence the vertex corresponding to $\Omega^{2i+l_i+1} S$ and the one corresponding to $\Omega^{2j+l_j+1} S$ coincide too, which implies that

$$\mathrm{Hom}_{D^b(A)}(D \otimes_B \Omega^{2i} V, D \otimes_B \Omega^{2j} V[-n]) \cong \mathrm{Hom}_A(\Omega^{2i+l_i+1} S, \Omega^{2j+l_j+1} S) \neq 0$$

for $i \neq j$. Since D is a two-sided tilting complex, this is a contradiction. Hence we only need to show that $U \neq U_j$ to show that $\mathrm{Hom}_A(\Omega^{2i+l_i} S, \Omega^{2j+l_j} S) = 0$ by Lemma 2.8. If $U = U_j$, then $\Omega^{2j+l_j} S = U$ is a simple module. Also the injective hull of $\Omega^{2j+l_j} S$ is on the degree $l_j = l_i + 1$ of $\mathrm{Res}_A^{A \otimes B^{op}} C$. Since by the simplicity of $\Omega^{2j+l_j} S$ we have $I(\Omega^{2j+l_j} S) \cong P(\Omega^{2j+l_j} S) \cong P(U)$, we have $P(U)$ is on the degree l_j of $\mathrm{Res}_A^{A \otimes B^{op}} C$. This is a contradiction since we construct a complex C by deleting $P(U) \otimes X'$ in degree $l_i + 1 = l_j$ and deleting $P(U) \otimes X$ in degree $l_j + 1$ from D.

Case 2.3. $C \otimes_B \Omega^{2i} V \cong D \otimes_B \Omega^{2i} V$ and $C \otimes_B \Omega^{2j} V \ncong D \otimes_B \Omega^{2j} V$.

We remark that the assumption implies that $D \otimes_B \Omega^{2i} V \cong \Omega^{2i+l_i} S\,[l_i]$, $D \otimes_B \Omega^{2j} V \cong \Omega^{2j+l_j+1} S\,[l_j + 1]$ and that $\operatorname{soc} \Omega^{2j+l_j+1} S \cong U$. Also, we remark that the assumption implies that $i \neq j$. Hence we show that $\mathrm{Hom}_A(\Omega^{2i+l_i} S, \Omega^{2j+l_j} S) = 0$. Since D is a tilting complex and since $l_j - l_i - n = 0$, for $i \neq j$ we have

$$
\begin{aligned}
\underline{\mathrm{Hom}}_A(\Omega^{2i+l_i}S, \Omega^{2j+l_j}S) &\cong \mathrm{Ext}^1_A(\Omega^{2i+l_i}S, \Omega^{2j+l_j+1}S) \\
&\cong \mathrm{Hom}_{D^b(A)}(\Omega^{2i+l_i}S, \Omega^{2j+l_j+1}S[1]) \\
&\cong \mathrm{Hom}_{D^b(A)}(\Omega^{2i+l_i}S, \Omega^{2j+l_j+1}S[l_j - l_i - n + 1]) \\
&\cong \mathrm{Hom}_{D^b(A)}(\Omega^{2i+l_i}S[l_i], \Omega^{2j+l_j+1}S[l_j - n + 1]) \\
&\cong \mathrm{Hom}_{D^b(A)}(D \otimes_B \Omega^{2i}V, D \otimes_B \Omega^{2j}V[-n]) \\
&= 0.
\end{aligned}
$$

Hence, if $\mathrm{Hom}_A(\Omega^{2i+l_i}S, \Omega^{2j+l_j}S) \neq 0$, then there is a non-zero homomorphism in

$$\mathrm{Hom}_A(\Omega^{2i+l_i}S, P(\mathrm{top}\,\Omega^{2j+l_j}S)) \cong \mathrm{Hom}_A(\Omega^{2i+l_i}S, P(\mathrm{soc}\,\Omega^{2j+l_j+1}S)).$$

In particular, the uniserial module $\Omega^{2i+l_i}S$ includes $\mathrm{soc}\,\Omega^{2j+l_j+1}S \cong U$ as a composition factor. Hence the vertex corresponding to $\Omega^{2i+l_i}S$ is adjacent to U. Moreover by the construction of C, there exists an integer t different from i and j satisfying that $D \otimes_B \Omega^{2t}V \cong \Omega^{2t+l'_t}S[l'_t]$ with $\mathrm{soc}\,\Omega^{2t+l'_t}S \cong U \cong \mathrm{soc}\,\Omega^{2j+l_j+1}S$ for some integer l'_t. Hence the uniserial modules $\Omega^{2i+l_i}S$, $\Omega^{2j+l_j+1}S$ and $\Omega^{2t+l'_t}S$, which are isomorphic to $D \otimes_B \Omega^{2i}V[-l_i]$, $D \otimes_B \Omega^{2j}V[-l_j - 1]$ and $D \otimes_B \Omega^{2t}V[-l'_t]$, correspond to the vertices adjacent to U respectively. Since the number of the vertices adjacent to U is just two, at least two of these uniserial modules $\Omega^{2i+l_i}S$, $\Omega^{2j+l_j+1}S$ and $\Omega^{2t+l'_t}S$ have a common corresponding vertex, which implies that there is a non-zero homomorphism between these two modules. Hence we have

$$\mathrm{Hom}_{D^b(A)}(D \otimes_B \Omega^{2x_1}V, D \otimes_B \Omega^{2x_2}V[-n']) \neq 0$$

for some $x_1, x_2 \in \{i, j, t\}$ with $x_1 \neq x_2$ and some $n' \geq 0$, but this contradicts to the fact that D is a two-sided tilting complex.

CASE 2.4. $C \otimes_B \Omega^{2i}V \not\cong D \otimes_B \Omega^{2i}V$ and $C \otimes_B \Omega^{2j}V \cong D \otimes_B \Omega^{2j}V$.

We remark that the assumption implies that $D \otimes_B \Omega^{2i}V \cong \Omega^{2i+l_i+1}S[l_i + 1]$, $D \otimes_B \Omega^{2j}V \cong \Omega^{2j+l_j}S[l_j]$ and that $\mathrm{soc}\,\Omega^{2i+l_i+1}S \cong U$. Assume $\mathrm{Hom}_A(\Omega^{2i+l_i}S, \Omega^{2j+l_j}S) \neq 0$. Then $\Omega^{2j+l_j}S$ has $\mathrm{top}\,\Omega^{2i+l_i}S \cong \mathrm{soc}\,\Omega^{2i+l_i+1}S \cong U$ as a composition factor. Also, by the construction of C, there exists an integer t different from i and j satisfying that $D \otimes_B \Omega^{2t}V \cong \Omega^{2t+l'_t}S[l'_t]$ with $\mathrm{soc}\,\Omega^{2t+l'_t}S \cong U$ for some integer l'_t. Since all of the three uniserial modules $\Omega^{2i+l_i+1}S$, $\Omega^{2j+l_j}S$ and $\Omega^{2t+l'_t}S$, isomorphic to $D \otimes_B \Omega^{2i}V[-l_i - 1]$, $D \otimes_B \Omega^{2j}V[-l_j]$ and $D \otimes_B \Omega^{2t}V[-l'_t]$ respectively, have U as composition factors, all the vertices corresponding to these uniserial modules are adjacent to the edge U. In particular, at least two of these uniserial modules must have a common corresponding vertex. Hence we have

$$\mathrm{Hom}_{D^b(A)}(D \otimes_B \Omega^{2x_1}V, D \otimes_B \Omega^{2x_2}V[-n']) \neq 0$$

for some $x_1, x_2 \in \{i, j, t\}$ with $x_1 \neq x_2$ and some $n' \geq 0$, but this contradicts to the fact that D is a two-sided tilting complex.

CASE 3. $l_j - l_i - n > 0$.

In this case, we have

$$\begin{aligned}\mathrm{Hom}_{D^b(A)}(\Omega^{2i+l_i}S, \Omega^{2j+l_j}S[l_j-l_i-n]) &\cong \mathrm{Ext}_A^{l_j-l_i-n}(\Omega^{2i+l_i}S, \Omega^{2j+l_j}S)\\ &\cong \underline{\mathrm{Hom}}_A(\Omega^{2i+l_i}S, \Omega^{2j+l_j-(l_j-l_i-n)}S)\\ &\cong \underline{\mathrm{Hom}}_A(S, \Omega^{2(j-i)+n}S).\end{aligned}$$

Hence we show that

$$\underline{\mathrm{Hom}}_A(S, \Omega^{2(j-i)+n}S) = \begin{cases} k & \text{if } i=j \text{ and } n=0,\\ 0 & \text{otherwise.}\end{cases}$$

If $C\otimes_B \Omega^{2i}V \not\cong D\otimes_B \Omega^{2i}V$, then we have $D\otimes_B \Omega^{2i}V \cong \Omega^{2i+l_i+1}S[l_i+1]$ or $D\otimes_B \Omega^{2i}V \cong \Omega^{2i+l_i+2}S[l_i+2]$, but we only show the statement under the first situation, the other being similar as in Case 2.

Case 3.1. $C\otimes_B \Omega^{2i}V \cong D\otimes_B \Omega^{2i}V$ and $C\otimes_B \Omega^{2j}V \cong D\otimes_B \Omega^{2j}V$.

In this case, since D is a two-sided tilting complex, it is clear that the complex C has the required properties.

Case 3.2. $C\otimes_B \Omega^{2i}V \not\cong D\otimes_B \Omega^{2i}V$ and $C\otimes_B \Omega^{2j}V \not\cong D\otimes_B \Omega^{2j}V$.

By the assumption, we have

$$D\otimes_B \Omega^{2i}V \cong \Omega^{2i+l_i+1}S[l_i+1] \text{ and } D\otimes_B \Omega^{2j}V \cong \Omega^{2j+l_j+1}S[l_j+1].$$

Hence we have

$$\begin{aligned}\underline{\mathrm{Hom}}_A(S, \Omega^{2(j-i)+n}S) &\cong \underline{\mathrm{Hom}}_A(\Omega^{2i+l_i+1}S, \Omega^{2j+l_j+1-(l_j-l_i-n)}S)\\ &\cong \mathrm{Ext}_A^{l_j-l_i-n}(\Omega^{2i+l_i+1}S, \Omega^{2j+l_j+1}S)\\ &\cong \mathrm{Hom}_{D^b(A)}(\Omega^{2i+l_i+1}S, \Omega^{2j+l_j+1}S[l_j-l_i-n])\\ &\cong \mathrm{Hom}_{D^b(A)}(\Omega^{2i+l_i+1}S[l_i+1], \Omega^{2j+l_j+1}S[l_j+1-n])\\ &\cong \mathrm{Hom}_{D^b(A)}(D\otimes_B \Omega^{2i}V, D\otimes_B \Omega^{2j}V[-n])\\ &= \begin{cases} k & \text{if } i=j \text{ and } n=0,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Case 3.3. $C\otimes_B \Omega^{2i}V \cong D\otimes_B \Omega^{2i}V$ and $C\otimes_B \Omega^{2j}V \not\cong D\otimes_B \Omega^{2j}V$.

We have $i \neq j$ and

$$D\otimes_B \Omega^{2i}V \cong \Omega^{2i+l_i}S[l_i] \text{ and } D\otimes_B \Omega^{2j}V \cong \Omega^{2j+l_j+1}S[l_j+1].$$

Hence since D is a two-sided tilting complex we have

$$\begin{aligned}\underline{\mathrm{Hom}}_A(S, \Omega^{2(j-i)+n}S) &\cong \underline{\mathrm{Hom}}_A(\Omega^{2i+l_i}S, \Omega^{2j+l_j+1-(l_j-l_i-n+1)}S)\\ &\cong \mathrm{Ext}_A^{l_j-l_i-n+1}(\Omega^{2i+l_i}S, \Omega^{2j+l_j+1}S)\\ &\cong \mathrm{Hom}_{D^b(A)}(\Omega^{2i+l_i}S, \Omega^{2j+l_j+1}S[l_j-l_i-n+1])\\ &\cong \mathrm{Hom}_{D^b(A)}(\Omega^{2i+l_i}S[l_i], \Omega^{2j+l_j+1}S[l_j+1-n])\\ &\cong \mathrm{Hom}_{D^b(A)}(D\otimes_B \Omega^{2i}V, D\otimes_B \Omega^{2j}V[-n])\\ &= 0.\end{aligned}$$

Case 3.4. $C \otimes_B \Omega^{2i} V \ncong D \otimes_B \Omega^{2i} V$ and $C \otimes_B \Omega^{2j} V \cong D \otimes_B \Omega^{2j} V$.

We have $i \neq j$ and

$$D \otimes_B \Omega^{2i} V \cong \Omega^{2i+l_i+1} S\,[l_i + 1] \text{ and } D \otimes_B \Omega^{2j} V \cong \Omega^{2j+l_j} S\,[l_j].$$

First, suppose $l_j - l_i - n > 1$. Then since D is a two-sided tilting complex and since $l_j - l_i - n - 1 > 0$, we have

$$\begin{aligned}
\underline{\mathrm{Hom}}_A(S, \Omega^{2(j-i)+n} S) &\cong \underline{\mathrm{Hom}}_A(\Omega^{2i+l_i+1} S, \Omega^{2j+l_j-(l_j-l_i-n-1)} S) \\
&\cong \mathrm{Ext}_A^{l_j-l_i-n-1}(\Omega^{2i+l_i+1} S, \Omega^{2j+l_j} S) \\
&\cong \mathrm{Hom}_{D^b(A)}(\Omega^{2i+l_i+1} S, \Omega^{2j+l_j} S\,[l_j - l_i - n - 1]) \\
&\cong \mathrm{Hom}_{D^b(A)}(\Omega^{2i+l_i+1} S\,[l_i + 1], \Omega^{2j+l_j} S\,[l_j - n]) \\
&\cong \mathrm{Hom}_{D^b(A)}(D \otimes_B \Omega^{2i} V, D \otimes_B \Omega^{2j} V[-n]) \\
&= 0.
\end{aligned}$$

Next, suppose $l_j - l_i - n = 1$. Then we have

$$\begin{aligned}
\mathrm{Hom}_A(\Omega^{2i+l_i+1} S, \Omega^{2j+l_j} S) &= \mathrm{Hom}_{D^b(A)}(\Omega^{2i+l_i+1} S, \Omega^{2j+l_j} S\,[l_j - l_i - n - 1]) \\
&\cong \mathrm{Hom}_{D^b(A)}(\Omega^{2i+l_i+1} S\,[l_i + 1], \Omega^{2j+l_j} S\,[l_j - n]) \\
&\cong \mathrm{Hom}_{D^b(A)}(D \otimes_B \Omega^{2i} V, D \otimes_B \Omega^{2j} V[-n]) \\
&= 0.
\end{aligned}$$

Hence we have

$$\begin{aligned}
\underline{\mathrm{Hom}}_A(S, \Omega^{2(j-i)+n} S) &= \underline{\mathrm{Hom}}_A(S, \Omega^{2(j-i)+l_j-l_i-1} S) \\
&\cong \underline{\mathrm{Hom}}_A(\Omega^{2i+l_i+1} S, \Omega^{2j+l_j} S) \\
&= 0.
\end{aligned}$$

□

We prepare the following lemma before we demonstrate Proposition 3.6.

Lemma 3.5. *Let D be a complex obtained by applying the operations in* Theorem 1 (i) *and* Theorem 1 (ii) *several times and let l'_i be the integer satisfying that $D \otimes_B \Omega^{2i} V \cong \Omega^{2i+l'_i} S\,[l'_i]$ and let $U_1, U_2, \cdots, U_n$ be all simple modules adjacent to the exceptional vertex. If $\mathrm{soc}\,\Omega^{2i+l'_i} S \in \{U_1, U_2, \cdots, U_n\}$ and D is a two-sided tilting complex, then we get the following.*

(1) *The vertex corresponding to $\Omega^{2i+l'_i-1} S$ is the exceptional vertex. In particular, $\mathrm{soc}\,\Omega^{2i+l'_i-1} S \in \{U_1, U_2, \cdots, U_n\}$.*

(2) *The vertex corresponding to $\Omega^{2i+l'_i-2} S$ is a non-exceptional vertex.*

(3) *Suppose $i \neq j$ and $\mathrm{soc}\,\Omega^{2i+l'_i} S$, $\mathrm{soc}\,\Omega^{2j+l'_j} S \in \{U_1, U_2, \cdots, U_n\}$. If $\mathrm{soc}\,\Omega^{2i+l'_i} S \ncong \mathrm{soc}\,\Omega^{2j+l'_j} S$, then $\mathrm{soc}\,\Omega^{2i+l'_i-1} S \ncong \mathrm{soc}\,\Omega^{2j+l'_j-1} S$.*

Proof. (1) Suppose $\mathrm{soc}\,\Omega^{2i+l'_i} S$ is adjacent to the exceptional vertex. Then we have $\mathrm{top}\,\Omega^{2i+l'_i-1} S$ is adjacent to the exceptional vertex too since $I(\Omega^{2i+l'_i} S) \cong P(\Omega^{2i+l'_i-1} S)$. Also we have

$$\begin{aligned}\mathrm{End}_A(\Omega^{2i+l'_i}S) &\cong \mathrm{End}_{D^b(A)}(\Omega^{2i+l'_i}S[l'_i]) \\ &\cong \mathrm{End}_{D^b(A)}(D\otimes_B \Omega^{2i}V) \\ &\cong \mathrm{End}_{D^b(B)}(\Omega^{2i}V) \\ &\cong \mathrm{End}_B(\Omega^{2i}V)\end{aligned}$$

since D is a two-sided tilting complex. By simplicity of $\Omega^{2i}V$, we have $\dim \mathrm{End}_B(\Omega^{2i}V) = 1$. If $\mathrm{soc}\,\Omega^{2i+l'_i-1}S$ is not adjacent to the exceptional vertex, then the vertex corresponding to the uniserial module $\Omega^{2i+l'_i}S$ is the exceptional vertex since $\mathrm{top}\,\Omega^{2i+l'_i-1}S \cong \mathrm{soc}\,\Omega^{2i+l'_i}S$ is adjacent to the exceptional vertex. This contradicts the fact that D is a two-sided tilting complex. Hence $\mathrm{soc}\,\Omega^{2i+l'_i-1}S$ is adjacent to the exceptional vertex. Since $\Omega^{2i+l'_i-1}S$ has the adjacent top and socle to the exceptional vertex, the vertex corresponding to the uniserial module $\Omega^{2i+l'_i-1}S$ is the exceptional vertex.

(2) Since the point corresponding to $\Omega^{2i+l'_i-1}S$ is the exceptional vertex by (1), we have the vertex corresponding to $\Omega^{2i+l'_i-2}S$ is a non-exceptional vertex by Remark 2.7.

(3) Suppose $\mathrm{soc}\,\Omega^{2i+l'_i-1}S \cong \mathrm{soc}\,\Omega^{2j+l'_j-1}S$. Since $\mathrm{soc}\,\Omega^{2i+l'_i}S$ is adjacent to the exceptional vertex, by (1), $\mathrm{soc}\,\Omega^{2i+l'_i-1}S \cong \mathrm{soc}\,\Omega^{2j+l'_j-1}S$ is adjacent to the exceptional vertex. Also, since $\mathrm{top}\,\Omega^{2i+l'_i-1}S \cong \mathrm{soc}\,\Omega^{2i+l'_i}S$ and $\mathrm{top}\,\Omega^{2j+l'_j-1}S \cong \mathrm{soc}\,\Omega^{2j+l'_j}S$, $\mathrm{top}\,\Omega^{2i+l'_i-1}S$ and $\mathrm{top}\,\Omega^{2j+l'_j-1}S$ are adjacent to the exceptional vertex. Hence the vertex corresponding to $\Omega^{2i+l'_i-1}S$ and the one corresponding to $\Omega^{2j+l'_j-1}S$ are the exceptional vertex by Remark 2.7, which implies that $\Omega^{2i+l'_i-1}S \cong \Omega^{2j+l'_j-1}S$ since $\mathrm{soc}\,\Omega^{2i+l'_i-1}S \cong \mathrm{soc}\,\Omega^{2j+l'_j-1}S$. Therefore we have

$$\mathrm{soc}\,\Omega^{2i+l'_i}S \cong \mathrm{top}\,\Omega^{2i+l'_i-1}S \cong \mathrm{top}\,\Omega^{2j+l'_j-1}S \cong \mathrm{soc}\,\Omega^{2j+l'_j}S.$$

□

Proposition 3.6. *Let D be a complex of $A\otimes B^{op}$-modules given by applying the operations in* Theorem 1 *to D_T several times, and suppose D is a two-sided tilting complex. Then a complex C given by applying the operation in* Theorem 1 (ii) *to D once is a two-sided tilting complex.*

Proof. Let D be a complex obtained by applying the operations in Theorem 1 several times to D_T, and suppose D is a two-sided tilting complex. Let C be a complex obtained by applying the operation in Theorem 1 (ii) to D once, and we show that C is a two-sided tilting complex. In the same way as the proof of Theorem 1 (i), we have $C\otimes_B \Omega^{2i}V \cong \Omega^{2i+l_i}S[l_i]$ for some integer l_i for each $0 \le i \le e-1$ and we show

$$\mathrm{Hom}_{D^b(A)}(\Omega^{2i+l_i}S, \Omega^{2j+l_j}S[l_j-l_i-n]) = \begin{cases} k & \text{if } i=j \text{ and } n=0, \\ 0 & \text{otherwise.}\end{cases}$$

We divide into 3 cases: l_j-l_i-n is positive, negative and zero.

Case 1. $l_j-l_i-n<0$.

We can show $\mathrm{Hom}_{D^b(A)}(\Omega^{2i+l_i}S, \Omega^{2j+l_j}S[l_j-l_i-n])$ has required properties as is the case with the proof of Theorem 1 (i).

Case 2. $l_j-l_i-n=0$.

We shall show under the assumption $l_j-l_i-n=0$,

$$\operatorname{Hom}_A(\Omega^{2i+l_i}S, \Omega^{2j+l_j}S) = \begin{cases} k & \text{if } i = j \text{ and } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

CASE 2.1. $C \otimes_B \Omega^{2i} V \cong D \otimes_B \Omega^{2i} V$ and $C \otimes_B \Omega^{2j} V \cong D \otimes_B \Omega^{2j} V$.

In this case, since D is a two-sided tilting complex, it is clear that the complex C has the required properties.

CASE 2.2. $C \otimes_B \Omega^{2i} V \not\cong D \otimes_B \Omega^{2i} V$ and $C \otimes_B \Omega^{2j} V \not\cong D \otimes_B \Omega^{2j} V$.

We remark that the assumption implies that $D \otimes_B \Omega^{2i} V \cong \Omega^{2i+l_i+2}S[l_i + 2]$ and that $D \otimes_B \Omega^{2j} V \cong \Omega^{2j+l_j+2}S[l_j + 2]$. Also since $I(\Omega^{2i+l_i+2}S)$ and $I(\Omega^{2j+l_j+2}S)$ are deleted terms, by the definition of C, $\operatorname{soc}\Omega^{2i+l_i+2}S$ and $\operatorname{soc}\Omega^{2j+l_j+2}S$ are adjacent to the exceptional vertex, and are not isomorphic to each other.

First, suppose $i = j$. To show $\operatorname{Hom}_A(\Omega^{2i+l_i}S, \Omega^{2i+l_i}S) = k$, we only need to show that the point corresponding to the uniserial module $\Omega^{2i+l_i}S$ is not on the exceptional vertex by Lemma 2.8. Since $\operatorname{soc}\Omega^{2i+l_i+2}S$ is adjacent to the exceptional vertex, we have the vertex corresponding to $\Omega^{2i+l_i}S$ is a non-exceptional vertex by Lemma 3.5 (2).

Next suppose $i \neq j$, and we show $\operatorname{Hom}_A(\Omega^{2i+l_i}S, \Omega^{2j+l_j}S) = 0$. Since $\operatorname{soc}\Omega^{2i+l_i+2}S$ and $\operatorname{soc}\Omega^{2j+l_j+2}S$ are not isomorphic to each other and are adjacent to the exceptional vertex, by Lemma 3.5, we have $\operatorname{soc}\Omega^{2i+l_i+1}S \not\cong \operatorname{soc}\Omega^{2j+l_j+1}S$ and these two simple modules are adjacent to the exceptional vertex. Also, by the isomorphisms $\operatorname{top}\Omega^{2i+l_i}S \cong \operatorname{soc}\Omega^{2i+l_i+1}S$ and $\operatorname{top}\Omega^{2j+l_j}S \cong \operatorname{soc}\Omega^{2j+l_j+1}S$ we have $\operatorname{top}\Omega^{2i+l_i}S$ and $\operatorname{top}\Omega^{2j+l_j}S$ are adjacent to the exceptional vertex and are not isomorphic to each other. In particular, since two uniserial modules $\Omega^{2i+l_i}S$ and $\Omega^{2j+l_j}S$ have mutually non-isomorphic top which are adjacent to the exceptional vertex and the corresponding vertices are not exceptional vertex, the two vertices corresponding to these two uniserial modules do not coincide. Hence we only need to show that $\operatorname{top}\Omega^{2i+l_i}S \not\cong \operatorname{soc}\Omega^{2j+l_j}S$ to show $\operatorname{Hom}_A(\Omega^{2i+l_i}S, \Omega^{2j+l_j}S) = 0$ by Lemma 2.8. But if $\operatorname{soc}\Omega^{2j+l_j+1}S$ is not an end of the Brauer tree or equivalently $I(\operatorname{soc}\Omega^{2j+l_j+1}S)$ is not uniserial, then $\operatorname{soc}\Omega^{2j+l_j}S$ is not adjacent to the exceptional vertex because the vertex corresponding to $\Omega^{2j+l_j+1}S$ is the exceptional vertex by Lemma 3.5. Also, since $\operatorname{soc}\Omega^{2i+l_i+1}S$ is adjacent to the exceptional vertex, $\operatorname{top}\Omega^{2i+l_i}S \cong \operatorname{soc}\Omega^{2i+l_i+1}S$ is adjacent to the exceptional vertex. Hence $\operatorname{top}\Omega^{2i+l_i}S$ which is adjacent to the exceptional vertex can not be isomorphic to $\operatorname{soc}\Omega^{2j+l_j}S$. Also if $\operatorname{soc}\Omega^{2j+l_j+1}S$ is an end of the Brauer tree then $\Omega^{2j+l_j}S$ is a simple module since $I(\operatorname{soc}\Omega^{2j+l_j+1}S)$ is a uniserial module. Hence we have

$$\operatorname{soc}\Omega^{2j+l_j}S \cong \operatorname{top}\Omega^{2j+l_j}S \cong \operatorname{soc}\Omega^{2j+l_j+1}S \not\cong \operatorname{soc}\Omega^{2i+l_i+1}S \cong \operatorname{top}\Omega^{2i+l_i}S.$$

CASE 2.3. $C \otimes_B \Omega^{2i} V \cong D \otimes_B \Omega^{2i} V$ and $C \otimes_B \Omega^{2j} V \not\cong D \otimes_B \Omega^{2j} V$.

We remark that the assumption implies that $D \otimes_B \Omega^{2i} V \cong \Omega^{2i+l_i}S[l_i], D \otimes_B \Omega^{2j} V \cong \Omega^{2j+l_j+2}S[l_j+2]$ and that $\operatorname{soc}\Omega^{2j+l_j+2}S$ is adjacent to the exceptional vertex. Also, we remark that the assumption implies that $i \neq j$. Hence we show that $\operatorname{Hom}_A(\Omega^{2i+l_i}S, \Omega^{2j+l_j}S) = 0$. Since D is a two-sided tilting complex,

$$\begin{aligned}
&\mathrm{Hom}_{D^b(A)}(D \otimes_B \Omega^{2i}V, D \otimes_B \Omega^{2j}V[-n]) = 0 \\
\Leftrightarrow\ &\mathrm{Hom}_{D^b(A)}(\Omega^{2i+l_i}S, \Omega^{2j+l_j+2}S\,[l_j - l_i - n + 2]) = 0 \\
\Leftrightarrow\ &\mathrm{Ext}^2_A(\Omega^{2i+l_i}S, \Omega^{2j+l_j+2}S) = 0 \\
\Leftrightarrow\ &\underline{\mathrm{Hom}}_A(\Omega^{2i+l_i}S, \Omega^{2j+l_j}S) = 0,
\end{aligned}$$

here the second equivalence comes from $l_j - l_i - n = 0$. By the last equation, if there exists a non-zero homomorphism $\varphi : \Omega^{2i+l_i}S \to \Omega^{2j+l_j}S$, then it factors through $P(\Omega^{2j+l_j}S)$.

$$\begin{tikzcd}
\Omega^{2i+l_i}S \arrow[rr, "\varphi"] \arrow[dr] & & \Omega^{2j+l_j}S \\
& P(\Omega^{2j+l_j}S) \arrow[ur] &
\end{tikzcd}$$

By this decomposition, we have that $\Omega^{2i+l_i}S$ has $\mathrm{soc}\,P(\Omega^{2j+l_j}S) \cong \mathrm{top}\,P(\Omega^{2j+l_j}S) \cong \mathrm{top}\,\Omega^{2j+l_j}S$ as a composition factor, which is adjacent to the exceptional vertex by the definition of C. Also by the construction of C again, since $\mathrm{top}\,P(\Omega^{2j+l_j}S)$ is adjacent to the exceptional vertex, there exists an integer $0 \le t \le e-1$ satisfying

$$D \otimes_B \Omega^{2t}V \cong \Omega^{2t+l'_t}S\,[l'_t] \text{ and } \mathrm{soc}\,\Omega^{2t+l'_t}S \cong \mathrm{top}\,\Omega^{2j+l_j}S,$$

here the vertex corresponding to $\Omega^{2t+l'_t}S$ is not the exceptional vertex. Since $\mathrm{soc}\,\Omega^{2i+l_i}S$ is not adjacent to the exceptional vertex, we have $i \neq t$ and the vertex corresponding to $\Omega^{2i+l_i}S$ is not the exceptional vertex. Since these two vertices must be adjacent to the edge corresponding to $\mathrm{top}\,\Omega^{2j+l_j}S$ and since they are not exceptional, they must coincide. Hence it holds

$$\mathrm{Hom}_{D^b(A)}(D \otimes_B \Omega^{2t}V, D \otimes_B \Omega^{2i}V[-n']) \neq 0 \text{ or } \mathrm{Hom}_{D^b(A)}(D \otimes_B \Omega^{2i}V, D \otimes_B \Omega^{2t}V[-n']) \neq 0$$

for some $n' \ge 0$ since $D \otimes_B \Omega^{2t}V \cong \Omega^{2t+l'_t}S\,[l'_t]$ and since $D \otimes_B \Omega^{2i}V \cong \Omega^{2i+l_i}S\,[l_i]$. This contradicts to the fact that D is a two-sided tilting complex. Therefore we have $\mathrm{Hom}_A(\Omega^{2i+l_i}S, \Omega^{2j+l_j}S) = 0$.

CASE 2.4. $C \otimes_B \Omega^{2i}V \ncong D \otimes_B \Omega^{2i}V$ and $C \otimes_B \Omega^{2j}V \cong D \otimes_B \Omega^{2j}V$.

We remark that the assumption implies that $D \otimes_B \Omega^{2i}V \cong \Omega^{2i+l_i+2}S\,[l_i+2]$, that $D \otimes_B \Omega^{2j}V \cong \Omega^{2j+l_j}S\,[l_j]$ and that $\mathrm{top}\,\Omega^{2i+l_i}S$ is adjacent to the exceptional vertex. Also, we remark that the assumption implies that $i \neq j$. Hence we show that $\mathrm{Hom}_A(\Omega^{2i+l_i}S, \Omega^{2j+l_j}S) = 0$. Suppose $\mathrm{Hom}_A(\Omega^{2i+l_i}S, \Omega^{2j+l_j}S) \neq 0$. Then $\Omega^{2j+l_j}S$ has $\mathrm{top}\,\Omega^{2i+l_i}S$ as a composition factor. In particular the vertex corresponding to $\Omega^{2j+l_j}S$ is adjacent to the edge corresponding to $\mathrm{top}\,\Omega^{2i+l_i}S$. Now for $\mathrm{top}\,\Omega^{2i+l_i}S$ there exists an integer $0 \le t \le e-1$ such that $D \otimes_B \Omega^{2t}V \cong \Omega^{2t+l'_t}S\,[l'_t]$ and that $\mathrm{soc}\,\Omega^{2t+l'_t}S \cong \mathrm{top}\,\Omega^{2i+l_i}S$. For this integer t, we have $t \neq j$ since $D \otimes_B \Omega^{2t}S \ncong C \otimes_B \Omega^{2t}S$ and $D \otimes_B \Omega^{2j}S \cong C \otimes_B \Omega^{2j}S$. By Lemma 3.5 (1), $\mathrm{soc}\,\Omega^{2t+l'_t-1}S$ is adjacent to the exceptional vertex. Hence for $\Omega^{2t+l'_t}S$, we have $\mathrm{top}\,\Omega^{2t+l'_t}S$ is not adjacent to the exceptional vertex and not simple, or have $\Omega^{2t+l'_t}S$ is simple. In any case, we have the vertex corresponding to $\Omega^{2t+l'_t}S$ is on an end of the edge $\mathrm{top}\,\Omega^{2i+l_i}S$ and it is not exceptional vertex. Also the vertex corresponding to $\Omega^{2j+l_j}S$ is on the edge $\mathrm{top}\,\Omega^{2i+l_i}S$ and is not exceptional vertex because $\mathrm{soc}\,\Omega^{2j+l_j}S$ is not adjacent to the exceptional vertex. Hence the vertex corresponding to $\Omega^{2t+l'_t}S$ coincides with the one corresponding to $\Omega^{2j+l_j}S$ (not necessarily the points coincide). Hence we have

$$\mathrm{Hom}(\Omega^{2t+l'_t}S, \Omega^{2j+l_j}S) \neq 0 \text{ and } \mathrm{Hom}(\Omega^{2j+l_j}S, \Omega^{2t+l'_t}S) \neq 0.$$

Therefore it holds

$$\mathrm{Hom}(D \otimes_B \Omega^{2t}V, D \otimes_B \Omega^{2j}S[-n']) \neq 0 \text{ or } \mathrm{Hom}(D \otimes_B \Omega^{2j}V, D \otimes_B \Omega^{2t}S[-n']) \neq 0$$

for some $n' \geq 0$ but this contradicts to the fact that D is a two-sided tilting complex.

Case 3. $l_j - l_i - n > 0$.
In this case, we have

$$\begin{aligned}\mathrm{Hom}_{D^b(A)}(\Omega^{2i+l_i}S, \Omega^{2j+l_j}S[l_j - l_i - n]) &\cong \mathrm{Ext}_A^{l_j-l_i-n}(\Omega^{2i+l_i}S, \Omega^{2j+l_j}S)\\ &\cong \underline{\mathrm{Hom}}_A(\Omega^{2i+l_i}S, \Omega^{2j+l_j-(l_j-l_i-n)}S)\\ &\cong \underline{\mathrm{Hom}}_A(S, \Omega^{2(j-i)+n}S).\end{aligned}$$

Hence we show that

$$\underline{\mathrm{Hom}}_A(S, \Omega^{2(j-i)+n}S) = \begin{cases} k & \text{if } i = j \text{ and } n = 0,\\ 0 & \text{otherwise.}\end{cases}$$

Case 3.1. $C \otimes_B \Omega^{2i}V \cong D \otimes_B \Omega^{2i}V$ and $C \otimes_B \Omega^{2j}V \cong D \otimes_B \Omega^{2j}V$.
In this case, since D is a two-sided tilting complex, it is clear that the complex C has the required properties.

Case 3.2. $C \otimes_B \Omega^{2i}V \ncong D \otimes_B \Omega^{2i}V$ and $C \otimes_B \Omega^{2j}V \ncong D \otimes_B \Omega^{2j}V$.
We have

$$D \otimes_B \Omega^{2i}V \cong \Omega^{2i+l_i+2}S[l_i + 2] \text{ and } D \otimes_B \Omega^{2j}V \cong \Omega^{2j+l_j+2}S[l_j + 2]$$

as in the Case 2.2. Hence we have

$$\begin{aligned}\underline{\mathrm{Hom}}_A(S, \Omega^{2(j-i)+n}S) &\cong \underline{\mathrm{Hom}}_A(\Omega^{2i+l_i+2}S, \Omega^{2j+l_j+2-(l_j-l_i-n)}S)\\ &\cong \mathrm{Ext}_A^{l_j-l_i-n}(\Omega^{2i+l_i+2}S, \Omega^{2j+l_j+2}S)\\ &\cong \mathrm{Hom}_{D^b(A)}(\Omega^{2i+l_i+2}S, \Omega^{2j+l_j+2}S[l_j - l_i - n])\\ &\cong \mathrm{Hom}_{D^b(A)}(\Omega^{2i+l_i+2}S[l_i + 2], \Omega^{2j+l_j+2}S[l_j + 2 - n])\\ &\cong \mathrm{Hom}_{D^b(A)}(D \otimes_B \Omega^{2i}V, D \otimes_B \Omega^{2j}V[-n])\\ &= \begin{cases} k & \text{if } i = j \text{ and } n = 0,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Case 3.3. $C \otimes_B \Omega^{2i}V \ncong D \otimes_B \Omega^{2i}V$ and $C \otimes_B \Omega^{2j}V \cong D \otimes_B \Omega^{2j}V$.
We have

$$D \otimes_B \Omega^{2i}V \cong \Omega^{2i+l_i+2}S[l_i + 2], D \otimes_B \Omega^{2j}V \cong \Omega^{2j+l_j}S[l_j] \text{ and } i \neq j$$

as in the Case 2.3.

First, suppose $l_j - l_i - n \geq 3$. Then since $l_j - l_i - n - 2 \geq 1$ and since D is a two-sided tilting complex, we have the following isomorphisms:

$$\begin{aligned}
\underline{\mathrm{Hom}}_A(S, \Omega^{2(j-i)+n}S) &\cong \underline{\mathrm{Hom}}_A(\Omega^{2i+l_i+2}S, \Omega^{2j+l_j-(l_j-l_i-n-2)}S)\\
&\cong \mathrm{Ext}_A^{l_j-l_i-n-2}(\Omega^{2i+l_i+2}S, \Omega^{2j+l_j}S)\\
&\cong \mathrm{Hom}_{D^b(A)}(\Omega^{2i+l_i+2}S, \Omega^{2j+l_j}S[l_j-l_i-n-2])\\
&\cong \mathrm{Hom}_{D^b(A)}(\Omega^{2i+l_i+2}S[l_i+2], \Omega^{2j+l_j}S[l_j-n])\\
&\cong \mathrm{Hom}_{D^b(A)}(D\otimes_B \Omega^{2i}V, D\otimes_B \Omega^{2j}V[-n])\\
&= 0.
\end{aligned}$$

Next, suppose $l_j - l_i - n = 2$. Then since D is a two-sided tilting complex, we have

$$\begin{aligned}
\mathrm{Hom}_A(\Omega^{2i+l_i+2}S, \Omega^{2j+l_j}S) &= \mathrm{Hom}_{D^b(A)}(\Omega^{2i+l_i+2}S[l_i+2], \Omega^{2j+l_j}S[l_j-n])\\
&= \mathrm{Hom}_{D^b(A)}(D\otimes_B \Omega^{2i}V, D\otimes_B \Omega^{2j}V[-n])\\
&= 0.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\underline{\mathrm{Hom}}_A(S, \Omega^{2(j-i)+n}S) &= \underline{\mathrm{Hom}}_A(S, \Omega^{2j-2i+l_j-l_i-2}S)\\
&= \underline{\mathrm{Hom}}_A(\Omega^{2i+l_i+2}S, \Omega^{2j+l_j}S)\\
&= 0.
\end{aligned}$$

Finally, suppose $l_j - l_i - n = 1$. Since the vertex corresponding to the uniserial module $\Omega^{2i+l_i+1}S$ is the exceptional vertex by Lemma 3.5, any composition factor of $\Omega^{2i+l_i+1}S$ is adjacent to the exceptional vertex. Hence if $\mathrm{Hom}_A(\Omega^{2i+l_i+1}S, \Omega^{2j+l_j}S) \neq 0$, then $\mathrm{soc}\,\Omega^{2j+l_j}S$ is adjacent to the exceptional vertex. However since $\Omega^{2j+l_j}S[l_j]$ is isomorphic to $C\otimes_B \Omega^{2j}V \cong D\otimes_B \Omega^{2j}V$ this is a contradiction to the construction of C. Hence we have $\mathrm{Hom}_A(\Omega^{2i+l_i+1}S, \Omega^{2j+l_j}S) = 0$. In particular, we have

$$\begin{aligned}
\underline{\mathrm{Hom}}_A(S, \Omega^{2(j-i)+n}S) &\cong \underline{\mathrm{Hom}}_A(S, \Omega^{2(j-i)+l_j-l_i-1}S)\\
&\cong \underline{\mathrm{Hom}}_A(\Omega^{2i+l_i+1}S, \Omega^{2j+l_j}S)\\
&= 0.
\end{aligned}$$

Case 3.4. $C\otimes_B \Omega^{2i}V \cong D\otimes_B \Omega^{2i}V$ and $C\otimes_B \Omega^{2j}V \ncong D\otimes_B \Omega^{2j}V$.
We have

$$D\otimes_B \Omega^{2i}V \cong \Omega^{2i+l_i}S[l_i],\ D\otimes_B \Omega^{2j}V \cong \Omega^{2j+l_j+2}S[l_j+2] \text{ and } i \neq j$$

as in the Case 2.4. Hence since D is a two-sided tilting complex we have

$$\begin{aligned}
\underline{\mathrm{Hom}}_A(S, \Omega^{2(j-i)+n}S) &\cong \underline{\mathrm{Hom}}_A(\Omega^{2i+l_i}S, \Omega^{2j+l_j+2-(l_j-l_i-n+2)}S)\\
&\cong \mathrm{Ext}_A^{l_j-l_i-n+2}(\Omega^{2i+l_i}S, \Omega^{2j+l_j+2}S)\\
&\cong \mathrm{Hom}_{D^b(A)}(\Omega^{2i+l_i}S, \Omega^{2j+l_j+2}S[l_j-l_i-n+2])\\
&\cong \mathrm{Hom}_{D^b(A)}(\Omega^{2i+l_i}S[l_i], \Omega^{2j+l_j+2}S[l_j+2-n])\\
&\cong \mathrm{Hom}_{D^b(A)}(D\otimes_B \Omega^{2i}V, D\otimes_B \Omega^{2j}V[-n])\\
&= 0.
\end{aligned}$$

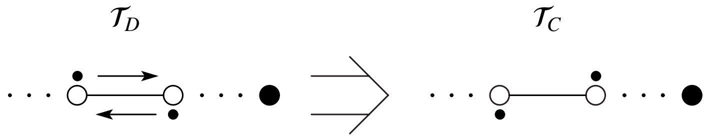
□

4. Example and Proof of Theorem 2

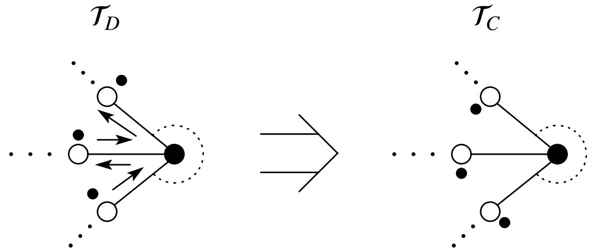
In this section, we will show that the two-sided tilting complexes given in Section 3 are isomorphic to some Rickard-Schaps tree-to-star complexes in $D^b(A)$. Moreover, we state how the corresponding pointings on the Brauer tree change when we apply the operations in Theorem 1 to the two-sided tilting complexes.

We fix the following notation.

Notation 4.1. We associate two-sided tilting complexes obtained in Theorem 1 to pointed Brauer trees. Let D_T be a two-sided tilting complex isomorphic to the Rickard tree-to-star complex T in $D^b(A)$. First we associate D_T to the pointed Brauer tree which corresponds to the Rickard tree-to-star complex, that is, $\mathcal{T}_{D_T}$ in Notation 2.11. Second suppose we have a two-sided tilting complex D obtained by applying operations in Theorem 1 (i) and Theorem 1 (ii) to D_T several times and the pointed Brauer tree associated to $\mathcal{T}_D$. If a two-sided tilting complex C is obtained by applying an operation in Theorem 1 (i) to D, and let U be the simple module with respect to the deleted terms, we define a pointed Brauer tree $\mathcal{T}_C$ associated to C to be a pointed Brauer tree given by moving the points on both end of the edge U along the reverse Green's walk.



If a two-sided tilting complex C is obtained by applying an operation in Theorem 1 (ii) to D, we define a pointed Brauer tree $\mathcal{T}_C$ associated to C to be a pointed Brauer tree given by moving the all points on the non-exceptional vertices of edges adjacent to the exceptional vertex along the reverse Green's walk twice.



Remark 4.2. By Remark 2.21 for any pointed Brauer tree of the Brauer tree $\mathcal{T}$ we can get it by applying "the moving points operations" of Notation 4.1 to $\mathcal{T}_{D_T}$. Therefore if we find operations on two-sided sided tilting complexes corresponding to foldings, starting with the two-sided tilting complex D_T, we can construct a two-sided tilting complex corresponding to any Rickard-Schaps tree-to-star complex.

By using Notation 4.1, we rewrite Theorem 2 as follows.

Theorem 4.3. *Let C be a two-sided tilting complex of $A \otimes B^{op}$-modules, let $\mathcal{T}_C$ be a pointed Brauer tree in* Notation 4.1*, and let T_C be the Rickard-Schaps tree-to-star complex associated to the pointed Brauer tree $\mathcal{T}_C$. Then we have $C \cong T_C$ in $D^b(A)$.*

Example 4.4. We know that the two-sided tilting complex D_T described in Example 3.1 is isomorphic to the Rickard-Schaps tree-to-star complex in Example 2.16. By Theorem 4.3 the two-sided tilting complex C_1 described in Example 3.1 is isomorphic to the Rickard-Schaps tree-to-star complex in Example 2.17 and the two-sided tilting complex C_2 described in Example 3.1 is isomorphic to the Rickard-Schaps tree-to-star complex in Example 2.18.

By using this notation, we give a correspondence between two-sided tilting complexes obtained in Theorem 4.3 and Rickard-Schaps tree-to-star complexes.

To prove Theorem 4.3, we fix the following notation.

Notation 4.5. Let $\mathcal{T}_C$ be a pointed Brauer tree associated to a two-sided tilting complex C of $A \otimes B^{op}$-modules, and let $T_C(S_i)$ be an indecomposable summand of the Rickard-Schaps tree-to-star complex associated to $\mathcal{T}_C$ corresponding to an edge S_i including the projective module $P(S_i)$. For a point r_i of $\mathcal{T}_C$ corresponding to the edge S_i, that is, the farther point on the edge S_i from the exceptional vertex, we denote by $U_C(S_i)$ or by $U_C(r_i)$ a uniserial module of Notation 2.4 associated to the point, that is, the uniserial module with its structure given by turning around the vertex adjacent to the point in the cyclic ordering from the point, and we denote by $l_C(S_i)$ or by $l_C(r_i)$ the degree in which $P(S_i)$ is for $T_C(S_i)$.

Remark 4.6. We remark the following.

(i) Let $\mathcal{T}_D$ be a pointed Brauer tree and suppose we get a pointed Brauer tree $\mathcal{T}_C$ by moving two points r_1 and r_2 of $\mathcal{T}_D$ on the both ends of an edge not adjacent to the exceptional vertex along the reverse Green's walk. Then $\Omega^{-1} U_{\mathcal{T}_D}(r_i) \cong U_{\mathcal{T}_C}(r_i)$ for $i = 1, 2$.

(ii) Let $\mathcal{T}_D$ be a pointed Brauer tree and suppose we get a pointed Brauer tree $\mathcal{T}_C$ by moving all points r_i of $\mathcal{T}_D$ on the edges adjacent to the exceptional vertex twice along the reverse Green's walk. Then $\Omega^{-2} U_{\mathcal{T}_D}(r_i) \cong U_{\mathcal{T}_C}(r_i)$ for all i.

We prepare the following two lemmas before we prove Theorem 4.3.

Lemma 4.7. *Let Γ be a basic symmetric algebra, and let $P_\bullet$ be a one-sided tilting complex of Γ-modules with endomorphism ring a basic algebra Λ. For any simple Λ-module U, if a two-sided tilting complex C of $\Gamma \otimes \Lambda^{op}$-modules satisfies the following conditions, then C is isomorphic to $P_\bullet$ in $D^b(\Gamma)$:*

$$\operatorname{Hom}_{D^b(\Gamma)}(P_\bullet, C \otimes_\Lambda U[n]) \cong \begin{cases} k & n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $F : D^b(\Gamma) \to D^b(\Lambda)$ be an equivalence induced by the one-sided tilting complex $P_\bullet$. By the assumption, we have $F(C \otimes_\Lambda U)$ is a one-dimensional Λ-module. In particular, it is a simple Λ-module. Also let $G : D^b(\Lambda) \to D^b(\Gamma)$ be the equivalence induced by the two-sided tilting complex C. Then since any simple Λ-module U is sent to $C \otimes_\Lambda U$ by G, U is sent to a simple Λ-module by $F \circ G$. Hence the restriction of $F \circ G$ to Λ-mod induces a Morita equivalence Λ-mod $\to$ Λ-mod. In particular $(F \circ G)(\Lambda) = \Lambda$. Therefore we have

$$F(C \otimes_\Lambda \Lambda) = (F \circ G)(\Lambda) = \Lambda = F(P_\bullet).$$

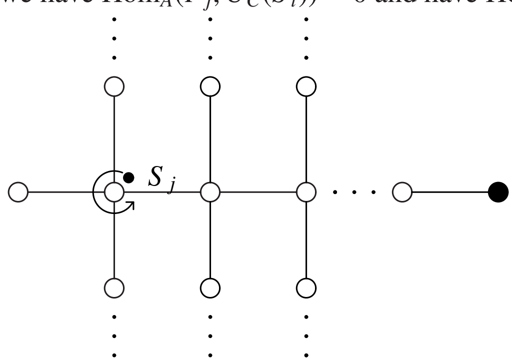
Since F is an equivalence we conclude $C \otimes_\Lambda \Lambda = P_\bullet$ in $D^b(\Gamma)$. □

Lemma 4.8. *Let $S_0, S_1, \cdots, S_{e-1}$ be all simple A-modules. Under the* Notation 4.5*, we have*

$$\mathrm{Hom}_{K^b(A)}(T_C(S_i), U_C(S_j)[l]) = \begin{cases} k & i = j, l = l_C(S_i), \\ 0 & \text{otherwise}. \end{cases}$$

Proof. Fix an edge S_j.

First we assume $i = j$ and $l = l_C(S_j)$, and we show that $\mathrm{Hom}_{K^b(A)}(T_C(S_j), U_C(S_j)[l_C(S_j)]) \cong k$. Since the simple module S_j appears only once as a composition factor of $U_C(S_j)$, we have $\mathrm{Hom}_A(P_j, U_C(S_j)) \cong k$, where we put $P_j = P(S_j)$. Also, for a pointed Brauer tree $\mathcal{T}_C$, we defined the vertex corresponding to the edge S_j by the farther vertex of the edge S_j from the exceptional vertex. Hence $U_C(S_j)$ does not have any simple module S_t with $d(S_t) < d(S_j)$ as a composition factor (see the following figure), which implies that for any such simple module S_t we have $\mathrm{Hom}_A(P_j, U_C(S_t)) = 0$ and have $\mathrm{Hom}_A(P_t, U_C(S_j)) = 0$.



Also for such simple module S_t, by the construction of $T_C(S_t)$, it includes the only projective modules associated to simple modules of distance smaller than the one of S_j, and does not include P_j. Therefore we have

$$\mathrm{Hom}_{K^b(A)}(T_C(S_j), U_C(S_j)[l_C(S_j)]) \cong \mathrm{Hom}_A(P_j, U_C(S_j)) \cong k,$$

and have for any simple module S_t with $d(S_t) < d(S_j)$ and $l \in \mathbb{Z}$, or for $S_t = S_j$ and $l \neq l_C(S_j)$ we have

$$\mathrm{Hom}_{K^b(A)}(T_C(S_t), U_C(S_j)[l]) = 0.$$

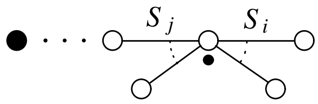
Also for any $l \in \mathbb{Z}$ and for an edge S'_t on the different interval from the one on which S_j is, we have similarly

$$\mathrm{Hom}_{K^b(A)}(T_C(S'_t), U_C(S_j)[l]) = 0.$$

We have shown that $\mathrm{Hom}_{K^b(A)}(T_C(S_i), U_C(S_j)[l])$ has the required properties in case $d(S_i) \leq d(S_j)$ and in case S_i is on the different interval from the one on which S_j is. Hence fixing S_j, we show that $\mathrm{Hom}_{K^b(A)}(T_C(S_i), U_C(S_j)[l]) = 0$ for S_i on the same interval as S_j with $d(S_i) > d(S_j)$ and for all $l \in \mathbb{Z}$. If $d(S_i) > d(S_j) + 1$, then $\mathrm{Hom}_A(P_i, U_C(S_j)) = 0$ by the earlier argument. Hence it suffices to show $\mathrm{Hom}_{K^b(A)}(T_C(S_i), U_C(S_j)[l]) = 0$ for any simple module S_i on the same interval as S_j with $d(S_i) = d(S_j) + 1$ since a summand $T_C(S_i)$ of T_C is obtained by adding projective module P_i. We denote a numbering of S_j and S_i given by the pointed Brauer tree $\mathcal{T}_C$ by $n_C(S_j)$ and $n_C(S_i)$(see Algorithm 2.13).

Case 1. $n_C(S_i) > n_C(S_j)$

We remark that by the assumption $n_C(S_i) > n_C(S_j)$, a point corresponding to S_j must be in a sector from S_j to S_i.



Hence the uniserial module $U_C(S_j)$ has composition series as below

$$\begin{bmatrix} \vdots \\ S_i \\ \vdots \\ S_j \\ \vdots \end{bmatrix}.$$

First we assume $l = l_C(S_j)$:

$$\begin{tikzcd} P_j \arrow[r] \arrow[d] & P_i \arrow[d] \\ U_C(S_j) \arrow[r] & 0. \end{tikzcd}$$

Since $U_C(S_j)$ is a uniserial module whose all composition factors are different and has the structure as above, for any non-zero homomorphism $\psi : P_j \to U_C(S_j)$ there exists a homomorphism $\varphi : P_i \to U_C(S_j)$ such that it makes the following diagram commutative.

$$\begin{tikzcd} P_j \arrow[r] \arrow[d, "\psi"'] & P_i \arrow[dl, "\varphi"] \\ U_C(S_j) & \end{tikzcd}$$

Therefore we have $\mathrm{Hom}_{K^b(A)}(T_C(S_i), U_C(S_j)[l_C(S_j)]) = 0$.

Next for $l \neq l_C(S_j)$ we show $\mathrm{Hom}_{K^b(A)}(T_C(S_i), U_C(S_j)[l]) = 0$. However all composition factors of $U_C(S_j)$ are adjacent to S_j and can not be a simple module of distance $d(S_j) - 1$, hence it suffices to show that $\mathrm{Hom}_{K^b(A)}(T_C(S_i), U_C(S_j)[l_C(S_j) - 1]) = 0$.

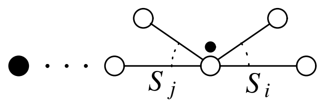
$$\begin{tikzcd} P_j \arrow[r] \arrow[d] & P_i \arrow[d] \\ 0 \arrow[r] & U_C(S_j) \end{tikzcd}$$

By the structure of $U_C(S_j)$ the composition of a non-zero homomorphism P_j to P_i and a non-zero homomorphism P_i and $U_C(S_j)$ is non-zero. To make above diagram commutative, the vertical map must be zero map. Therefore we have $\mathrm{Hom}_{K^b(A)}(T_C(S_i), U_C(S_j)[l_C(S_j) - 1]) = 0$.

Case 2. $n_C(S_i) < n_C(S_j)$

By the assumption $n_C(S_i) < n_C(S_j)$, a point corresponding to S_j must be in a sector from

S_i to S_j.



Hence the uniserial module $U_C(S_j)$ has composition series as below

$$\begin{bmatrix} \vdots \\ S_j \\ \vdots \\ S_i \\ \vdots \end{bmatrix}.$$

By the similar argument of Case 1, we have it suffices to show that $\mathrm{Hom}_{K^b(A)}(T_C(S_i), U_C(S_j)[l_C(S_j)]) = 0$ and $\mathrm{Hom}_{K^b(A)}(T_C(S_i), U_C(S_j)[l_C(S_j) + 1]) = 0$. But the dual proof of Case 1 shows that $\mathrm{Hom}_{K^b(A)}(T_C(S_i), U_C(S_j)[l_C(S_j)]) = 0$ and $\mathrm{Hom}_{K^b(A)}(T_C(S_i), U_C(S_j)[l_C(S_j) + 1]) = 0$. □

Lemma 4.9. *Let D_T be a two-sided tilting complex of $A \otimes B^{op}$-modules isomorphic to the Rickard tree-to-star complex in $D^b(A)$ and let C be a two-sided tilting complex of $A \otimes B^{op}$-modules obtained by applying operations in* Theorem 1 (i) *and* Theorem 1 (ii) *to D_T several times. For each point r in $\mathcal{T}_C$ let $U_C(r)$ be a uniserial module in* Notation 4.5. *Then, for each uniserial module $\Omega^{2j+l_j}S$ satisfying that $C \otimes_B \Omega^{2j}V \cong \Omega^{2j+l_j}S[l_j]$ for some l_j, $\Omega^{2j+l_j}S \cong U_C(r_j)$ and $l_j = l_C(r_j)$ for some point r_j.*

Proof. If $C = D_T$, then it is clear by [3, Proposition 5.3], [3, Lemma 5.5] and [3, Corollary 5.6]. Let D be a two-sided tilting complex of $A \otimes B^{op}$-modules obtained by applying operations in Theorem 1 (i) and Theorem 1 (ii) to D_T several times, and satisfying the required properties. Then it suffices to show that a two-sided tilting complex C obtained by applying the operation in Theorem 1 (i) or Theorem 1 (ii) to D just once satisfies the required properties.

First, suppose C is obtained by applying the operation in Theorem 1 (i) to D just once, and denote the simple module which the deleted term is associated to by U. Moreover we denote deleted terms by $P(U) \otimes P(\Omega^{2i}V)^*$ and $P(U) \otimes P(\Omega^{2j}V)^*$ where $P(U) \otimes P(\Omega^{2i}V)^*$ is a direct summand of the leftmost non-zero term, and where $P(U) \otimes P(\Omega^{2i}V)^*$ is a direct summand of the second leftmost non-zero term. Then we have isomorphisms

$$C \otimes_B \Omega^{2i}V \cong \Omega^{2i+l_i}S[l_i] \text{ and } C \otimes_B \Omega^{2j}V \cong \Omega^{2j+l_j}S[l_j]$$

for some l_i and l_j. We consider the case $i \neq j$. Then we have $l_j = l_i - 1$. Also for the pointed Brauer tree $\mathcal{T}_D$, we denote the points on the both ends of U by r_1 and r_2 where r_1 is the farther point from the exceptional vertex. By the assumption, we have

$$\Omega^{2i+l_i+1}S \cong U_D(r_1), l_i + 1 = l_D(r_1)$$

and

$$\Omega^{2j+l_j+1}S \cong U_D(r_2), l_j + 1 = l_D(r_2).$$

Now for moved points r_1 and r_2 in $\mathcal{T}_C$ from those in $\mathcal{T}_D$ along the reverse Green's walk, we have $U_C(r_1) \cong \Omega^{-1}U_D(r_1)$ and $U_C(r_2) \cong \Omega^{-1}U_D(r_2)$ by Remark 4.6. Hence we have the following isomorphisms:

$$\Omega^{2i+l_i}S \cong \Omega^{-1}U_D(r_1) \cong U_C(r_1) \text{ and } \Omega^{2j+l_j}S \cong \Omega^{-1}U_D(r_2) \cong U_C(r_2).$$

A movement along the reverse Green's walk of the points on the both ends of U implies that folding of $P(U)$ in the one-sided tilting complex corresponding to the pointed Brauer tree $\mathcal{T}_D$, that is, -2 shift of $P(U)$. Hence we have $l_C(r_2) = l_C(U) = l_D(U)-2 = l_D(r_1)-2 = l_i+1-2 = l_i-1 = l_j$ since the point in $\mathcal{T}_D$ corresponding to the edge U is r_1, and the one in $\mathcal{T}_C$ is r_2. Also the point r_2 in $\mathcal{T}_D$ and r_1 in $\mathcal{T}_C$ do not correspond to U, we have $l_C(r_1) = l_D(r_2) = l_j + 1 = l_i$. Also if $i = j$ then the point r_1 is on an end of the Brauer tree $\mathcal{T}_D$ and $\Omega^{-2}U_D(r_2) \cong U_C(r_1)$. Also we have $C \otimes_B \Omega^{2i}V \cong \Omega^{2i+l_i}S[l_i]$ and $D \otimes_B \Omega^{2i}V \cong \Omega^{2i+l_i+2}S[l_i + 2]$ in this case. The similar argument shows that the uniserial module $\Omega^{2i+l_i}S$ and the integer l_i have the required properties.

Next, suppose C is obtained by applying the operation to D in Theorem 1 (ii) just once, and denote all projective modules associated to simple modules adjacent to the exceptional vertex by $P(U_1), P(U_2), \cdots, P(U_{t-1})$ and $P(U_t)$. By the construction of C, we have $C \otimes_B \Omega^{2j}V \cong \Omega^{2j+l_j}S[l_j]$ and $D \otimes_B \Omega^{2j}V \cong \Omega^{2j+l_j+2}S[l_j + 2]$ for $1 \le j \le t$. Moreover, by the assumption, we have $\Omega^{2j+l_j+2}S \cong U_D(r_j)$ and $l_j + 2 = l_D(r_j)$ for $1 \le j \le t$. Since a pointed Brauer tree $\mathcal{T}_C$ is obtained by moving all the points of $\mathcal{T}_D$ on the edges adjacent to the exceptional vertex, that is, moving them along the reverse Green's walk twice, we have $U_C(r_j) \cong \Omega^{-2}U_D(r_j)$ by Remark 4.6. Therefore we have $\Omega^{2j+l_j}S \cong \Omega^{-2}U_D(r_j) \cong U_C(r_j)$. Also, the one-sided tilting complex corresponding to $\mathcal{T}_C$ is obtained by -2 shifts of $P(U_1), P(U_2), \cdots, P(U_{t-1})$ and $P(U_t)$ in the one-sided tilting complex corresponding to $\mathcal{T}_D$. Therefore we have $l_C(r_j) = l_D(r_j) - 2 = l_j$ for $1 \le j \le t$. □

Proof of Theorem 4.3. By Lemma 4.7 it suffices to show that

$$\mathrm{Hom}_{K^b(A)}\Big(\bigoplus_{0\le i\le e-1} T_C(S_i), C \otimes_B \Omega^{2j}V[n]\Big) = \begin{cases} k & n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 4.9

$$\mathrm{Hom}_{K^b(A)}\Big(\bigoplus_{0\le i\le e-1} T_C(S_i), C \otimes_B \Omega^{2j}V[n]\Big) \cong \mathrm{Hom}_{K^b(A)}\Big(\bigoplus_{0\le i\le e-1} T_C(S_i), U_C(r_j)[l_C(r_j) + n]\Big)$$

for some point r_j in $\mathcal{T}_C$. Since by Lemma 4.8

$$\mathrm{Hom}_{K^b(A)}(T_C(S_i), U_C(r_j)[l_C(r_j) + n]) \cong \begin{cases} k & r_i \text{ corresponds to } S_i \text{ and } n = 0, \\ 0 & \text{otherwise,} \end{cases}$$

we have the statement. □

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