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ALMOST RELATIVE INJECTIVE MODULES

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Abstract

The concept of a module M being almost N-injective, where N is some module, was introduced by Baba (1989). For a given module M, the class of modules N, for which M is almost N-injective, is not closed under direct sums. Baba gave a necessary and sufficient condition under which a uniform, finite length module U is almost V-injective, where V is a finite direct sum of uniform, finite length modules, in terms of extending properties of simple submodules of V. Let M be a uniform module and V be a finite direct sum of indecomposable modules. Some conditions under which M is almost V-injective are determined, thereby Baba's result is generalized. A module M that is almost M-injective is called an almost self-injective module. Commutative indecomposable rings and von Neumann regular rings that are almost self-injective are studied. It is proved that any minimal right ideal of a von Neumann regular, almost right self-injective ring, is injective. This result is used to give an example of a von Neumann regular ring that is not almost right self-injective.

Introduction

Let M_R, N_R be two modules. As defined by Baba [4], M is said to be *almost N-injective*, if for any homomorphism $f\colon A \to M$, $A \leqslant N$, either f extends to a homomorphism $g\colon N \to M$ or there exist a decomposition $N = N_1 \oplus N_2$ with $N_1 \neq 0$ and a homomorphism $h\colon M \to N_1$ such that $hf(x) = \pi(x)$ for any $x \in A$, where $\pi\colon N \to N_1$ is a projection with kernel N_2. A module M that is almost M-injective, is called an *almost self-injective module*. For a module M, the class of those modules N for which M is almost N-injective, is not closed under direct sums. Let $\{U_k\colon 0 \leq k \leq n\}$ be a finite family of uniform modules of finite composition lengths, and $U = \bigoplus \sum_{k=1}^{n} U_k$. Baba [4] has given a characterization for U_0 to be almost U-injective in terms of the property of simple submodules of U being contained in uniform summands of U. Let M be a uniform module and V be a finite direct sum of indecomposable modules. In Section 1, we investigate conditions under which M is almost V-injective. The main result is given in Theorem 1.12 and it generalizes the result by Baba. An alternative short proof of a result by Harada [10] is given in Theorem 1.16. It is well known that a (commutative) integral domain R is almost self-injective if and only if it is a valuation domain. Let R be a commutative ring having no non-trivial idempotent and Q be its classical quotient ring. In Section 2, it is proved that R_R is almost self-injective

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if and only if for any elements $a, b \in R$ with $ann(a) \subseteq ann(b)$, either $bR \subseteq aR$ or $aR < bR$ with $a = bc$ for some regular element c, and Q_R is injective and uniform. It follows that any commutative, indecomposable ring R that is almost self-injective but not self-injective, is local. In Section 3, von Neumann regular rings R with R_R almost self-injective are studied. A characterization of such rings is given in Theorem 3.1. It is proved that any von Neumann regular ring R that is either commutative or right CS is almost right self-injective. In Theorem 3.4, it is proved that any minimal right ideal of a von Neumann regular ring R that is almost right self-injective, is injective. This result is used to give an example of a von Neumann regular ring that is not almost right self-injective.

Preliminaries

All rings considered here are with unity and all modules are unital right modules unless otherwise stated. Let M a module. Then $E(M)$, $J(M)$ denote its injective hull, radical respectively. The symbols $N \leqslant M$, $N < M$, $N \subset_e M$ denote that N is a submodule of M, N is a submodule different from M, N is an essential submodule of M respectively. A module M whose ring of endomorphisms $End(M)$ is local, is called an *LE module*. A module M such that its complement submodules are summands of M, is called a CS module (or a module satisfying condition (C_1)). If a module M is such that for any two summands A, B of M with $A \cap B = 0$, $A + B$ is a summand of M, then it is said to satisfy condition (C_3). A module M satisfying conditions (C_1), (C_3) is called a *quasi-continuous module*. The terminology used here is available in standard text books like [3], [6].

1. Direct sums of uniform modules

DEFINITION 1.1. Let M_R and N_R be any two modules. Then M is said to be *almost N-injective*, if given any R-homomorphism $f\colon A \to M$, $A \leqslant N$ either f extends to an R-homomorphism from N to M or there exist a decomposition $N = N_1 \oplus N_2$ with $N_1 \neq 0$, and an R-homomorphism $h\colon M \to N_1$ such that $hf(x) = \pi(x)$ for any $x \in A$, where $\pi : N \to N_1$ is a projection with kernel N_2.

One can easily prove the following two results. (See [2])

Proposition 1.2. (i) *A module M_R is almost N_R-injective, if and only if for any R-homomorphism $f : L \to M$, $L < N$ which is maximal with respect to the property that it cannot be extended from N to M, there exist a decomposition $N = N_1 \oplus N_2$ with $N_1 \neq 0$, and an R-homomorphism $h\colon M \to N_1$ such that $hf(x) = \pi(x)$ for any $x \in L$, where $\pi : N \to N_1$ is a projection with kernel N_2.*
(ii) *If a module M is almost N-injective and N is indecomposable, then any R-homomorphism $f : L \to M$, $L \subset_e N$ with* $\ker f \neq 0$ *extends to an R-homomorphism from N to M.*

Proposition 1.3. *Let A_R, B_R any two modules and $f: L \to B$, $L < A$ be an R-homomorphism that is maximal with respect to the property that it cannot be extended from A to B. If C is a summand of A and $L \cap C < C$, then $f_1 = f \mid L \cap C$ from $L \cap C$ to B is a maximal homomorphism that cannot be extended from C to B.*

The following is well known. (See [12])

Proposition 1.4. *Let M_R, N_R be any two modules such that M is almost N-injective.*
(i) *Any summand K of M is almost N-injective.*
(ii) *If W is a summand of N, then M is almost W-injective.*
(iii) *If $N = N_1 \oplus N_2$ and M is not N-injective, then M is either not N_1-injective or not N_2-injective.*

Lemma 1.5. *Let M_R and N_R be any two modules such that M is almost N-injective, and $f: L \to M$, $L < N$ be a maximal homomorphism which cannot be extended from N to M. Let $N = N_1 \oplus N_2$ with $N_1 \neq 0$ and $h: M \to N_1$ be a homomorphisms such that $hf(x) = \pi(x)$ for $x \in L$, where $\pi: N \to N_1$ is a projection with kernel N_2. Then the following hold.*
(i) *f is monic on $L \cap N_1$ and $f(L \cap N_1)$ is a closed submodule of M.*
(ii) *$\ker h$ is a complement of $f(N_1 \cap L)$.*
(iii) *$f(N_2 \cap L) \subseteq \ker h$.*
(iv) *If M is a CS module, then $f(N_1 \cap L)$ and $\ker h$ are summands of M.*

Proof. (i) Now $hf(x) = x$ for any $x \in L \cap N_1$, which gives $f(L \cap N_1) \cap \ker h = 0$. We get a complement H of $\ker h$ containing $f(L \cap N_1)$. Then $h \mid H$ is monic and $N_1 \cap L \subseteq h(H) \subseteq N_1$. Define $\lambda: h(H) \to H$, $\lambda(h(y)) = y$ for any $y \in H$. Then λ extends $f \mid (L \cap N_1)$. By Proposition 1.3, $h(H) = L \cap N_1$. Which proves that $f(N_1 \cap L) = H$. Hence $f(L \cap N_1)$ is a closed submodule of M and is a complement of $\ker h$.

(ii) Let K be a complement of $f(N_1 \cap L)$ containing $\ker h$. Then $\ker h \subset_e K$. Let $x \in K$. Suppose $h(x) \neq 0$. As $h(x) \in N_1$, there exists an $r \in R$ such that $0 \neq h(xr) \in L \cap N_1$. Thus $h(xr) = h(y)$ for some $y \in f(L \cap N_1)$, $xr - y \in \ker h \subseteq K$. Which gives $y \in K \cap f(L \cap N_1) = 0$. Therefore, $h(xr) = h(y) = 0$, which is a contradiction. Hence $K = \ker h$.

The last two parts are obvious. □

Theorem 1.6. *Let M_R be a quasi-continuous module and N_R any module. Then M is almost N-injective if and only if for any homomorphism $f: L \to M$, $L < N$ which is maximal such that it cannot be extended to a homomorphism from N to M, the following hold.*
(i) *There exist decompositions $N = N_1 \oplus N_2$, $M = M_1 \oplus M_2$ with $N_1 \neq 0$.*
(ii) *f is monic on $L \cap N_1$ and $f(N_1 \cap L) = M_1$.*

(iii) $f(N_2 \cap L) \subseteq M_2$.
(iv) $L = (L \cap N_1) \oplus (L \cap N_2)$.

Proof. (i) Let M be almost N-injective. By Lemma 1.5, there exist a decomposition $N = N_1 \oplus N_2$ and a homomorphism h: $M \to N_1$ such that $N_1 \neq 0$, f is monic on $N_1 \cap L$, $M_1 = f(N_1 \cap L)$ and $M_2 = \ker h$ are summands of M, and $hf(x) = \pi(x)$ for $x \in L$, where π: $N \to N_1$ is a projection with kernel N_2. As M_1, M_2 are complements of each other and M satisfies (C_3), we get $M = M_1 \oplus M_2$. Thus $h(M) = h(M_1)$.

(ii) It is proved in Lemma 1.5.

(iii) Let $z \in L$. Then $z = x_1 + x_2$ for some $x_1 \in N_1$, $x_2 \in N_2$. Then $x_1 = hf(z) \in h(M_1) = hf(N_1 \cap L) = N_1 \cap L$, which also gives $x_2 \in N_2 \cap L$. Hence $L = (L \cap N_1) \oplus (L \cap N_2)$.

Conversely, let the above conditions hold. Define h: $M \to N_1$ as follows. Let $y \in M$. Then $y = y_1 + y_2$ for some $y_1 \in M_1$, $y_2 \in M_2$. Now $y_1 = f(x_1)$ for some $x_1 \in N_1 \cap L$. Set $h(y) = x_1$. □

Corollary 1.7. *Let M_R be a uniform module and N_R any module.*
(i) *M is almost N-injective if and only if for any homomorphism f: $L \to M$, $L < N$ which is maximal such that it cannot be extended from N to M, there exists a decomposition $N = N_1 \oplus N_2$ such that $f(N_1 \cap L) = M$, $N_2 = \ker f$ and $L = (L \cap N_1) \oplus N_2$.*
(ii) *M is almost N-injective if and only if for any homomorphism f: $L \to M$, $L < N$ which is maximal such that it cannot be extended from N to M, there exists a decomposition $N = N_1 \oplus N_2$ such that f is monic on $N_1 \cap L$, $f(N_1 \cap L) = M$ and $L = (L \cap N_1) \oplus N_2$.*
(iii) *Let D be an (commutative) integral domain and F be its quotient field. Then D is almost F_D-injective.*

Proof. Clearly, M is quasi-continuous. (i) Suppose M is almost N-injective. By Theorem 1.6, $N = N_1 \oplus N_2$, $N_1 \neq 0$, f is monic on $N_1 \cap L$, $f(N_1 \cap L) = M$, and $f(N_2 \cap L) = 0$. As $f \mid N_2 \cap L = 0$, it can be extended from N_2 to M, therefore by Proposition 1.3, $N_2 = N_2 \cap L$. Hence $L = (N_1 \cap L) \oplus N_2$. The converse is immediate from Theorem 1.6.

(ii) Suppose the given condition holds. We get a homomorphism λ: $N_2 \to (N_1 \cap L)$ such that for any $x \in N_2$, $\lambda(x) = y$, whenever $f(x) = f(y)$. Then $N_2' = \{x - \lambda(x)\colon x \in N_2\} \subseteq \ker f$ and $N = N_1 \oplus N_2'$. After this (i) proves the result.

(iii) Let f: $L \to D$, $L < F_D$ be a homomorphism that cannot be extended from F to D. Then $F \neq D$. However F_D is injective, so f extends to an automorphism g of F_D. Let $K = g^{-1}(D)$. Then $K = cD$ for some $c \in F$ such that $g(c) = 1$. Clearly, $L \subseteq K$. $g(K) = D$. The maximality of f gives $L = K$. By (i), D is almost F_D-injective. □

Lemma 1.8. *Let M_R be uniform module and be almost N_R-injective. If N has a uniform summand N_1 such that M is not N_1-injective, then for any uniform submodule V of N, there exists a proper summand K_2 of N such that $K_2 \cap V \neq 0$.*

If $N = N_1 \oplus N_2$ with N_2 also uniform, then K_2 is uniform.

Proof. Now M is almost N_1-injective. So there exists a maximal R-monomorphism $\lambda\colon T \to M$, $T < N_1$, which cannot be extended from N_1 to M. By Corollary 1.7. $\lambda(T) = M$. Now $N = N_1 \oplus N_2$ for some $N_2 < N$. This gives a maximal R-homomorphism $f\colon L \to M$, $L < N$ which extends λ and $N_2 = \ker f$. We can take $V \subseteq T \oplus N_2$. We need only to discuss the case, when $V \cap N_1 = 0 = V \cap N_2$. We take $V = xR$, $x = x_1 + x_2$ with $x_1 \in T$, $x_2 \in N_2$. We get an isomorphism $g\colon x_2R \to x_1R$, $g(x_2) = x_1$. Define a mapping $\mu\colon x_1R \oplus x_2R \to M$, $\mu(x_1r_1 + x_2r_2) = f(x_1r_1 - g(x_2r_2)) = f(x_1(r_1 - r_2))$. It is one-to-one on x_1R and it equals f on x_1R. So we have a maximal extension $\eta\colon K \to M$, $K \leqslant N$, of μ, which also extends $f \mid T$. As $\lambda = f \mid T$ has no extension from N_1 to M, $K < N$. By Corollary 1.7, we have $N = K_1 \oplus K_2$ such that with $K_2 = \ker \eta$. As $x_1 + x_2 \in \ker \mu \subseteq \ker \eta$, we get $x_1 + x_2 \in K_2$, which shows that $V \cap K_2 \neq 0$. The last part is obvious. □

REMARK. In the above proof, K_2 need not be uniform.

Theorem 1.9. *Let M_R be uniform, N_R a module that is not indecomposable and M be almost T-injective for any proper summand T of N. Then M is almost N-injective if and only if given any uniform summand K of N and uniform submodule V of N such that M is not K-injective and V embeds in K, there exists a proper summand K' of N such that $K' \cap V \neq 0$*

Proof. If M is almost N-injective, by Lemma 1.8, M satisfies the given condition. Conversely, let the given condition hold. Let $f\colon L \to M$, $L < N$ be a maximal homomorphism that cannot be extended from N to M. By the hypothesis, there exists a decomposition $N = N_1 \oplus N_2$ with $0 < N_1 < N$. Set $f_1 = f \mid N_1 \cap L$. Suppose $f_1\colon N_1 \cap L \to M$ cannot be extended from N_1 to M. As M is almost N_1-injective, $N_1 = N_{11} \oplus N_{12}$, such that f_1 is monic on $N_{11} \cap L$, $f(N_{11} \cap L) = M$ and $N_{12} = \ker f_1$.

CASE 1. $N_2 = N_2 \cap L$. We get an R-homomorphism $\lambda\colon N_2 \to N_{11}$ such that for any $x \in N_2$, $\lambda(x) = y \in (N_{11} \cap L)$ whenever $f(x) = f(y)$, i.e. $f(x - y) = 0$. Set $K_2 = \{x - \lambda(x)\colon x \in N_2\}$. Then $K_2 \subseteq \ker f$, $N = N_{11} \oplus N_{12} \oplus N_2 = N_{11} \oplus N_{12} \oplus K_2 = N_{11} \oplus \ker f$. In this case we finish.

CASE 2. $N_2 \cap L < N_2$. Then we also have $N_2 = N_{21} \oplus N_{22}$ such that $f_2 = f \mid N_{21}$ is monic on N_{21}, $f(N_{21} \cap L) = M$ and $N_{22} = \ker f_2$. As $f(N_{11} \cap L) = M = f(N_{21} \cap L)$, we have an isomorphism $\lambda\colon N_{21} \cap L \to N_{11} \cap L$ such that for any $x \in (N_{21} \cap L)$, $y \in (N_{11} \cap L)$, $\lambda(x) = y$ if and only if $f(x) = f(y)$. Then $V = \{x - \lambda(x)\colon x \in N_{21} \cap L\} \subseteq N_{11} \oplus N_{21}$, V is embeddable in N_{11} and $V \subseteq \ker f$.

Now N_{11}, N_{21} are uniform. If $K = N_{11} \oplus N_{21} < N$, then by the hypothesis, M is almost K-injective. Therefore $K = U_1 \oplus U_2$ such that U_1 is uniform, f is monic on $U_1 \cap L$ and $U_2 \subseteq \ker f$, which gives $N = U_1 \oplus \ker f$, as already seen $N_{12} \oplus N_{22} \subseteq \ker f$.

Now suppose $N = N_{11} \oplus N_{21}$. By the hypothesis, $N = U_1 \oplus U_2$ such that $0 < U_2 < N$ and $V \cap U_2 \neq 0$ for the V defined above. As U_2 is uniform, $\ker f \cap U_2 \neq 0$. Thus $f \mid U_2$ is not monic, it follows from Corollary 1.7 that $f \mid U_2 \cap L$ can be extended from U_2 to M. Therefore $U_2 \subset L$. Which gives $U_1 \cap L < U_1$, f is monic on $U_1 \cap L$ and $f(U_1 \cap L) = M$. We get a homomorphism $\mu\colon U_2 \to U_1$ such that $\mu(x) = y$ for any $x \in U_2$, $y \in U_1 \cap L$ whenever $f(x) = f(y)$. Then $V_2 = \{x - \mu(x)\colon x \in U_2\} \subseteq \ker f$. We get $N = U_1 \oplus \ker f$.

Hence in any case $N = U \oplus \ker f$ for some uniform submodule U, f is monic on $U \cap L$ and $f(U \cap L) = M$. By Corollary 1.7, M is almost N-injective. □

Lemma 1.10. *Let $N_R = N_1 \oplus N_2$, where N_i are indecomposable and their rings of endomorphisms are local. Let M_R be uniform and almost N-injective, $f\colon L \to M$, $L < N$ be a maximal homomorphism that cannot be extended from N to M and $N_1 \cap L < N_1$.*

(i) *If $g\colon W \to N_1 \cap L$, $W \leqslant N_2 \cap L$ is a non-zero homomorphism, then either g extends from N_2 to N_1 or g is monic and g^{-1} on $g(W)$ extends from N_1 to N_2.*

(ii) *If V is a uniform submodule of N such that $V \subseteq (N_1 \cap L) \oplus (N_2 \cap L)$ and it naturally embeds in N_2, then there exists a proper summand U of N containing V.*

(iii) *For any uniform submodule $V_1$of N, there exists a proper summand U of N such that $V_1 \cap U \neq 0$.*

Proof. (i) Now $N_1 \cap L < N_1$ and $f \mid (N_1 \cap L)$ cannot be extended from N_1 to M. As M is almost N_1-injective, by Corollary 1.7, f is monic on $N_1 \cap L$ and $f(N_1 \cap L) = M$, which gives that N_1 is uniform. Let $W_1 = (N_1 \cap L) + W$. Define $f'\colon W_1 \to M$, $f'(x + y) = f(x - g(y))$, $x \in N_1 \cap L$, $y \in W$. Then $\ker f' = \{x + y\colon y \in W,\ x = g(y)\} \neq 0$. We get a maximal homomorphism $f_1\colon L_1 \to M$, $L_1 \leqslant N$ which extends f' and $f \mid N_1 \cap L$. Then $L_1 < N$ and $N = U_1 \oplus U_2$, where U_1 is uniform and $U_2 = \ker f_1$. In particular, $\ker f' \subseteq U_2$. By Krull–Schmidt–Azumaya theorem, we can get $N = N_1 \oplus U_2$ or $N = N_2 \oplus U_2$.

CASE 1. $N = N_1 \oplus N_2 = N_1 \oplus U_2$. Let $\pi_i\colon N \to N_i$ be associated projections. Then $\pi_2(U_2) = N_2$. Let $\lambda = \pi_2 \mid U_2$. We have $\lambda^{-1}\colon N_2 \to U_2$. Let $y \in W$. By definition $g(y) + y \in (N_1 \cap L) \oplus (N_2 \cap L)$ and $g(y) + y \in \ker f' \subseteq U_2$. Thus $\lambda(g(y) + y) = y$, which gives $\lambda^{-1}(y) = g(y) + y$. Under the projection $\pi_1\colon N \to N_1$, $\pi_1\lambda^{-1}(y) = g(y)$. Thus $\pi_1\lambda^{-1}\colon N_2 \to N_1$ extends g.

CASE 2. $N = N_1 \oplus N_2 = N_2 \oplus U_2$. Then $\pi_1(U_2) = N_1$. Let $\lambda_1 = \pi_1 \mid U_2$. Then $\lambda_1(g(y) + y) = g(y)$, and as λ_1 is monic, $g(y) = 0$ if and only if $y = 0$, i.e. g monic. Now $\lambda_1^{-1}(g(y)) = g(y) + y$, $\pi_2\lambda_1^{-1}(g(y)) = y$. Thus $\pi_2\lambda_1^{-1}\colon N_1 \to N_2$ extends g^{-1} on $g(W)$.

(ii) Suppose V is a uniform submodule of N such that $V \subseteq (N_1 \cap L) \oplus (N_2 \cap L)$ and V naturally embeds in N_2. Let $W = \pi_2(V)$. We get a homomorphism $g\colon W \to N_1 \cap L$, $g(\pi_2(x)) = \pi_1(x)$, $x \in V$. If g extends to an R-homomorphism g' from N_2 to N_1, then $U = \{x + g'(x)\colon x \in N_2\}$ is a summand of N containing V. If g does not extend from N_2 to N_1, by Case 2, g is monic and g^{-1} on $g(W)$ extends to a homomorphism $g'\colon N_1 \to N_2$. In this case $U' = \{x + g'(x)\colon x \in N_1\}$ contains V and is a summand of N isomorphic to N_1.

Take any uniform submodule V_1 of N such that $V_1 \cap N_1 = 0$. Then V_1 embeds in N_2. As $L \cap N_2 \subset_e N_2$, there exists a non-zero $x = x_1 + x_2 \in V_1$ with $x_1 \in N_1$, $x_2 \in N_2 \cap L$. Once again as $N_1 \cap L \subset_e N_1$, we can choose x to be also have $x_1 \in N_1 \cap L$. Then $V = xR \subseteq (N_1 \cap L) \oplus (N_2 \cap L)$, which, by (ii), is contained in a proper summand K of N. Clearly, $V_1 \cap K \neq 0$. □

Theorem 1.11. *Let $N_R = N_1 \oplus N_2$, where N_i are indecomposable and their rings of endomorphisms are local. Let M_R be uniform. Then M is almost N-injective if and only if either M is N-injective or M is almost N_i-injective for $i = 1, 2$, but is not N_j-injective for some j, say for $j = 1$, and any uniform submodule V of N has non-zero intersection with some indecomposable summand of N.*

Proof. In view of Lemma 1.10, we only need to prove the converse. Suppose the given conditions holds. Let $f\colon L \to M$, $L < N$ be a maximal R-homomorphism that cannot be extended from N to M.

Let $L \cap N_1 < N_1$. Then f is monic on $L \cap N_1$, $f(L \cap N_1) = M$, which gives that $V = \{x - y\colon x \in N_1 \cap L,\ y \in N_2 \cap L \text{ and } f(x) = f(y)\} \neq 0$, $V \subseteq \ker f$ and it embeds in N_2. Suppose $f \mid (N_2 \cap L)$ is monic. Then V naturally embeds in N_1, therefore V is uniform. By the hypothesis, $N = U_1 \oplus U_2$ with $V \cap U_2 \neq 0$. As M is almost U_2-injective and $\ker f \cap U_2 \neq 0$, $U_2 \subseteq L$. Then $L \cap U_1 < U_1$ and f is monic on U_1, $f(U_1 \cap L) = M$. We get $K = \{x - y\colon x \in U_1 \cap L,\ y \in U_2 \text{ and } f(x) = f(y)\} \cong U_2$ and $K \subseteq \ker f$. Trivially, $N = U_1 \oplus \ker f$. If $f \mid N_2 \cap L$ is not monic, then $N_2 \subseteq L$, as above we get $N = N_1 \oplus \ker f$.

Let $L \cap N_1 = N_1$. Then $L \cap N_2 < N_2$ and once again, we continue as before. Hence M is almost N-injective. □

Theorem 1.12. *Let M_R be a uniform module and $N_R = N_1 \oplus N_2 \oplus \cdot \oplus N_k$ a finite direct sum of modules whose rings of endomorphisms are local. Then M is almost N-injective if and only if M is almost N_i-injective for every i, and if for some i, M is not N_i-injective, then for every $j \neq i$, $N_i \oplus N_j$ has the property that for any uniform submodule V of $N_i \oplus N_j$, there exists a proper summand U of $N_i \oplus N_j$ such that $U \cap V \neq 0$.*

Proof. In view of Theorem 1.11, we only need to prove the converse. Let $f\colon L \to M$, $L < N$ be a maximal homomorphism that cannot be extended from N to M. Then for some i, say for $i = 1$, $f_1 = f \mid (N_1 \cap L)$: $(N_1 \cap L) \to M$ cannot be extended from N_1 to M. As M is almost N_1-injective, f_1 is monic and $f(N_1 \cap L) = M$. Consider any $j \neq 1$ and $f_j = f \mid (N_j \cap L)$. By Theorem 1.11, M is $N_1 \oplus N_j$-injective. By Corollary 1.7, $N_1 \oplus N_j = U_1 \oplus U_2$ for some uniform submodules U_1 and $U_2 \subseteq \ker f$. Thus $U_2 \subseteq L$ and $L \cap U_1 < U_1$. This proves that in the decomposition $N_R = N_1 \oplus N_2 \oplus \cdots \oplus N_k$, we can replace $N_1 \oplus N_j$ by a $U_1 \oplus U_2$ with $U_2 \subseteq \ker f$. This proves that $N = V \oplus \ker f$ for some uniform submodule V. By Corollary 1.7, M is almost N-injective. □

The above theorem generalizes the following result by Baba [4].

Theorem 1.13. *Let U_k be a uniform module of finite composition length for $k = 0, 1, \ldots, n$. Then the following two conditions are equivalent.*
(1) *U_0 is almost $\bigoplus \sum_{k=1}^{n} U_k$-injective.*
(2) *U_0 is almost U_k-injective for $k = 1, 2, \ldots, n$ and if $soc(U_0) \cong soc(U_k) \cong soc(U_l)$ for some $k, l \in \{1, 2, \ldots, n\}$ with $k \neq l$, then*
  (i) *U_0 is U_k and U_l-injective or*
  (ii) *$U_k \oplus U_l$ is extending for simple modules, in the sense that any simple submodule of $U_k \oplus U_l$ is contained in a uniform summand of $U_k \oplus U_l$.*

The following is known.

Lemma 1.14. *Let $\{N, V_i\}$ be a family of modules over a ring R. Then $M = \bigoplus \sum_{i=1}^{n} V_i$ is almost N-injective if and only if every V_i is almost N-injective.*

Lemma 1.15. *Let U_1, U_2 be two uniform modules such that U_2 is almost U_1-injective. Let V be a uniform submodule of $N = U_1 \oplus U_2$ such that $V \cap U_2 = 0$. Then there exists a uniform summand K of N isomorphic to U_1 or U_2, which contains V. Any uniform submodule of N has non-zero intersection with some uniform summand of N.*

Proof. Let $\pi_i : N \to U_i$ be associated projections. The hypothesis gives a homomorphism $\sigma : \pi_1(V) \to \pi_2(V)$, $\sigma(\pi_1(x)) = \pi_2(x)$ for any $x \in V$. We get a maximal homomorphism $\eta : L \to U_2$, $L \leqslant U_1$ extending σ. Then either $L = U_1$, or η is monic and $\eta(L) = U_2$. In the former case, take $K = \{y + \eta(y) : y \in U_1\}$ and in the later case, take $K = \{y + \eta(y) : y \in L\}$. The second part is immediate. □

We get an alternative proof of the following result by Harada [10].

Theorem 1.16. *Let $M = M_1 \oplus M_2 \oplus \cdots \oplus M_k$, where each M_i has its ring of endomorphisms local. Then the following are equivalent.*
(i) *M is almost self-injective.*
(ii) *For any i, j, M_i is almost M_j-injective.*

Proof. Suppose M is almost self-injective. Then each M_i is almost self-injective. Therefore each M_i is uniform. As M_i is almost M-injective, by Lemma 1.14, condition (ii) holds. Fix an i, $1 \leq i \leq k$. Consider any $1 \leq r, s \leq k$. By the hypothesis, M_s is almost M_r-injective. By Lemma 1.15, given any uniform submodule V of $W = M_r \oplus M_s$, there exists a uniform summand K of W such that $V \cap K \neq 0$. By Theorem 1.12, M_i is almost M-injective. As M is a direct sum of M_i's, it follows from Lemma 1.14 that M is almost self-injective. □

2. Commutative rings

Proposition 2.1. *Let R be any commutative indecomposable ring and Q be its quotient ring. If R is almost self-injective. Then the following hold.*
(i) *If $a, b \in R$ and $ann(a) \subseteq ann(b)$, then $bR \subseteq aR$, or $aR < bR$ and $a = bc$ for some regular element $c \in R$.*
(ii) *If $a, b \in R$ are regular, then either $aR \subseteq bR$ or $bR \subseteq aR$.*
(iii) *Q_R is injective and uniform.*
Conversely, if R satisfies conditions (i) *and* (iii), *then R is almost self-injective.*

Proof. Let a, b be two elements of R such that $ann(a) \subseteq ann(b)$. We have a homomorphism $\sigma : aR \to bR$, $\sigma(a) = b$. If σ extends to an endomorphism η of R_R, then $b = ac$, where $c = \eta(1)$, which gives $bR \subseteq aR$. Suppose σ does not extend to an endomorphism of R_R. Then $b \notin aR$. As R_R is uniform, by Corollary 1.7, there exists a maximal extension η: $L \to R$. $L < R$ of σ such that it is monic and $\eta(L) = R$. Thus $L = cR$ where c is such that $\eta(c) = 1$. This c is regular, non-unit and $a = bc$. This proves (i). Now (ii) is immediate from (i).

(iii) Let σ: $A \to Q_R$, $A < R_R$ be a homomorphism. Suppose $\sigma(A) \subseteq R$. If it extends to an $\eta \in End(R_R)$ and $\eta(1) = c$, then multiplication by c gives an endomorphism of Q_R extending σ. Otherwise for some regular element $c \in R$ we have an η: $cR \to R$ with $\eta(c) = 1$, which extends σ. Then $c^{-1} \in Q$ and multiplication by c^{-1} gives an R-endomorphism of Q_R extending σ. This proves that if $\sigma(A) \subseteq R$, then σ extends to an endomorphism of Q_R.

Suppose $\sigma(A) \nsubseteq R$. Let S be the set of regular elements of R. Then $Q = R_S$. Set $B = \sigma(A)$. Let $B' = B \cap R$. Then $B \subseteq B'_S$. Let $A' = \sigma^{-1}(B')$ and $\sigma_1 = \sigma \mid A'$. Then $\sigma_1(A') = B' \subseteq R$. Therefore σ_1 extends to an endomorphism η of Q_R. Let $x \in A$. Then $\sigma(x) = yc^{-1}$ for some regular element $c \in R$, $y \in B'$. Which gives $\sigma(xc) = y$, $xc \in A'$, $\eta(xc) = y$, If $\eta(x) = z$, then $y = zc$, $\sigma(x) = z$. Hence η extends σ. This proves that Q_R is injective. It also gives that $Q_R = E(R_R)$. As R is uniform, Q_R is uniform.

Conversely, let R satisfy the given conditions. Let f: $A \to R_R$, $A < R$ be a homomorphism that cannot be extended in $End(R_R)$. By (iii), σ extends to an R-endomorphism η of Q. It follows from (ii) that if an $x \in Q$ is regular, then $x \in R$ or $x^{-1} \in R$. Now $\eta(1) = ac^{-1}$ for some $a, c \in R$ with c regular.

CASE 1. a is regular. It follows from (ii) that $\eta(1) \in R$ or $\eta(1)^{-1} \in R$. In the former case, $\eta \mid R$ is an extension in $End(R_R)$ of σ. Suppose $\eta(1)^{-1} \in R$, but $\eta(1) \notin R$.

Then for any $x \in A$, $\sigma(x) = \eta(1)x$, gives $x = \sigma(x)\eta(1)^{-1}$. So that $A \subseteq \eta(1)^{-1}R < R$. We have an isomorphism $\lambda \colon \eta(1)^{-1}R \to R$ with $\lambda(\eta(1)^{-1}) = 1$. Then λ extends σ.

CASE 2. a is not regular. By (i), $a = cr$ for some $r \in R$, therefore $\eta(1) = ac^{-1} = r$ and $\eta \mid R$ is an extension in $End(R_R)$ of σ. Hence R is almost self-injective. □

Theorem 2.2. *Let R be a commutative, indecomposable, almost self-injective ring. Let Q be the quotient ring of R.*
(i) *Either $Q = R$ or there exists a prime ideal P in R such that $Q = R_P$.*
(ii) *R is a local ring.*

Proof. Suppose $Q \neq R$. Then R has a regular element that is not a unit. Let $a \in R$ be regular but not a unit. We claim that $A = \bigcup_{k=1}^{\infty} a^k R$ is the unique maximal prime ideal such that $a \notin A$. And we also prove that any element in $R \setminus A$ is regular. Let $b \in R \setminus A$. Then for some k, $b \notin a^k R$. It follows from Proposition 2.1 that b is regular and $a^k R < bR$. Thus A is a prime ideal of R. As $a^2 R < aR$, $a \notin A$. Let P' be a maximal prime ideal in R such that $a \notin P'$. Suppose $P' \not\subseteq A$. Then there exists a $b \in P'$ such that $b \notin A$. Then, as seen above, $a^k \in bR \subseteq P'$ for some $k \geq 1$, which gives $a \in P'$, which is a contradiction. Hence $A = P'$. Thus to each regular non-unit $a \in R$, is associated a unique maximal prime ideal $P_a = \bigcap_{k=1}^{\infty} a^k R$ such that $a \notin P_a$. Every element of $R \setminus P_a$ is regular. It follows from Proposition 2.1 (ii) that the family of P_a is linearly ordered. Let P be the intersection of these P_a's. Then $R \setminus P$ is the set of all regular elements in R. Hence $Q = R_P$.

Let P' be a prime ideal of R other than P. Suppose $P' \not\subseteq P$. As $R \setminus P$ consists of regular elements, there exists a regular element $a \in P'$. Then $P_a \subseteq P'$, so $P \subseteq P_a \subseteq P'$. Let P_1, P_2 be two prime ideals not contained in P. Suppose $P_1 \not\subseteq P_2$. Then there exists an $a \in P_1 \setminus P_2$. As $a \notin P$, it is regular. Let $b \in P_2$. By Proposition 2.1 (i), $b \in a^k R$ for any $k \geq 1$. It follows that $P_2 \subseteq P_a$. Trivially, $P_a \subseteq P_1$. Hence $P_2 \subseteq P_1$. It follows that the family F of those prime ideals of R that are not contained in P is linearly ordered and each member of F contains P. Hence R is local. □

An indecomposable, commutative, almost self-injective ring need not be a valuation ring.

EXAMPLE 1. Let F be a field and $Q = F[x, y]$ with $x^2 = 0 = y^2$. Then $Q = F + Fx + Fy + Fxy$ is a local, self-injective ring. Choose F to be the quotient field of a valuation domain $T \neq F$. Set $R = T + Fx + Fy + Fxy \subset Q$. Any $0 \neq a \in F$ is such that either $a \in T$ or $a^{-1} \in T$, $J(Q) = Fx + Fy + Fxy \subset R$ and is nilpotent. Any element of R not in $J(Q)$ is regular and is of the form au with $a \in T$ and u a unit in R. By using this it follows that Q is the classical quotient ring of R. Let A be a non-zero ideal of R. Then $\{a \in F \colon axy \in A\}$ is a non-zero T-submodule of F, which shows that R_R is uniform. The ideals $Fx + Fxy$, $Fy + Fxy$ in R are not comparable. Therefore R_R is not uniserial. If $A \not\subseteq J(Q)$, then some $au \in A$ with $0 \neq a \in F$, u a unit in R, so $a \in A$; which gives $J(Q) = aJ(Q) \subset A$.

Let $\sigma\colon A \to R$, $A < R_R$ be an R-homomorphism. Now $A' = \{\alpha v\colon \alpha \in F,\ v \in A\}$ is an ideal of Q containing A, $\eta\colon A' \to Q$, such that for any $c \in F$, $v \in A$, $\eta(cv) = c\sigma(v)$ is a Q-homomorphism. As Q is self-injective, there exists an $\omega \in Q$ such that $\eta(cv) = \omega cv$ for any $cv \in A'$. If $\omega \in R$, obviously σ extends to an endomorphism of R_R. Suppose $\omega \notin R$. Then $\omega = c^{-1}u$ for some non-zero $c \in T$ which is not a unit in T, and u is a unit in R. Thus $g = cu^{-1} \in R$. For any $v \in A$, $\sigma(v) = g^{-1}v \in R$, $v = g\sigma(v) \in gR$. Thus $A < gR$ and $\lambda\colon gR \to R$, $\lambda(g) = 1$, extends σ. Hence R is a local ring that is almost self-injective and R_R is not uniserial.

Lemma 2.3. *Let A, B be two rings such that A is local and M be an $(A.B)$-bimodule. Let $R = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$. Then $e_{11}R$ is uniform if and only if M_B is uniform and ${}_AM$ is faithful.*

Proof. Let $e_{11}R$ be uniform. Let $x = a_{11}e_{11} + a_{12}e_{12}$, $y = b_{11}e_{11} + b_{12}e_{22}$ be two non-zero elements in $e_{11}R$. Then for some $r = r_{11}e_{11} + r_{12}e_{12} + r_{22}e_{22}$, $s = s_{11}e_{11} + s_{12}e_{12} + s_{22}e_{22} \in R$, $xr = ys \neq 0$. Which gives $a_{11}r_{11} = b_{11}s_{11}$, $a_{11}r_{12} + a_{12}r_{22} = b_{11}s_{12} + b_{12}s_{22}$.

CASE 1. $a_{11} = 0$ and $b_{11} = 0$. Then $a_{12}r_{22} = b_{12}s_{22} \neq 0$, which gives that M_B is uniform.

CASE 2. $a_{11} \neq 0$, $b_{11} = 0$, $a_{12} = 0$, $b_{12} \neq 0$. Then $a_{11}r_{12} = b_{12}s_{22} \neq 0$. Therefore $a_{11}M \neq 0$. Hence ${}_AM$ is faithful.

Conversely, let M_B be uniform and ${}_AM$ be faithful. Then $e_{12}M$ is a uniform right ideal of R, and for any $x \neq 0$ in $e_{11}R$, $xR \cap e_{12}M \neq 0$. Hence $e_{11}R$ is uniform. □

The above lemma helps to get examples of non-commutative, almost self-injective rings.

EXAMPLE 2. Let A be a valuation domain and K be its quotient field. Let $R = \begin{bmatrix} A & K \\ 0 & B \end{bmatrix}$, where B is a valuation ring contained in K such that K is a quotient field of B. By Lemma 2.3, $e_{11}R$ is uniform. Let $f\colon L \to e_{11}R$, $L < e_{11}R$ be a maximal homomorphism that cannot be extended to an endomorphism of $e_{11}R$. Now $e_{12}K$ is a quasi-injective R-module and $f(L \cap e_{12}K) \subseteq e_{12}K$. Therefore $f \mid (L \cap e_{12}K)$ can be extended to an R-endomorphism g of $e_{12}K$. As f is monic on $L \cap e_{12}K$, f is monic. Then $f'\colon L + e_{12}K \to e_{11}R$, $f'(x + y) = f(x) + g(y)$ for any $x \in L$, $y \in e_{12}K$ extends f. Which gives $e_{12}K \subseteq L$, $L = (e_{11}A \cap L) \oplus (e_{12}K)$ as an abelian group. Now $f(e_{12}) = e_{12}b$ for some $b \in K$. Then $f(e_{12}c) = e_{12}cb$ for every $c \in K$. Let $x = a_{11}e_{11} + a_{12}e_{12} \in L$ with $a_{11} \neq 0$. Then $e_{11}a_{11} \in L$ and $f(e_{11}a_{11}) = e_{11}a_{11}u$ for some $u \in K$. We get $f(e_{12}a_{11}) = f(e_{11}a_{11})e_{12} = e_{12}u$. On the other hand, $f(e_{12}a_{11}) = e_{12}a_{11}b$. Hence $u = a_{11}b$. Thus $f(x) = xb = (e_{11}b)x$ for every $x \in L$. If $b \in A$, f can be extended to an R-endomorphism of $e_{11}R$ given by left multiplication by $e_{11}b$. Suppose $b \notin A$. Then $b^{-1} \in A$. Then the R-endomorphism h of $e_{11}R$ given by left multiplication by $e_{11}b^{-1}$ is such that $hf(z) = z$ for every $z \in L$. Hence $e_{11}R$ is almost self-injective.

Any R-homomorphism $\lambda\colon L \to e_{11}R$, $L < e_{22}R$ is such that $f(L) \subseteq e_{12}K$. As $e_{22}R = e_{22}B$, λ can be extended from $e_{22}R$ to $e_{12}R$. It follows that $e_{11}R$ is $e_{22}R$-injective. Let $f\colon L \to e_{22}R$, $L < e_{11}R$ be a non-zero homomorphism. Now $L \cap e_{12}K \neq 0$. As $e_{22}R = e_{22}B$ it follows that for some $b \in K$, $g = f \mid L \cap e_{12}K$ is such that $g(e_{12}x) = e_{22}xb$ for any $e_{12}x \in L \cap e_{12}K$, therefore f is monic. If an $x = a_{11}e_{11} \in L$, then $f(x) = 0$. This proves that $L \subseteq e_{12}K$. Then $h\colon e_{22}R \to e_{11}R$, $h(e_{11}x) = e_{12}xb^{-1}$, $x \in B$ is such that $hf(u) = u$ for every $u \in L$. Hence $e_{22}R$ is almost $e_{11}R$-injective. By Theorem 1.16, R_R is almost self-injective.

By using Theorem 1.16, one can easily prove that the ring $T_n(D)$ of upper triangular matrices over a division ring D is almost right self-injective.

3. Von Neumann regular rings

Theorem 3.1. *Let R be a von Neumann regular ring. Then R is almost right self-injective if and only if for any maximal homomorphism $\sigma\colon A \to R_R$, $A < R_R$ which cannot be extended to an R-endomorphism of R_R, there exist non-zero idempotents e, $f \in R$, such that $eR \subseteq A$, $\sigma \mid eR$ is a monomorphism, $\sigma(eR) = fR$, $\sigma(A \cap (1-e)R) \subseteq (1-f)R$.*

Proof. Let R be almost right self-injective, Let $\sigma : L \to R_R$ be a maximal R-homomorphism that cannot be extended to an endomorphism of R_R. By definition, $R = eR \oplus (1-e)R$ and there exists an R-homomorphism $h\colon R_R \to eR$ such that $hf(x) = ex$ for every $x \in L$. There exists $u^2 = u \neq 0$ in $L \cap eR$ such that $eR = uR \oplus (e-u)R$, and $e-u$ is an idempotent orthogonal to u. Let $\pi : eR \to uR$ be a projection with kernel $(e-u)R$. Then $\pi h\sigma(x) = ux$. So we take $e = u$ and $h = \pi h$. As $h(R) = eR$, $R = gR \oplus (1-g)R$ for some idempotent $g \in R$ such that $\ker h = (1-g)R$. Now $h(R) = eR = h\sigma(eR)$, we get $R = \sigma(eR) \oplus \ker h$. Thus, there exists an idempotent $f \in R$, such that $R = fR \oplus (1-f)R$, $\sigma(eR) = fR$, $\ker h = (1-f)R$ and $h \mid fR$ is the inverse of $\sigma \mid eR$. Clearly, for any $x \in (1-e)R \cap A$, $h\sigma(x) = 0$ gives $\sigma(x) \in (1-f)R$.

Conversely, let R satisfy the given conditions. Let $\sigma : L \to R_R$ be a maximal homomorphism that cannot be extended to an endomorphism of R_R. Then there exist non-zero idempotents e, $f \in R$ such that $L = eR \oplus (L \cap (1-e)R)$, σ is monic on eR, $\sigma(eR) = fR$, $\sigma((1-e)R \cap L) \subseteq (1-f)R$. We define $h\colon R \to eR$ as follows. Let $y \in R$. Then $y = fy + (1-f)y$. Now $fy = \sigma(ex)$ for some uniquely determined $ex \in eR$. Set $h(y) = ex$. If follows that for any $x \in L$, $h\sigma(x) = ex$. Hence R is almost right self-injective. □

Corollary 3.2. *Any von Neumann regular ring R that is right CS, is almost right self-injective.*

Proof. Let $\sigma: A \to R$, $A < R_R$ be a non-zero R-homomorphism. As $\ker\sigma$ is not large in A, there exists a non-zero idempotent $e \in A$ such that $eR \cap \ker\sigma = 0$. Then $\sigma(eR) = fR$ for some idempotent $f \in R$. Let B be a complement of eR in R_R containing $\ker\sigma$. As R is right CS, $B = bR$. We get $R = eR \oplus B$. Hence we can take e to be such that $B = (1-e)R$. Now $A = eR \oplus (A \cap (1-e)R)$. Let $a \in A \cap (1-e)R$ such that $\sigma(a) \in fR$. Then for some $x \in eR$, $\sigma(x) = \sigma(a)$, $x - a \in \ker\sigma$, $x \in B$, so $x = 0$. Hence $\sigma(A \cap (1-e)R) \cap fR = 0$. Let C be a complement of fR containing $\sigma(A \cap (1-e)R)$, We again have $R = fR \oplus C$. We get an idempotent $g \in R$ such that $fR = gR$, $C = (1-g)R$. By Proposition 2.1, R is almost right self-injective. □

REMARK. Any von Neumann regular ring that is right CS is right continuous. In [7], examples of continuous commutative von Neumann regular rings that are not self-injective are given. Hence a von Neumann regular almost right self-injective need not be right self-injective.

Proposition 3.3. *Any von Neumann regular ring in which all idempotents are central, is almost self-injective.*

Proof. Let $\sigma: A \to R$, $A < R_R$ be a non-zero homomorphism. We get a non-zero idempotent $e \in A$ such that $f \mid eR$ is monic. Let $\sigma(e) = x$, then $x = xe = ex$ gives $\sigma(eR) \subseteq eR$. Suppose $\eta = \sigma \mid eR$. Now $\sigma(eR) = xR = fR$ for some idempotent $f \in eR$. Therefore $x = xf$, $\eta(e-f) = xf(e-f) = 0$, $e = f$. Hence $\sigma(eR) = eR$. It also follows that $\sigma(A \cap (1-e)R) \subseteq A \cap (1-e)R$. Hence R is almost right self-injective. □

The following result determines a class of von Neumann regular rings that are not almost right-injective.

Theorem 3.4. *Let R be an almost right self-injective, von Neumann regular ring.*
(i) *Any complement of a minimal right ideal of R is principal*
(ii) *Any minimal right ideal of R is injective.*

Proof. Let A be a minimal right ideal of R. Then $A = eR$ for some indecomposable idempotent $e \in R$. Let C be a complement of eR. We get a maximal homomorphism $\sigma: L \to R_R$, $L \leqslant R_R$ such that $eR \oplus C \subseteq L$, σ is identity on eR, and is zero on C.

CASE 1. $L = R$. Then $R_R = fR \oplus \ker\sigma$, But $C \subseteq \ker\sigma$, therefore fR is uniform, hence minimal. As $e \notin \ker\sigma$, we get $R_R = eR \oplus \ker\sigma$. We get $C = \ker\sigma$, a principal right ideal.

CASE 2. $L < R_R$. By Theorem 3.1, there exist non-zero idempotents $f \in L$, $g \in R$ such that $\sigma \mid fR$ is monic, $\sigma(fR) = gR$, $\sigma(L \cap (1-f)R) \subseteq (1-g)R$. Now $C \subseteq \ker\sigma \subseteq (1-f)R$. Thus fR is simple, as in Case 1. $L = eR \oplus ((1-f)R \cap L)$ and $eR \nsubseteq (1-f)R$. As $C \subset_e (1-f)R$, we get $C = (1-f)R$.

Suppose A is not injective. Let $E = E(A)$. We get $x \in E \setminus A$. Then $A < xR$. Let $C = ann_R(x)$, As xR is non-singular, C is a closed right ideal of R and its complement in R_R is uniform. If C were principal, we would get $R = B \oplus C$ with B simple, which is not possible, as xR is not simple. Hence C is not principal. Let H be a complement of C. As H is uniform, it is simple. This contradicts (i), Hence A is injective. □

EXAMPLE 3. Let F be any field and R be the ring of column finite matrices over F, indexed by the set $\mathcal{N}$ of positive integers. This ring is right self-injective. Let S be subring of R consisting of matrices that are also row finite. Then R is a maximal right quotient ring of S. Consider the matrix unit e_{11}. Then $e_{11}S$ is a minimal right ideal of S. However $e_{11}S < e_{11}R$ and $e_{11}R$, as a right S-module is injective hull of $e_{11}S$. Hence S is not almost right self-injective.

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