Gupta, R. N.
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# SELF-INJECTIVE QUOTIENT RINGS AND INJECTIVE QUOTIENT MODULES*

RAM NIWAS GUPTA

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## 1. Introduction

Until further notice, we assume that $R$ is a ring (with unity) and $S$ is a multiplicatively closed set of regular elements of $R$ such that $R$ satisfies the multiplicity condition with respect to $S$ (for every $a$, $s$ in $R$, $s$ in $S$, there exist $a_1$, $s_1$ in $R$, $s_1$ in $S$ such that $a\,s_1=s\,a_1$). Let $Q$ denote the (Asano's) quotient ring $R_S$ of $R$. If 1 denotes the identity of $R$, $1'$ the identity of $Q$, then $1.1'=1.s.s^{-1}=(1.s)\,s^{-1}=s.s^{-1}=1'$. Also $1.1'=1$, because $1'$ is the identity of a bigger ring $Q$. So that the identities of the two rings coincide.

Let $M$ be a (unital) right $R$-module. $M$ is said to be $S$-free if $m\,s=0$, $m\in M$, $s\in S$ implies $m=0$. $M$ is said to be $S$-divisible iff $Ms=M\ \forall s\in S$. If $M$ is both $S$-free and $S$-divisible $R$-module, then the module composition $M\times R\to M$ can be extended to $M\times Q\to M$ in one and only one way such that $M$ becomes a $Q$-module, by defining $m(a.s^{-1})=m'$ where $m'$ is such that $m's=m.a$ (note that $m'$ exists because of $S$-divisibility and is unique because of $S$-freeness). This composition is well defined, because if $a.s^{-1}=a_1.s_1^{-1}$, suppose $m'.s=m.a$, $m''.s_1=m.a_1$. Now there exist $s_2$, $s_3$, $s_2\in S$ such that $s.s_2=s_1.s_3$. Then $a\,s_2=a_1s_3$. $m.a.s_2=m.a_1.s_3$. $m'.s.s_2=m''.s_1.s_3$ implies $m'=m''$. It is not very difficult to check that $M$ is $Q$ module with this composition.

However if $M$ is $S$-free $R$-module, then there exists a $Q$-module $M'$ such that $M_R\subset M'_R$ and $M'=MQ=\{m.s^{-1};\ m\in M,\ s\in S\}$. This module $M'$ is unique upto isomorphism over $M$. There are various construction for this module $M'$ available in the literature.

**Asano's construction.** In $M\times Q$ define $(m, q)\sim(m', q')$ if there exists $s\in S$ such that $q.s\in R$, $q's\in R$ and $m(q\,s)=m'(q'\,s)$. The relation $\sim$ can be verified to be an equivalence relation. In $M'$=the set of equivalence classes of $M\times Q$ define '$+$' and '$\cdot$' as follows: $\overline{(m, q)}+\overline{(m', q')}=\overline{(m(q\,s)+m(q\,s), s^{-1})}$ where $s\in S$ is such that $q\,s\in R$ and $q's\in R$. $\overline{(m, q)}.q'=\overline{(m, q\,q')}$. It can be verified that these compositions are well defined and $M'$ is $Q$ module. The mapping

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$\sigma: m \to \overline{(m, 1)}$ is a $R$-isomorphism of $M_R$ into $M'_R$ so that identifying $\sigma M$ with $M$, we find that

$$\overline{(m, q)} = \overline{(m, a s^{-1})} = \overline{(m, a)} s^{-1} = \overline{(ma, 1)} s^{-1} = (m, a) s^{-1},$$

so that

$$M' = MQ = \{m s^{-1}: m \in M, s \in S\}.$$

**Another construction of M'.** Let $M' = M \otimes_R Q$. $M'$ is a $Q$-module. The mapping $\sigma: m \to m \otimes 1$ embeds $M_R$ into $M'_R$ and

$$\sum_{i=1}^{n} (m_i \otimes q_i) = \sum_{i=1}^{n} (m_i \otimes a_i s_i^{-1}) = \sum_{i=1}^{n} (m_i \otimes a_i s_i^{-1}) s s^{-1},$$

where $s$ is such that $a_i s_i^{-1} s \in R$ $i = 1, 2, \cdots, n$, so that

$$\sum_{i=1}^{n} (m_i \otimes q_i) = (\sum_{i=1}^{n} m_i (a_i s_i^{-1} s)) \otimes s^{-1} = (m \otimes 1) s^{-1}$$
$$= (\sigma m) s^{-1}, \quad \text{where} \quad m = \sum_{i=1}^{n} m_i (a_i s_i^{-1} s).$$

Note that if $M$ is $S$-free $R$ module, then $MQ = M$ iff $M$ is $S$-divisible.

The starting point of this paper is the following result: If $M$ is a $Q$-module, the $M_Q$ is injective iff $M_R$ is injective. This result is used to prove that if $M$ is a $Q$ module then injective dimension of $M_Q$ = injective dimension of $M_R$. The following corollary follows:

The right Global dim. of $Q \leqslant$ right Global dim. of $R$.

If $M$ is any $Q$-module, then the injective hulls $E(M_Q)$ and $E(M_R)$ are seen to coincide. The following result is also proved: Every $S$-free $S$-divisible module over $R$ is injective if and only if $Q$ is semi-simple Artinian ring.

Similar results about quasi-injective modules have also been proved.

If $M$ be an $S$-free $R$ module, then the $Q$ module $M'$ (mentioned above) when regarded as a right $R$ moudle is an essential extension of $M$. Necessary and sufficient condition on $M$ such that $M'$ becomes the injective hull of $M$ are obtained. This result is applied to characterize rings whose classical quotient rings are qusi-Frobenius rings.

These results have also above been applied to obtain necessary conditions and sufficient conditions on a ring under which the Utumi's ring of quotients of the ring is an Asano's quotient ring. Necessary and sufficient conditions, when $R_r^{\Delta} = 0$, follow.

Hereditary orders in semi-simple Artinian rings have been characterized. These rings are found to 'resemble' Dedekind domains, which are precisely hereditary orders in commutative fields. A principal right ideal semi-prime ring is found to be a hereditary ring.

It is proved that if $R$ is a semi-prime Goldie ring, and $Q$ its semi-simple

Artinian classical right quotient ring, then $Q_R$ is never projective except when $Q=R$.

Returning back to arbitrary $R$, with an Asano's quotient ring $Q=R_S$, if $M$ be a projective $R$ module, then trivially $M$ is $S$-free. It is proved that $M'(=MQ)$ is a projective $Q$-module. This result is used to rededuce our earlier result: right Global dim. $Q\leqslant$ right Global dim. $R$.

Finally, necessary and sufficient conditions on $R$ such that $Q(=R_S)$ becomes a hereditary ring are obtained. An immediate corollary thereof being: if $R$ is right hereditary, then $Q$ is also right hereditary.

## 2. Modules over Asano's quotient ring

**2.1. Theorem.** *Let $R$ be a ring and $S$ be a multiplicatively closed set of regular elements of $R$ such that $R$ satisfies the multiplicity condition with respect to $S$. Let $Q$ denote the Asano's quotient ring $R_S$ of $R$ with respect to $S$. Let $M$ be a module over $Q$. Then $M$ is clearly a module over $R$, $M$ is an injective $Q$-module if and only if $M$ is an injective $R$-module.*

Proof. Let $M$ be an injective $Q$ module. Let $f\colon I_R\to M_R$, $I$ a right ideal of $R$. We know that $IQ=\{as^{-1}\colon a\in I,\ s\in S\}$ Define $f'\colon IQ\to M$ as follows:

$$f'(as^{-1})=f(a)s^{-1}$$

$f'$ is well defined for if

$as^{-1}=a_1s_1^{-1}$, $a$, $a_1\in I$, $s$, $s_1\in S$, then there exist $s_2\in S$ such that (see Asano [1]) $s^{-1}s_2\in R$ and $s_1^{-1}s_2\in R$. Clearly $a(s^{-1}s_2)=a_1(s_1^{-1}s_2)$, therefore $f(a)(s^{-1}s_2)=f(a_1)(s_1^{-1}s_2)$.
Post multiplying by $s_2^{-1}$ we get $f(a)s^{-1}=f(a_1)s_1^{-1}$.
We check that $f'$ is a $Q$-homomorphism of $IQ$ into $M$.

$$f'(as^{-1}+a's'^{-1})=f'((as^{-1}+a's'^{-1})s_1s_1^{-1})$$

where $s_1$ is an element of $S$ such that $s^{-1}s_1$, $s'^{-1}s_1\in R$

$$\begin{aligned}&=(f(as^{-1}s_1+a's'^{-1}s_1))s_1^{-1}\\&=(f(a)s^{-1}s_1+f(a')s'^{-1}s_1)s_1^{-1}\\&=f(a)s^{-1}+f(a')s'^{-1}\\&=f'(as^{-1})+f'(a's'^{-1})\end{aligned}$$

$$\begin{aligned}f'(as^{-1}\cdot rs_1^{-1})&=f'(ar_1s_2^{-1}s_1^{-1})\qquad\text{where } s^{-1}r=r_1s_2^{-1}\\&=f(ar_1)(s_1s_2)^{-1}\\&=f(a)r_1s_2^{-1}s_1^{-1}\\&=f(a)s^{-1}rs_1^{-1}\\&=f'(as^{-1})\cdot rs_1^{-1}\end{aligned}$$

$M$ being an injective $Q$-module there exists $m \in M$ such that

$$f'(x) = mx \ \forall x \in IQ .$$

But $f'(x)=f(x)$ $\forall x \in I$. Therefore $f(x)=mx$ $\forall x \in I$.

Conversely suppose $M_R$ is injective. Let $f: I \to M$, be $Q$-homomorphism, where $I$ is a right ideal of $Q$. Then let $J=I \cap R$. $J$ is a right ideal of $R$ and $JQ=I$. Let $f'$ denote the restriction of $f$ to $J$. Then $f'$ is clearly a $R$-homomorphism of $J$ into $M$. As $M$ is an injective $R$-module, there exists $m \in M$ such that $f'(x)=mx$ $\forall x \in J$. Any element of $I$ is of the form $xs^{-1}$, $x \in J$, $s \in S$.

$$f(xs^{-1}) = f'(x)s^{-1} = m(xs^{-1})$$

Hence $M$ is an injective $Q$-module.

**2.2. Theorem.** *Let $M$ be a module over $Q$, then $M_Q$ is quasi-injective iff $M_R$ is quasi-injective.*

Proof. Assume that $M$ is a quasi-injective module over $Q$. Let $f: N \to M$ be a $R$-homomorphism where $N$ is a $R$-submodule of $M$. We know that $NQ=\{ns^{-1}: n \in N, s \in S\}$
Define a mapping $f': NQ \to M$

$$f'(ns^{-1}) = f(n)s^{-1} .$$

It can be checked that $f'$ is well defined and $f'$ is a $Q$-homomorphism of $NQ$ into $M$. Also $f'$ coincides with $f$ on $N$. As $M$ is $Q$-quasi-injective, therefore there exists an extension $f''$ of $f'$ such that $f'' \in \mathrm{Hom}_Q(M, M)$. Clearly $f'' \in \mathrm{Hom}_R(M, M)$ and $f''$ is an extension of $f$.

Conversely suppose that $M_R$ is quasi-injective. Let $f: N \to M$ be a $Q$-homonorphism, where $N$ is $Q$-submodule of $M$. $N$ is also $R$-submodule of $M$. There exists $g \in \mathrm{Hom}_R(M, M)$ such that $g$ coincides with $f$ on $N$. We prove that $g$ is infact a $Q$-homomorphism.

$$g(m.rs^{-1})s = g(mrs^{-1}s) = g(mr) = g(m)r .$$

Therefore

$$g(mrs^{-1}) = g(m)rs^{-1} \ \forall r \in R, \ s \in S, \ m \in M .$$

**2.3. Theorem.** *Every $S$-divisible $S$-free module over $R$ is injective iff $Q$ $(=R_S)$ is a semi-simple Artinian ring.*

Proof. Let every $S$-free $S$-divisible module over $R$ be injective. Then in order to prove that $Q$ is semi-simple Artinian, we shall prove every module over $Q$ is injective. Let $M$ be any $Q$ module, then $M$ is $S$-free and $S$-divisible $R$-module ($ms=0$ implies $mss^{-1}=0$ $m=0$, $Ms \supset (Ms^{-1})s=M \supset Ms$, $M=Ms$ $\forall s$

$\in S$). Therefore $M$ is an injective $R$-module. By theorem 2.1 $M$ is an injective $Q$ module. Hence $Q$ is semi-simple Artinian, see Cartan-Eilenberg [3, page 11, Theorem 4.2].

Conversely let $Q$ be a semi-simple Artinian ring, then if $M$ be a $S$-free and $S$-divisible $R$-module, $M$ can be regarded as a $Q$-module, the module composition $M\times Q\to M$ being such that it extends the original module composition $M\times R\to M$, in one and only one way (see introduction). $Q$ being semi-simple Artinian, any module over $Q$ is injective, therefore $M$ is an injective $Q$-module. Consequently $M$ is an injective $R$-module by theorem 2.1.

**2.4.** DEFINITION. Let $M$ be a module over $R$. $M$ is said to be a torsion free module if $mx=0$, $m\in M$, $x$ regular in $R$ implies $m=0$. $M$ is said to be a divisible module if $Mx=M$ $^\forall$regular element $x$ in $M$.

**2.5. Corollary.** (Levy, 1963) *If $R$ be a ring having a classical right quotient ring $Q$, then every torsion free divisible $R$-module over $R$ is injective if and only if $Q$ is semi-simple Artinian.*

**2.6. Theorem.** *Let $R$ be a ring and $Q(=R_S)$ be an Asano's quotient ring of $R$ with respect to a set $S$ of regular elements of $R$. Then every $S$-free $S$-divisible module over $R$ is quasi-injective iff $Q$ is semi-simple Artinian.*

Proof. In the proof we use the following result of Faith and Utumi: [6, page 169, Cor. 2.4]. A ring $Q$ is semi-simple Artinian if and only if every module over $Q$ is quasi-injective.

Assume that every $S$-free $S$-divisible $R$-module is quasi-injective. Let $M$ be any module over $Q$. Then $M$ is $S$-free, $S$-divisible module over $R$. Therefore $M$ is a quasi-injective $R$-module. Therefore by theorem 2.2 $M$ is a quasi-injective $Q$ module. Hence $Q$ is semi-simple Artinian.

Converse is proved in the previous theorem 2.3.

**2.7. Corollary.** *If $R$ is a ring with a right classical quotient ring $Q$, then every torsion-free divisible $R$-module is quasi-injective if and only if $Q$ is semi-simple Artinian.*

**2.8. Theorem.** *Let $Q(=R_S)$ be an Asano's quotient ring of $R$. Let $M$ be a $Q$-module, then if $E$ is the injective hull of $M_Q$ then $E_R$=the injective hull of $M_R$.*

Proof. $E_Q$ being an injective module, $E_R$ is an injective module by 2.1. $E_R$ is an essential extension of $M_R$, because if $0\neq m\in E$, then $mQ\cap M\neq 0$, $0\neq mrs^{-1}\in M$ for some $r$ in $R$, $s$ in $S$, $0\neq mr\in M$. Therefore $mR\cap M\neq 0$ $^\forall m\neq 0$ in $E$. Hence $E_R$ is an injective hull of $M_R$.

**2.9. Quasi-injective hull.** The concept of quasi-injective hull of a module was introduced by Johnson and Wong [15] who proved that if $M$ is any

module, then $M'=\Lambda M$, where $\Lambda=\mathrm{Hom}_R(E, E)$, $E$ being the injective hull of $M$ is the unique minimal quasi-injective essential extension of $M$. Faith and Utumi [6] proved that $M$ is infact a unique minimal quasi-injective extension by observing that complement (closed) submodules of a quasi-injective module are quasi-injective, see Faith and Utumi [6, Corollary 2.2].

**2.10. Theorem.** *If $Q$ $(=R_S)$ be an Asano's quotient ring of $R$ and $M$ be a $Q$-module, then quasi-injective hull of $M_Q$=quasi-injective hull of $M_R$.*

Proof. Let $E_Q$ be an injective hull of $M_Q$. Then $E_R$ is an injective hull of $M_R$ by 2.8. Now note that $\mathrm{Hom}_Q(E, E)=\mathrm{Hom}_R(E, E)$, because if $f\in\mathrm{Hom}_R(E, E)$, then $f(mrs^{-1})s=f(mr)=f(m)r$ $\forall r\in R$, $s\in S$, therefore $f(mrs^{-1})=f(m)rs^{-1}$. Let $\Lambda=\mathrm{Hom}_Q(E, E)=\mathrm{Hom}_R(E, E)$. The quasi-injective hull of $M_Q$ is $\Lambda M=$ the quasi-injective hull of $M_R$.

**2.11.** DEFINITION. Injective resolution of a module.
Injective dimension of a module.

Let $R$ be an arbitrary ring. Let $M_R$ be a module over $R$. An exact sequence

$$0\to M\xrightarrow{\cdots} M_0\xrightarrow{d_0} M_1\xrightarrow{d_1} M_2\xrightarrow{d_2} M_3\to\cdots\to M_{n-1}\xrightarrow{d_{n-1}} M_n\to\cdots$$

where each $M_1$ is injective is called an injective resolution of $M$. The least integer $n$ such that kernel $d_n$ is injective is called the injective dimension of $M$. If no such integer $n$ exists, then the injective dimension of $M$ is defined to be $\infty$.

It has to be noticed that the injective dimension of a module is independent of the injective resolution.

**2.12. Theorem.** *Let $Q$ $(=R_S)$ be an Asano's quotient ring of a ring $R$. If $M$ be a module over $Q$, then injective dimension $(M_Q)$=injective dimension $(M_R)$.*

Proof. Let

$$0\to M\xrightarrow{\cdots} M_0\xrightarrow{d_0} M_1\xrightarrow{d_1} M_2\to\cdots\to M_{n-1}\xrightarrow{d_{n-1}} M_n\to\cdots$$

be an injective resolution of the module $M_Q$. Then by 2.1 this is also an injective resolution of the module $M_R$. If ker. $d_n$ is never $Q$ injective, then the ker. $d_n$ is never $R$-injective by 2.1, so that if dim. $M_Q=\infty$, then dim. $M_R=\infty$. If however dim. $M=n$, then ker. $d_n$ is $Q$ injective and therefore it is $R$-injective. Ker. $d_m$, $m<n$ cannot be $R$-injective, because if it is $R$-injective, then it will be $Q$ injective by 2.1, which is a contradiction. Hence injective dimension $M_Q=$ injective dimension $M_R$.

**2.13.** DEFINITION. If $R$ is any ring, then right Global dimension of $R=$ supremum of the injective dimension of all $R$ modules.

**2.14. Theorem.** *If $Q(=R_S)$ be an Asano's quotient ring of $R$, then right Global dimension of $Q \leqslant$ right Global dimension of $R$.*

Proof. Immediate from 2.12.

**3.** Let $R$ be a ring and $Q(=R_S)$ be the Asano's quotient ring of $R$ with respect to a set $S$ of regular elements of $R$. If $M$ be an $S$-free module over $R$, then two known constructions for the 'quotient module' $M'(=MQ)$ have been outlined in the introduction. A still new construction for this module $M$, which seems to be more natural is given below. But before we give this construction we observe one lemma which in essence is due to Levy [18].

**3.1. Lemma.** *If $S$ be a multiplicatively closed set of regular elements of a ring $R$, then $R_S$ exists (i.e. $R$ satisfies the multiplicity condition with respect to $S$) if and only if for every $R$-module $M$, $T(M)=\{m \in M,\ ms=0$ for some $s \in S\}$ is a submodule of $M$.*

Proof. This result can be proved exactly as in, Levy [18, theorem 1.4, page 134].

**3.2. Lemma.** *Every injective module is divisible.*

Proof. Let $M$ be an injective $R$-module. Let $a$ be a regular element of $R$. Let $m \in M$. The mapping $f$: $aR \to M$ such that $f(ar)=mr\ ^{\forall}r \in R$ can be realized by an element $m'$ of $M$. $m'ar=mr\ ^{\forall}r \in R$. Hence $m'a=m$. Hence $Ma=M$.

We are now ready to give another construction of the quotient module $MQ$.

**3.3.** Let $R$ be a ring and $Q(=R_S)$ be the Asano's right quotient ring of $R$ with respect to $S$. Let $M$ be a $S$-free right $R$ module. Let $E$ denote the injective hull of $M_R$. $E$ is $S$-divisible by 3.2. $E$ is $S$-free, because $T(E)=\{m \in E$: $ms=0$ for some $s \in S\}$ is a submodule of $E$ by 3.1. But $T(E) \cap M=0$ because $M$ is $S$-free. Therefore $T(E)=0$. Therefore $E$ is an $S$-free $S$-divisible module, therefore $E$ becomes a $Q$-module (see introduction) Let $M'=MQ \subset E$. Then $M'$ is the required module over $Q$ such that $M_R \subset M'_R$ and $M'=MQ$.

In view of this construction of $MQ$, a natural question arises when is $MQ=E(M)$? The following theorem answers this question.

**3.4. Theorem.** *Let $R$ be a ring and $Q(=R_S)$ be an Asano's quotient ring of $R$. Let $M$ be an $S$-free module over $R$. Then the 'quotient' module $M'(=MQ)$ is injective $Q$-module (or injective $R$-module see 2.1) iff every $R$-homomorphism $f$: $I \to M$, $I$ a right ideal of $R$ can be extended to a $R$-homomorphism $g$: $J \to M$, where $J$ is a right ideal of $R$ containing $I$ and containing an element of $S$.*

Proof. Let $M_R$ satisfy the condition. We prove that $M'=MQ$ is an

injective $R$-module. Let there be a $R$-homomorphism $f: I \to MQ$, where $I$ is a right ideal of $R$.

Set $I' = \{x \in I: f(x) \in M\}$. $I'$ is a right ideal of $R$, $I' \subset I$. We claim that for each $r \in I$, there exists $s \in S$ such that $rs \in I'$. Let $r \in I$. Then $f(r) \in MQ$. Therefore $f(r) = ms^{-1}$, where $m \in M$ and $s \in S$. Then $f(rs) = f(r)s = ms^{-1}s = m \in M$. $rs \in I'$. Let $f'$ denote the restriction of $f$ to $I$. There exists by hypothesis $g: J \to M$, where $J$ is a right ideal of $R$ containing $I$ and containing an element of $S$ and $g(x) = f'(x)$ $\forall x \in I'$. As $J$ contains an element of $S$, $JQ = Q$. There exists an extension $g': JQ \to MQ$ of $g$ defined as follows:

$$g'(js^{-1}) = g(j)s^{-1}.$$

The mapping $g'$ is a $Q$-homomorphism. Let $g'(1) = m$. Let $r \in I$. There exists $s \in S$ such that $rs \in I'$.

$$f(r)s = f(rs) = f'(rs) = g(rs) = g'(rs) = g'(1).rs = (m.r.).s$$

Therefore $f(r) = mr$ $\forall r \in I$.

Conversely, suppose $MQ$ be an injective $R$-module (injective $Q$-module). Let there be a $R$-homomorphism $f: I \to M$, where $I$ is a right ideal of $R$. As $MQ$ is $R$-injective and $M \subset MQ$, therefore there exists an element $m.s^{-1}$ of $MQ$ such that

$$ms^{-1}x = f(x) \quad \forall x \in I.$$

Let $J = sR + I$. Define $g: J \to M$ as follows:

$$g(sr + x) = ms^{-1}(sr + x) = mr + ms^{-1}x \quad \forall r \in R,\ x \in I.$$

Surely $g$ is a $R$-homomorphism of $J$ into $M$. $J$ contains $I$ and an element $s$ of $S$. Also $g(x) = ms^{-1}x = f(x)$ $\forall x \in I$.

## Self injective quotient rings

**3.5. Corollary.** *A quotient ring $Q$ $(= R_S)$ of $R$ is self injective ($Q_R$ is injective) if and only if for every $R$-homomorphism $f: I \to R$, $I$ a right ideal of $R$, there exists a $R$-homomorphism $g: J \to R$, where $J$ is a right ideal of $R$ containing $I$ and an element of $S$ and $g$ is such that*

$$g(x) = f(x) \quad \forall x \in I.$$

**3.6. Corollary.** *Let $R$ be a ring having a classical right quotient ring $Q$. The ring $Q$ is self injective ($Q_R$ is injective or $Q_R$ is the injective hull of $R_R$) iff every $R$-homomorphism*

$$f: I \to R,$$

*where $I$ is a right ideal of $R$ can be extended to a $R$-homomorphism*

$$g: J \to R$$

*where $J$ is a right ideal of $R$ containing $I$ and a regular element of $R$.*

**3.7.** REMARK. Semi-prime right Goldie rings form a class of rings which possess self injective classical quotient rings. One is tempted to verify whether this condition is really satisfied by the class of rings.

If $f: I \to R$, then let $J$ be a complement of $I$ in $R$, so that $I \oplus J$ is an essential right ideal of $R$. $I \oplus J$ contains a regular element of $R$. Goldie [9, Theorem 3.9]. One can trivially extend $f$ to $f'$: $(I+J) \to R$ by defining $f(i+j)=f(i)$ $^{\forall} i \in I$, $j \in J$.

If $Q$ is a classical quotient of a ring $R$, we say that $R$ is an order in $Q$.

In view of the corollary 3.6 and the characterization of orders in perfect rings, orders in semi-primary rings, orders in Artinian rings and Noetherian orders in Artinian rings given in [10] and [14] we have the following four results:

### Orders in self-injective perfect rings

**3.8. Theorem.** *A ring $R$ is an order in a (left) perfect self injective ring iff*

(*i*) *$N(R)$, the upper nil radical of $R$ is left T-nilpotent.*

(*ii*) *$R/N(R)$ is a right Goldie ring.*

(*iii*) *$a_\alpha R_\alpha$ is an essential right ideal of $R_\alpha$, for every right regular element $a_\alpha$ in $R_\alpha$ and for every ordinal number $\alpha$, where $R_\alpha = R/T_\alpha$, $a_\alpha = a + T_\alpha(R)$, $T_\alpha(R)$, an ideal of $R$ defined as follows:*

$$T_0(R) = 0,\ T_{\alpha+1}(R) = \{x: x \in N(R),\ xN(R) \subset T_\alpha(R)\},$$

*for an ordinal number of the type $\alpha+1$.*

$$T_\alpha(R) = \bigcup_{\beta<\alpha} T_\beta(R),$$

*for a limit ordinal $\alpha$.*

(*iv*) *If $a+N(R)$ is regular in $R/N(R)$ then $a$ is regular in $R$.*

(*v*) *Every $R$-homomorphism $f: I \to R$, $I$ a right ideal of $R$ can be extended to a $R$-homomorphism $g: J \to R$, where $J$ is a right ideal of $R$ containing $I$ and a regular element of $R$.*

### Orders in self-injective semi-primary rings

**3.9. Theorem.** *A ring $R$ is an order in a self-inejctive semi-primary ring if and only if*

(*i*) *$N(R)$, the upper nil radical of $R$ is nilpotent.*

(*ii*) *$R/N(R)$ is a right Goldie ring.*

(*iii*) *$a_i R_i$ is essential right ideal of $R_i$ for every right regular element $a_i$ in $R_i$ and for every integer $i \geqslant 0$, where $R_i = R/T_i$, $a_i = a + T_i$, $T_i$ being a two sided ideal of $R$ defined as follows: $T_0=(0)$, $T_i = l(N(R)^i) \cap N(R)$ for $i \geqslant 1$.*

(*iv*) *If $a+N(R)$ is regular in $R/N(R)$, then $a$ is regular in $R$.*
(*v*) *Every $R$-homomorphism $f: I\to R$, $I$ a right ideal of $R$, can be extended to a $R$-homomorphism $g: J\to R$, where $J$ is a right ideal of $R$ containing $I$ and a regular element of $R$.*

DEFINITION. A ring $R$ such that $I^{lr}=I$ $\forall$ right ideal $I$ and $L^{lr}=L$ $\forall$ left ideal of $L$ and $R$ satisfies the d.c.c. on left and right ideals is called a quasi-Frobenius ring.

It is well known that a ring $R$ is quasi-Frobenius iff $R$ is Artinian and right self-injective see Faith [5], Eilenberg-Nakayama [4].

**Orders in qusai-Frobenius rings**

**3.10. Theorem.** *A ring $R$ is an order in a quasi-Frobenius ring iff*
(*i*) *$N(R)$, the upper nil radical of $R$ is nilpotent.*
(*ii*) *$R/N(R)$ is a right Goldie ring.*
(*iii*) *$R/T_i$ has no infinite direct sum of right ideals for every $i\geqslant 0$, where $T_i$ is defined as follows:*

$$T_0 = (0),\ T_i = l(N(R)^i)\cap N(R) \qquad i\geqslant 1\,.$$

(*iv*) *If $a+N(R)$ is regular in $R$, then $a$ is regular in $R$.*
(*v*) *Every $R$-homomorphism $f: I\to R$, $I$ a right ideal of $R$, can be extended to $g: J\to R$, where $J$ is a right ideal of $R$ containing $I$ and a regular element of $R$.*

**3.11. Theorem.** *A Noetherian ring $R$ is an order in a quasi-Frobenius ring if and only if*
(*i*) *If $a+N(R)$ is regular in $R/N(R)$, then $a$ is regular in $R$.*
(*ii*) *Every $R$-homomorphism $f: I\to R$, $I$ a right ideal of $R$, can be extended to a $R$-homomorphism $g: J\to R$, where $J$ is a right ideal of $R$ containing $I$ and a regular element of $R$.*

(In connection with Theorem 3.10, the author wishes to point out that when this manuscript was ready, the author received an unpublished paper entitled 'Orders in quasi-Frobenius Rings' from Professor J.P. Jans wherein the following result is proved:

A ring $R$ is an order in a quasi-Frobenius ring iff
(i) $R$ has no infinite direct sum of right ideals.
(ii) $A_r(E(R), R)$ satisfies a.c.c. where $E(R)$ is the injective hull of $R_R$, $A_r(E(R), R)=\{S^{\perp}: S\subset E(R)\}$, $S^{\perp}=\{x\in R,\ Sx=0\}$ for subsets $S$ of $E(R)$.
(iii) $T(E(R)/R)=E(R)/R$, where for any $R$-module $M$, $T(M)=\{m\in M: ma=0$ for some regular element $a$ in $R\}$.
(iv) If $M$ is a finitely generated (cyclic) $R$-module such that $T(M)=(0)$, then $E(M)$, the injective hull of $M\subset(E(R))^n$.)

Semi-prime right Goldie rings belong to the class of rings $R$ for which

$Q(R)$, the classical right quotient ring of $R$ exists, $Q_R=E_R$, the injective hull of $R_R$ and $E_R$ is $\Sigma$-injective, see Faith [5, page 189, Corollary 3]. (An injective module $M_R$ is said to be $\Sigma$-injective iff the direct sum of arbitrarily many copies of $M_R$ is an injective $R$-module). The following theorem precisely determines this class of rings:

**3.12. Theorem.** *For a ring $R$, $Q(R)$ exists, $Q_R=E_R$, the injective hull of $R_R$, $E_R$ is $\Sigma$-injective iff $R$ is an order in a quasi-Frobenius ring.*

Proof. If $R$ is an order in a quasi-Frobenius ring, then let $Q=Q(R)$, $Q$ is quasi-Frobenious. $Q_Q$ is injective. Therefore $Q_R$ is injective by 2.1. $Q_R=E_R$, the injective hull of $R_R$. Now $Q_Q$ is $\Sigma$-injective, because $Q$ is quasi-Frobenius ($Q$ is quasi-Frobenious iff every projective module over $Q$ is an injective $Q$ module, see Faith [5]). Therefore $Q_R$ is $\Sigma$-injective by 2.1. Hence $E_R$ is $\Sigma$-injective. Conversely let us assume $Q(R)$ exists, $Q_R=E_R$ and $E_R$ is $\Sigma$-injective, then $Q_R$ is injective and $Q_R$ is $\Sigma$-injective. Then $Q$ is self injective and $Q_Q$ is $\Sigma$-injective by 2.1. Hence any free module over $Q$ is injective. Therefore any projective $Q$ module is $Q$ injective. Hence $Q$ is quasi-Frobenius.

**3.13. Theorem.** *If $R$ has a classical right quotient ring $Q$, then $Q$ is quasi-Frobenius iff every projective $Q$ module is $R$-injective.*

Proof. Let $Q$ be quasi-Frobenius. Let $M$ be a projective module over $Q$, then by Faith [5], $M$ is an injective $Q$ module. Therefore by 2.1 $M$ is an injective $R$ module. Conversely suppose that any projective module over $Q$ is an injective $R$ module, then any projective module over $Q$ is an injective $Q$ module. Hence $Q$ is quasi-Frobenius, Faith [5].

Faith and Walker [8] proved that a ring $Q$ is quasi-Frobenius iff every injective module over $Q$ is projective. From this and with the help of 2.1 we obtain the following theorem:

**3.14. Theorem.** *Let $R$ be a ring having a classical right quotient ring $Q$. Then $Q$ is quasi-Frobenius iff every torsion free injective $R$-module is a projective $Q$-module.*

### Utumi's quotient ring and Johnson's quotient ring

**3.15. Theorem.** *Let $R$ be a ring such that $R$ has a multiplicatively closed set $S$ of regular elements with respect to which $R$ satisfies the multiplicity condition and every $f: I\to R$, $I$ a right ideal of $R$, can be extended to $g: J\to R$, where $J$ is a right ideal of $R$ containing $I$ and an element of $S$, then $Q$ $(=R_S)$ is the Utumi's quotient ring of $R$.*

Proof. Lambek [16] proved that $Q$ the Utumi's ring of quotients of $R$ has the following characterization. $Q=$Bicommutant of $E_R$ where $E_R$ is the injective

hull of $R_R$. In our case by 3.5 $Q_Q$ is injective. Therefore by 2.1, $Q_R$ is injective. Also $R_R \subset {}'Q_R$. Therefore $Q_R$ is the injective hull of $R_R$. Now $\mathrm{Hom}_R(Q_R, Q_R) \cong Q$. Also $\mathrm{Hom}_Q({}_QQ, {}_QQ) \cong Q$. Therefore $Q$ is the Utumi's quotient ring of $R$.

Conversely we have the following result:

**3.16. Theorem.** *If $R$ is such that $Q_R$, where $Q$ is the Utumi's quotient ring of $R$, is the injective hull of $R_R$ (see, Lambek [17, page 95]) and $Q$ is an Asano's quotient ring of $R$, then there exists a multiplicatively closed set $S$ of regular elements of $R$ such that $R$ satisfies the multiplicity condition with respect to $S$ and every $R$-homomorphism $f\colon I \to R$, $I$ a right ideal of $R$, can be extended to $g\colon J \to R$, where $J$ is a right ideal of $R$ containing $I$ and an element of $S$.*

Proof. Let $S=\{x \in R\colon x$ invertibe in $Q\}$. Then $Q=R_S$ and the rest follows, because $Q_R$ is injective by 3.5.

Combining the previous two theorems and noting that the Utumi's quotient ring of a ring $R$ coincides with $\hat{R}$, the Johnson's maximal right quotient ring of $R$, if $R_r{}^{\Delta}=0$, and that $\hat{R}_R$, where $\hat{R}$ denotes the Johnson's maximal quotient ring of $R$, is the injective hull of $R_R$, we notice the following result:

**3.17. Theorem.** *If $R$ is a ring with $R_r{}^{\Delta}=0$, then $\hat{R}_r$, the Johnson's quotient ring of $R$ is an Asano's quotient ring of $R$ iff there exists a multiplicatively closed set $S$ regular elements of $R$ such that (i) $R$ satisfies the multiplicity condition with respect to $S$ and (ii) every $R$-homomorphism $f\colon I \to R$, $I$ a right ideal of $R$, can be extended to a $R$-homomorphism $g\colon J \to R$, where $J$ is a right ideal of $R$ containing $I$ and an element of $S$. In this case $\hat{R}_r = R_S$.*

**3.18. Corollary.** *If $R$ is a ring such that $R_r{}^{\Delta}=0$, and $Q(R)$ the classical right quotient ring of $R$ exists, then $\hat{R}=Q(R)$ iff every $f\colon I \to R$, $I$ a right ideal of $R$, can be extended to $g\colon J \to R$, where $J$ contains $I$ and a regular element of $R$.*

**4.** If $R$ is a commutative integral domain and $Q$ its field of quotients, then $Q_R$ is projective implies $Q=R$, see Tsi-Che-Te [20, page 174]. The aim of the present section is to generalize this result from commutative integral domain to an arbitrary semi-prime right Goldie ring.

**4.1. Theorem.** *Let $R$ be a semi-prime ring with the Goldie conditions on right ideals and let $Q$ be its semi-simple Artinian classical right quotient ring. If $Q_R$ is projective, then $R=Q$.*

**Lemma.** *Let $R$ be a prime ring with right Goldie conditions and $Q$ its classical right quotient ring. If there exists $0 \neq q$ in $Q$ such that $q \cdot Q \subset R$, then $R=Q$.*

Proof. By a theorem of Faith and Utumi [7, Theorem 3, page 56], there exists a complete set $M=\{e_{ij}\colon i, j=1, 2, \cdots, n\}$ of matrix units in $Q$ with the following property: If $D$ is the centralizer of $M$ in $Q$, then $D$ is a division ring and

$$Q = \sum_{i,j=1}^{n} De_{ij} \supset R \supset \sum_{i,j=1}^{n} Fe_{ij}$$

where $F$ is a right Ore-domain contained in $R \cap D$ and $D$ is the right quotient field of $F$. (It is to be noted that $F$ in general does not contain identity, see example given by Faith and Utumi [7, (C) page 60]) Let

$$q = \sum_{i,j=1}^{n} \alpha_{ij} e_{ij}, \quad \alpha_{ij} \in D .$$

Suppose $\alpha_{kl} \neq 0$. Fix a non-zero element $a$ of $F$. $a e_{rk} \in R$ for every $r=1, 2, \cdots, n$. Also because $qQ \subset R$,

$$(\sum_{i,j=1}^{n} \alpha_{ij} e_{ij})(a \cdot \alpha_{kl})^{-1} d e_{ls} \in R$$

for every $s=1, \cdots, n$ and every $d \in D$. Therefore

$$a e_{rk} (\sum_{i,j=1}^{n} \alpha_{ij} e_{ij})(a \alpha_{kl})^{-1} d e_{ls} \in R$$

for every $r, s=1, 2, \cdots, n$ and every $d \in D$. But

$$a e_{rk} (\sum_{i,j=1}^{n} \alpha_{ij} e_{ij})(a \alpha_{kl})^{-1} d e_{ls} = d e_{rs} .$$

Therefore $\sum_{i,j=1}^{n} D e_{rs} = Q \subset R$. Hence $Q=R$.

**4.2. Lemma.** *If $Q$ be an Asano's right quotient ring of $R$ and $f \in Hom_R(Q, R)$ then $f(q)=f(1).q$ for every $q \in Q$.*

Proof. Let $q=a b^{-1}$, $a$, $b$ in $R$, $b$ regular. Then

$$f(q)b = f(a b^{-1})b = f(a b^{-1} b) = f(a) = f(1)a .$$

Therefore $f(q)=f(1)a b^{-1}=f(1)q$.

Proof of the Theorem. A module $M_R$ is projective iff there exist subsets $\{m_i\}_{i \in I}$ of $M$ and $\{f_i\}_{i \in I}$ of $\mathrm{Hom}_R(M, R)$, such that for each $m \in M$ $f_i(m)=0$ for almost all $i$ and $m=\sum_{i \in I} m_i f_i(m)$, see Bass [2, (4.8) page 477].

As $Q_R$ is projective there exists subsets $\{b_i\}_{i \in I}$ of $Q$ and $\{f_i\}_{i \in I}$ of $\mathrm{Hom}_R(Q, R)$ with the above properties. Now

$$1 = b_1 f_1(1) + b_2 f_2(1) + \cdots\cdots + b_n f_n(1)$$

for finite subset $(1, 2, \cdots, n)$ of I. By the above lemma 4.2 $f_k(1)$ $Q \subset R$ for each $k=1, 2, \cdots, n$. Let $e_1, e_2, \cdots, e_m$ the central idempotents in $Q$ such that $e_1 Q$, $e_2 Q$, $\cdots$, $e_m Q$ are the simple components of the semi-simple Artinian ring $Q$. Clearly for each $i=1, \cdots, m$, there exists $k(i)$, $1 \leqslant k(i) \leqslant n$ such that $f_{k(i)}(1) e_i \neq 0$. But

$$(\underset{f(i)}{f(1)}e_i)e_iQ \subset e_iR \cap R .$$

But $e_iQ$ is the classical quotient ring of $e_iR$ $R$ and $e_iQ$ is a simple Artinian ring, therefore by the lemma $e_iQ=e_iR\subset R$. Therefore $e_iQ \cap R$ for every $i$. Therefore $Q=R$.

## 5. Hereditary orders in semi-simple Artinian rings

In this section right hereditary semi-prime right Goldie-rings have been characterized.

**5.1. Theorem.** *If $R$ be a semi-prime right Goldie ring, and $Q$ be its classical right quotient ring, then $R$ is hereditary if and only if for every essential right ideal $I$ of $R$, there exist $b_1, b_2, \cdots, b_n$ in $I$ and $\alpha_1, \alpha_2, \cdots, \alpha_n$ in $Q$ such that $\sum_{i=1}^{n} b_i\alpha_i=1$ and $\alpha_i I\subset R$ for every $i=1, 2, \cdots, n$.*

Proof. Assume $R$ is hereditary. Let $I$ be an essential right ideal of $R$. $I$ is a projective right $R$ module. By the characterization of projective modules mentioned in the proof of Theorem 4.1, there exist subsets $\{b_j\}_{j\in J}$ of $I$ and $\{f_j\}_{j\in J}$ of $\mathrm{Hom}_R(I, R)$ such that for every $b\in I$, $f_j(b)=0$ for almost all values of $j$ and $b=\sum_{j\in J} b_jf_j(b)$. Now $I$ being an essential right ideal of $R$, $I$ contains a regular element, Goldie [9, Theorem 3.9]. Therefore $IQ=Q$. Each $f_j$ has a unique extension $f_j'$ in $\mathrm{Hom}_Q(Q, Q)$. Let $f_j'(1)=\alpha_j$. Then $f_j'(q)=f_j'(1.q)=f_j'(1)q=\alpha_j(q)$ for every $q$ in $Q$. Therefore $\alpha_jb=f_j(b)=f_j(b)$ for every $b\in I$. Consequently $\alpha_jI\subset R$. Now let $a$ be a regular element of $I$.

$$a=\sum_{j=1}^{n} b_jf_j(a)=(\sum_{j=1}^{n} b_j\alpha_j)a$$

for some finite subset $(1, 2, \cdots, n)$ of J. Therefore $1=\sum_{i=1}^{n} b_i\alpha_i$. The sets $(b_1, b_2, \cdots, b_n)$ and $(\alpha_1, \alpha_2, \cdots, \alpha_n)$ are the desired sets.

Conversely since every right ideal is a direct summand of an essential right ideal and a direct summand of a projective module is projective, it is sufficient to show that every essential right ideal is projective. Let $I$ be an essential right ideal of $R$. There exist $b_1, b_2, \cdots, b_n$ in $I$ and $\alpha_1, \alpha_2, \cdots, \alpha_n$ in $Q$ such that $\alpha_iI\subset R$ for every $i=1, 2, \cdots, n$ and $\sum_{i=1}^{n} b_i\alpha_i=1$. Define $f_1, f_2, \cdots, f_n$ such that $f_i(b)=\alpha_ib$, $\forall b\in I$, $f_i\in\mathrm{Hom}_R(I, R)$. Also $b=(\sum_{i=1}^{n} b_i\alpha_i)b=\sum_{i=1}^{n} b_i(\alpha_ib)=\sum_{i=1}^{n} b_if_i(b)$ $\forall b\in I$. Hence $I$ is projective by the characterization of a projective module mentioned in the proof of Theorem 4.1. (The 'only if' part of this result is due to Levy [18]).

**5.2. Corollary.** (Levy, 1963) *A hereditary semiprime right Goldie ring is a right Noetherian ring.*

Proof. Since every right ideal is a direct summand of an essential right ideal, it is sufficient to show that every essential right ideal is finitely generated. Let $I$ be an essential right ideal. There exist $b_1, b_2, \cdots, b_n$ in $I$ and $\alpha_1, \alpha_2, \cdots, \alpha_n$ in $Q$ such that $\sum_{i=1}^{n} b_i\alpha_i=1$ and $\alpha_i I \subset R$ $i=1, 2, \cdots, n$. Therefore

$$I = \sum_{i=1}^{n} b_i(\alpha_i I) \subset \sum_{i=1}^{n} b_i R \subset I .$$

Hence

$$I = \sum_{i=1}^{n} b_i R .$$

**5.3. Corollary.** *A commutative integral domain $R$ is hereditary if and only if it is a Dedekind domain.*

Proof. It is easy to make the following observations (a) An ideal $I$ of $R$ is invertible in the semi-group of fractionary ideals if and only if there exist $b_1$, $b_2, \cdots, b_n$ in $I$ and $\alpha_1, \alpha_2, \cdots, \alpha_n$, in $Q$, the field of quotients of $R$ such that $\sum_{i=1}^{n} b_i\alpha_i=1$ and $\alpha_i I \subset R$ for every $i=1, 2, \cdots, n$.

(b) Every ideal of $R$ is an essential ideal. These two observations along with the theorem 6.1 proves the corollary.

**5.4. Theorem.** *A semi-prime principal right ideal ring is a hereditary ring.*

Proof. A semi-prime principal right ideal ring is clearly a semi-prime Goldie ring. By a remark made in the previous theorem 5.1 it is sufficient to prove that every essential right ideal of $R$ is a projective $R$-module. Let $I$ be an essential right ideal of $R$, then $I=aR$ for some $a$ in $R$. The element $a$ is regular, see Processi and Small [19, page 81]. Therefore $R_R \cong (aR)_R$ under the mapping $r \leftrightarrow ar$. Hence $I\,(=aR)$ is a projective $R$-module.

**6.** We return to the study of modules over a ring $R$, which has an Asano's quotient ring $Q\,(=R_S)$. If $M_R$ is a projective $R$-module, then $M'\,(=MQ)$ is proved to be a projective $Q$-module. This result is used to prove that if $M$ is an $S$-free $R$-module then projective dimension $(MQ)_Q \leqslant$ projective dimension $M_R$.

**6.1. Category of $S$-free modules.** Let $\mathfrak{F}$ denote the category of all $S$-free $R$-modules (with $R$-homomorphisms as the maps).

Let $\mathfrak{Q}$ denote the category of $Q$-modules (with $Q$-homomorphisms as maps).

If $M \in \mathfrak{F}$, then there exists $M'$ (unique upto isomorphism over $M$) in $\mathfrak{Q}$ such that $M_R \subset M'_R$ and $M'=MQ$. If $M_1, M_2 \in \mathfrak{F}$ and $f\colon M_1 \to M_2$, then there exists unique $Q$-homomorphism $f'\colon M_1Q \to M_2Q$, which extends $f$. This map $f'$ is defined as follows:

$$f'(m_1 s^{-1}) = f(m_1)s^{-1} \quad \forall m_1 \in M_1,\ s \in S .$$

In fact the rule $T$:

$$T(M) = MQ$$
$$T(f) = f'$$

is an additive covariant functor from the category $\mathfrak{F}$ into the category $\mathfrak{H}$. $T$ is seen below to be an exact functor:
Let

$$0 \to M_1 \xrightarrow{j} M \xrightarrow{\pi} M_2 \to 0$$

be an exact sequence in $\mathfrak{F}$. Then the sequence

$$0 \to M_1Q \xrightarrow{j'} MQ \xrightarrow{\pi'} M_2Q \to 0$$

is also an exact sequence.

Exactness at $M_1Q$: $j'(m_1s^{-1}) = 0$ implies $j(m_1)s^{-1} = 0$,
$j(m_1) = 0$, $m_1 = 0$, $m_1s^{-1} = 0$.

Exactness at $MQ$: $\pi'j'(m_1s^{-1}) = \pi'(j'(m_1s^{-1})) = \pi'(j(m_1)s^{-1})$
$= (\pi j(m_1))s^{-1} = 0$.

If $\pi'(ms^{-1}) = 0$, then $\pi(m) = 0$, $m = j(m_1)$ for some $m_1$ in $M_1$.
$j'(m_1s^{-1}) = j(m_1)s^{-1} = ms^{-1}$.

Exactness at $M_2Q$: If $m_2s^{-1}$ be any element of $M_2Q$, then there exists $m \in M$ such that $\pi(m) = m_2$, $\pi'(ms^{-1}) = m_2s^{-1}$.

**6.2.** We observe in passing that a projective $R$-module is torsion free.

**6.3. Theorem.** *Let $R$ be a ring and $Q$ $(=R_S)$ be an Asano's right quotient ring of $R$. If $M$ is a projective $R$-module, then $MQ$ is a projective $Q$-module.*

Proof. Let

$$\begin{array}{ccccc} & & (MQ)_Q & & \\ & & \downarrow f & & \\ A_Q & \xrightarrow[\pi]{} & B_Q & \longrightarrow & 0 \end{array}$$

be a diagram of $Q$-modules with exact row. Let $f_1: M \to B$ denote the restriction of $f$ to $M$. $f_1$ is $R$-homomorphism of $M$ into $B$. We therefore have the diagram

$$\begin{array}{ccccc} & & M_R & & \\ & & \downarrow f_1 & & \\ A_R & \xrightarrow[\pi]{} & B_R & \longrightarrow & 0 \end{array}$$

of $R$-modules with exact row. Since $M_R$ is projective there exists $g: M \to A$ such that $\pi g = f_1$. Let $g': MQ \to AQ\ (=A)$ denote the extension of $g: M \to A$. Then

$$\pi g'(m s^{-1}) = \pi(g(m)s^{-1}) = \pi g(m)s^{-1} = f_1(m)s^{-1} = f(m.s^{-1}).$$

**6.4. Corollary.** *If $M$ is a $Q$-module such that $M_R$ is projective, then $M_Q$ is also projective.*

**6.5.** DEFINITION. Projective dimension of a module
Global dimension of a ring

Let $R$ be an arbitrary ring and $M$ be a module over $R$. An exact sequence

$$\cdots \to M_n \xrightarrow{d_n} M_{n-1} \to \cdots\cdots \to M_2 \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \xrightarrow{\cdots} M \to 0$$

where each $M_i$ is projective is called a projective resolution of $M$. The smallest positive integer $n$ such that kernel $d_n$ is projective is called the projective dimension of the module $M$. If no such integer $n$ exists, then the projective dimension of $M$ is infinity. (It is known that the projective dimension of $M$ is independent of the projective resolution). Right Global dimension of a ring $R$=supremum of the projective dimensions of all $R$-modules (see sec. 2).

It is well known that for any ring $R$, supremum of the injective dimensions of all $R$-modules=supremum of all the projective dimensions of all $R$-modules. Therefore the same term Right Global dimension is used for both the supremums.

**6.6. Theorem.** *Let $R$ be a ring and $Q(=R_S)$ be an Asano's right quotient ring of $R$. If $M$ is an $S$-free $R$-module, then projective dimension $(MQ)_Q \leqslant$ projective dimension $M_R$.*

Proof. Let

$$\cdots \to M_n \xrightarrow{d_n} M_{n-1} \to \cdots\cdots \to M_2 \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \xrightarrow{\cdots} M \to 0 \qquad \text{(i)}$$

be a projective resolution of $M_R$. If projective dimension $M_R = \infty$, we have nothing to prove. Let the projective dimension $M = n\ (< \infty)$. Then $K_n$, the kernel of $d_n$ is projective $R$-module.

From the projective resolution (i) of $M_R$ we get a projective resolution of $(MQ)_Q$

$$\cdots \to M_nQ \xrightarrow{d_n'} M_{n-1}Q \to \cdots\cdots \to M_1Q \xrightarrow{d_1'} M_0Q \xrightarrow{\cdots} MQ \to 0$$

(We note that the sequence is exact because we have noticed that the functor $T$ is an exact functor. Also each $M_iQ$ is a projective $Q$-module by theorem 6.3).

The kernel of $d_n$=(kernel $d_n$)$Q$ which is a projective $Q$-module by theorem 6.3. Hence

projective dimension $(MQ)_Q \leqslant$ projective dimension $M_R$.

**6.7. Corollary.** *If $M$ is a $Q$ module, then projective dimension $M_Q \leqslant$ projective dimension $M_R$.*

**6.8. Corollary.** *Right Global dimension $Q \leqslant$ right Global dimension $R$.*

## 7. Hereditary quotient rings

Let $R$ be a ring and $S$ be a multiplicatively closed set of regular elements of $R$ such that $R_S$ exists. Denote $R_S$ by $Q$. In this section we obtain necessary and sufficient conditions on $R$ such that $Q$ becomes a hereditary ring.

**7.1. Theorem.** *A quotient ring $Q$ of $R$ is hereditary if and only if for every exact sequence of $R$ modules*

$$M \xrightarrow{\pi} N \longrightarrow 0$$

*where $M$ is $S$-free and injective and $N$ is $S$-free, $N$ is injective.*

Proof. Let $Q$ be hereditary. Let

$$M \xrightarrow{\pi} N \longrightarrow 0$$

be an exact sequence of $R$-modules, where $M$ is $S$-free and injective and $N$ is $S$-free. Since every injective module is divisible (see 3.2), $M$ is $S$-divisible and therefore $M$ is $Q$-module. Also because $M$ is $S$-divisible, therefore $N$ is also $S$-divisible, therefore $N$ is also a $Q$-module. The map $\pi$ is $Q$-homomorphism (see 6.1, $\pi' = \pi$ in this case). Now since $M_R$ is injective, therefore $M_Q$ is injective. As $Q$ is hereditary $N_Q$ is injective and therefore $N_R$ is injective.

Conversely we prove that $Q$ is hereditary. It is sufficient to show that for every exact sequence of $Q$-modules

$$M \xrightarrow{\pi} N \longrightarrow 0$$

where $M$ is injective, $N$ is injective. Now regarding the above sequence as a sequence of $R$-modules we note that $M_R$ is $S$-free, $M_R$ is injective and $N$ is $S$-free. Therefore $N_R$ is injective. Hence $N_Q$ is injective.

**7.2. Corollary.** *If $R$ is hereditary, then a quotient ring $Q\,(=R_S)$ of $R$ is also hereditary.*

Proof. Since $R$ is hereditary, for any exact sequence of $R$-modules

$$M \longrightarrow N \longrightarrow 0$$

where $M$ is injective, $N$ is injective. Therefore by the theorem 7.1 $Q$ is hereditary.

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