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EXTENSIONS OF THE GENOCCHI POLYNOMIALS AND THEIR FOURIER EXPANSIONS AND INTEGRAL REPRESENTATIONS

Dedicated to my father Jia-Fu Luo, a teacher of Chinese, on his 70th birthday

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Abstract

We extend the Genocchi polynomials and investigate their Fourier expansions and integral representations. We obtain their formulas at rational arguments in terms of Hurwitz zeta function and show an explicit relationship with Gaussian hypergeometric functions. Some known results for the classical Genocchi polynomials are also deduced.

1. Introduction

The Genocchi polynomials $G_n(x)$ are usually defined by means of the following generating functions (see, for details, [4], [5], [12] and [14]):

$$\frac{2ze^{xz}}{e^z+1}=\sum_{n=0}^{\infty}G_n(x)\frac{z^n}{n!}\quad (|z|<\pi). \tag{1.1}$$

In particular, $G_n := G_n(0)$ for $n \geq 0$ are called Genocchi numbers, with $G_{2n+1}=0$ for $n \geq 1$ and, for example,

$$G_0=0,\quad G_1=1,\quad G_2=-1,\quad G_4=1,$$
$$G_6=-3,\quad G_8=17,\quad G_{10}=-155,\quad G_{12}=2073.$$

Some interesting analogues of the classical Bernoulli and Euler polynomials were investigated by Apostol ([2]), Luo and Srivastava ([9], [10], [11] and [15]), and these analogues are called the Apostol–Bernoulli and Apostol–Euler polynomials. We further extend the Genocchi polynomials as follows:

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The *Apostol–Genocchi polynomials* $\mathcal{G}_n(x;\lambda)$ in x are defined by means of the generating functions:

$$\frac{2ze^{xz}}{\lambda e^z+1}=\sum_{n=0}^{\infty}\mathcal{G}_n(x;\lambda)\frac{z^n}{n!}\quad (|z|<|\log(-\lambda)|). \tag{1.2}$$

We note that here $G_n(x)=\mathcal{G}_n(x;1)$. We set $\mathcal{G}_n(\lambda):=\mathcal{G}_n(0;\lambda)$, which we call the Apostol–Genocchi numbers (in fact, it is a function in λ).

In [7], we gave the Fourier expansions and integral representations for the classical Genocchi polynomials by using the Lipschitz summation formula. In the present paper, we further investigate the Fourier expansions for the Apostol–Genocchi polynomials based on the same method and provide their integral representations by using the Fourier expansions. We obtain a formula for the Apostol–Genocchi polynomials at rational arguments and give an explicit relationship between the Apostol–Genocchi polynomials and Gaussian hypergeometric functions. The corresponding formulas of [7] are some special cases of the results of this paper.

The paper is organized as follows: In the second section we derive the Fourier expansions for the Apostol–Genocchi polynomials. In the third section we show their integral representations. In the fourth section we give the formula at rational arguments in terms of the Hurwitz zeta function. In the fifth section we provide an explicit relationship between the Apostol–Genocchi polynomials and Gaussian hypergeometric functions. In the sixth section we deduce the corresponding results for the Genocchi polynomials. Some remarks are given in the seventh section, in particular, we derive the Euler formula $\zeta(2n)=((-1)^{n-1}(2\pi)^{2n}/(2(2n)!))B_{2n}$ in a different way.

2. Fourier expansions for the Apostol–Genocchi polynomials

In this section, we investigate the Fourier expansions for the Apostol–Genocchi polynomials by applying the Lipschitz summation formula.

First we recall the Lipschitz summation formula ([6]):

$$\sum_{n+\mu>0}\frac{e^{2\pi i(n+\mu)\tau}}{(n+\mu)^{1-\alpha}}=\frac{\Gamma(\alpha)}{(-2\pi i)^{\alpha}}\sum_{k\in\mathbb{Z}}\frac{e^{-2\pi ik\mu}}{(\tau+k)^{\alpha}}, \tag{2.1}$$

where $\alpha\in\mathbb{C}$, $\Re(\alpha)>1$ if $\mu\in\mathbb{Z}$ and $\Re(\alpha)>0$ if $\mu\in\mathbb{R}\setminus\mathbb{Z}$, $\tau\in H$, H denotes the complex upper half plane; Γ denotes the Gamma function.

Theorem 2.1. *For* $n > 0$, $0 \leq x \leq 1$; $\lambda \in \mathbb{C} \setminus \{0, -1\}$, *we have*

$$\mathcal{G}_n(x;\lambda) = \frac{2 \cdot n!}{\lambda^x} \sum_{k \in \mathbb{Z}} \frac{e^{(2k-1)\pi i x}}{[(2k-1)\pi i - \log \lambda]^n} \tag{2.2}$$

$$= \frac{2 \cdot n!\, i^n}{\lambda^x} \left[\sum_{k=0}^{\infty} \frac{\exp[(n\pi/2 - (2k+1)\pi x)i]}{[(2k+1)\pi i + \log \lambda]^n} + \sum_{k=0}^{\infty} \frac{\exp[(-n\pi/2 + (2k+1)\pi x)i]}{[(2k+1)\pi i - \log \lambda]^n} \right]. \tag{2.3}$$

Proof. By (1.2) and the generalized binomial theorem, we have

$$\sum_{k=0}^{\infty} \mathcal{G}_k(x;\lambda) \frac{(2\pi i \tau)^{k-1}}{k!} = \frac{2e^{2\pi i \tau x}}{\lambda e^{2\pi i \tau} + 1} = 2 \sum_{k=0}^{\infty} (-1)^k \lambda^k e^{2\pi i (k+x)\tau} \quad (|2\pi i \tau + \log \lambda| < \pi). \tag{2.4}$$

Differentiating on the both sides of (2.4) with respect to the variable τ and iterating $n-1$ times, and noting that $\mathcal{G}_0(x;\lambda) = \mathcal{G}_0(\lambda) = 0$ (see Section 5 below), we obtain

$$\sum_{k=n}^{\infty} \mathcal{G}_k(x;\lambda) \frac{(2\pi i)^{k-1} \tau^{k-n}}{k(k-n)!} = 2(2\pi i)^{n-1} \sum_{k=0}^{\infty} (-1)^k \lambda^k (k+x)^{n-1} e^{2\pi i (k+x)\tau}. \tag{2.5}$$

On the other hand, letting $\alpha = n$ $(n = 1, 2, \ldots)$, $\mu \mapsto x$, $\tau \mapsto \tau + (\log \lambda)/(2\pi i) + 1/2$ in (2.1), we find

$$(-1)^n (n-1)! \sum_{k \in \mathbb{Z}} \frac{e^{-(2k+1)\pi i x}}{[(2k+2\tau+1)\pi i + \log \lambda]^n} = \sum_{k=0}^{\infty} (-1)^k \lambda^{k+x} (k+x)^{n-1} e^{2\pi i (k+x)\tau}. \tag{2.6}$$

Combining (2.5) and (2.6), we get

$$\lambda^x \sum_{k=n}^{\infty} \mathcal{G}_k(x;\lambda) \frac{(2\pi i)^{k-1} \tau^{k-n}}{k(k-n)!} = (-1)^n (n-1)!\, 2(2\pi i)^{n-1} \sum_{k \in \mathbb{Z}} \frac{e^{-(2k+1)\pi i x}}{[(2k+2\tau+1)\pi i + \log \lambda]^n}. \tag{2.7}$$

Letting $\tau \to 0$ in (2.7), we obtain the assertion (2.2) of Theorem 2.1.

Noting that $i^{-n} = e^{-n\pi i/2}$ and $(-1)^n = e^{n\pi i}$, and via simple calculation, we see that the assertion (2.3) of Theorem 2.1 is a direct consequence of (2.2). This completes our proof. □

3. Integral representations for the Apostol–Genocchi polynomials

In this section, we give the integral representations for the Apostol–Genocchi polynomials. For convenience, we take $\lambda = e^{2\pi i\xi}$ $(\xi \in \mathbb{R},\ |\xi| < 1)$ in this section.

Theorem 3.1. *For $n > 0$, $0 \le x \le 1$; $|\xi| < 1/2$, $\xi \in \mathbb{R}$, we have*

(3.1)
$$\mathcal{G}_n(x; e^{2\pi i\xi}) = 2ne^{-2\pi ix\xi} \int_0^\infty \frac{M(n;x,t)\cosh(2\pi\xi t) + iN(n;x,t)\sinh(2\pi\xi t)}{\cosh 2\pi t - \cos 2\pi x} t^{n-1}\, \mathrm{d}t,$$

where

$$M(n;x,t) = \left[e^{\pi t}\cos\left(\pi x - \frac{n\pi}{2}\right) - e^{-\pi t}\cos\left(\pi x + \frac{n\pi}{2}\right)\right],$$
$$N(n;x,t) = \left[e^{\pi t}\sin\left(\pi x - \frac{n\pi}{2}\right) + e^{-\pi t}\sin\left(\pi x + \frac{n\pi}{2}\right)\right].$$

Proof. Setting $\lambda = e^{2\pi i\xi}$ and letting $k \mapsto -k$ in (2.2), we have

$$\mathcal{G}_n(x; e^{2\pi i\xi}) = \frac{2\cdot n!}{(-\pi i)^n} \sum_{k\in\mathbb{Z}} \frac{e^{-(2k+2\xi+1)\pi ix}}{(2k+2\xi+1)^n}. \tag{3.2}$$

Applying the integral formula

$$\int_0^\infty t^n e^{-at}\, \mathrm{d}t = \frac{n!}{a^{n+1}} \quad (n = 0, 1, \ldots;\ \Re(a) > 0) \tag{3.3}$$

in (3.2), we get

$$\begin{aligned}
\mathcal{G}_n(x; e^{2\pi i\xi}) &= \frac{2n}{(-\pi i)^n}\left\{\sum_{k=0}^\infty e^{-(2k+2\xi+1)\pi ix}\int_0^\infty t^{n-1}e^{-(2k+2\xi+1)t}\, \mathrm{d}t\right.\\
&\qquad \left. + (-1)^n \sum_{k=0}^\infty e^{(2k-2\xi+1)\pi ix}\int_0^\infty t^{n-1}e^{-(2k-2\xi+1)t}\, \mathrm{d}t\right\}\\
&= \frac{2n}{(-\pi i)^n}\left\{e^{-(2\xi+1)\pi ix}\int_0^\infty e^{-(2\xi+1)t}t^{n-1}\sum_{k=0}^\infty e^{-2(\pi ix+t)k}\, \mathrm{d}t\right.\\
&\qquad \left. + (-1)^n e^{-(2\xi-1)\pi ix}\int_0^\infty e^{(2\xi-1)t}t^{n-1}\sum_{k=0}^\infty e^{2(\pi ix-t)k}\, \mathrm{d}t\right\}
\end{aligned}$$

$$= \frac{2n}{(-\pi i)^n}\left\{\int_0^\infty \frac{e^{-(2\xi+1)\pi i x}}{e^{2t}-e^{-2\pi i x}} e^{(1-2\xi)t} t^{n-1}\,\mathrm{d}t + (-1)^n \int_0^\infty \frac{e^{(1-2\xi)\pi i x}}{e^{2t}-e^{2\pi i x}} e^{(1+2\xi)t} t^{n-1}\,\mathrm{d}t\right\}$$

$$= \frac{ne^{-2\pi i x\xi}}{\pi^n}\left\{\int_0^\infty \frac{e^{n\pi i/2}(e^{-2\pi i x}-e^{-2t})e^{\pi i x}}{\cosh 2t - \cos 2\pi x} e^{-(2\xi-1)t} t^{n-1}\,\mathrm{d}t + \int_0^\infty \frac{e^{-n\pi i/2}(e^{2\pi i x}-e^{-2t})e^{-\pi i x}}{\cosh 2t - \cos 2\pi x} e^{(2\xi+1)t} t^{n-1}\,\mathrm{d}t\right\}.$$

Here we use $(-1/i)^n = e^{n\pi i/2}$ and $(-1)^n = e^{-n\pi i}$. Making the transformation $t = \pi u$, after the simplification, we obtain the desired (3.1) immediately. This completes the proof. □

Below we give another integral representations for the Apostol–Genocchi polynomials.

Theorem 3.2. *For $n = 1, 2, \ldots;\ 0 \le x \le 1;\ |\xi| < 1/2,\ \xi \in \mathbb{R}$, we have*

(3.4)
$$\mathcal{G}_n(x; e^{2\pi i\xi}) = (-1)^{n-1}\frac{4ne^{-2\pi i x\xi}}{\pi^n} \times \int_0^1 \frac{M'(n;x,t)\cosh(2\xi\log t) - iN'(n;x,t)\sinh(2\xi\log t)}{t^4 - 2t^2\cos 2\pi x + 1}(\log t)^{n-1}\,\mathrm{d}t,$$

where

$$M'(n;x,t) = \left[\cos\left(\pi x - \frac{n\pi}{2}\right) - t^2\cos\left(\pi x + \frac{n\pi}{2}\right)\right],$$
$$N'(n;x,t) = \left[\sin\left(\pi x - \frac{n\pi}{2}\right) + t^2\sin\left(\pi x + \frac{n\pi}{2}\right)\right].$$

Proof. Substituting $\cosh 2\pi t = (e^{2\pi t} + e^{-2\pi t})/2$ into (3.1), we obtain

(3.5)
$$\mathcal{G}_n(x; e^{2\pi i\xi}) = 4ne^{-2\pi i x\xi}\int_0^\infty \frac{M(n;x,t)\cosh(2\pi\xi t) + iN(n;x,t)\sinh(2\pi\xi t)}{e^{2\pi t} + e^{-2\pi t} - 2\cos 2\pi x} t^{n-1}\,\mathrm{d}t.$$

Making the transformation $u = e^{-\pi t}$ in (3.5), we obtain formula (3.4) directly. This proof is complete. □

REMARK 1. For any integers l, we see easily that $\mathcal{G}_n(x; e^{2\pi i(l+\xi)}) = \mathcal{G}_n(x; e^{2\pi i\xi})$. Therefore, the Apostol–Genocchi polynomials $\mathcal{G}_n(x; e^{2\pi i\xi})$ are periodic functions in ξ with period 2π. In view of this reason, we say that the variable ξ may take any real numbers in Theorem 3.1 and Theorem 3.2.

REMARK 2. We may also prove Theorem 2.1 by Theorem 3.1 in an inverse process.

4. Explicit formulas for the Apostol–Genocchi polynomials at rational arguments

In this section, we obtain the explicit formulas for the Apostol–Genocchi polynomials at rational arguments by applying the Fourier expansion. Here let $\mathbb{Z}_0^- = \{0, -1, -2, \ldots\}$ denote the set of nonpositive integers.

The Hurwitz–Lerch zeta function $\Phi(z, s, a)$ defined by (cf., e.g., [16, p. 121, et seq.])

$$\Phi(z, s, a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \quad (a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C} \text{ when } |z| < 1;\ \Re(s) > 1 \text{ when } |z| = 1) \tag{4.1}$$

contains, as its *special* cases, not only the Riemann and Hurwitz zeta functions:

$$\zeta(s) := \Phi(1, s, 1) = \zeta(s, 1) = \frac{1}{2^s - 1}\zeta\left(s, \frac{1}{2}\right) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{4.2}$$

and

$$\zeta(s, a) := \Phi(1, s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \quad (\Re(s) > 1;\ a \notin \mathbb{Z}_0^-) \tag{4.3}$$

and Lerch zeta function (or periodic zeta function):

$$l_s(\xi) := \sum_{n=1}^{\infty} \frac{e^{2n\pi i\xi}}{n^s} = e^{2\pi i\xi}\Phi(e^{2\pi i\xi}, s, 1) \quad (\xi \in \mathbb{R};\ \Re(s) > 1), \tag{4.4}$$

but also such other functions as the polylogarithmic function:

$$\mathrm{Li}_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z\Phi(z, s, 1) \quad (s \in \mathbb{C} \text{ when } |z| < 1;\ \Re(s) > 1 \text{ when } |z| = 1) \tag{4.5}$$

and Lipschitz–Lerch zeta function (cf. [16, p. 122, Equation 2.5 (11)]):

$$\phi(\xi, a, s) := \sum_{n=0}^{\infty} \frac{e^{2n\pi i\xi}}{(n+a)^s} = \Phi(e^{2\pi i\xi}, s, a) =: L(\xi, s, a) \quad (a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ \Re(s) > 0 \text{ when } \xi \in \mathbb{R} \setminus \mathbb{Z};\ \Re(s) > 1 \text{ when } \xi \in \mathbb{Z}), \tag{4.6}$$

which was first studied by Rudolf Lipschitz (1832–1903) and Matyáš Lerch (1860–1922) in connection with Dirichlet's famous theorem on primes in arithmetic progressions.

Recently, Srivastava made use of the Apostol's formula [2, p. 164]

$$\phi(\xi, a, 1-n) = \Phi(e^{2\pi i\xi}, 1-n, a) = -\frac{\mathcal{B}_n(a; e^{2\pi i\xi})}{n} \quad (n \in \mathbb{N}), \tag{4.7}$$

and Lerch's functional equation [2, p. 161, (1.4)]

$$\begin{aligned}\phi(\xi, a, 1-s) = \frac{\Gamma(s)}{(2\pi)^s}\Bigg\{&\exp\left[\left(\frac{1}{2}s - 2a\xi\right)\pi i\right]\phi(-a, \xi, s)\\ &+ \exp\left[\left(-\frac{1}{2}s + 2a(1-\xi)\right)\pi i\right]\phi(a, 1-\xi, s)\Bigg\}\\ &(s \in \mathbb{C};\ 0 < \xi < 1),\end{aligned} \tag{4.8}$$

to yield the following formula of Apostol–Bernoulli polynomials at rational arguments [15]:

$$\begin{aligned}\mathcal{B}_n\left(\frac{p}{q}; e^{2\pi i\xi}\right) = -\frac{n!}{(2q\pi)^n}\Bigg\{&\sum_{j=1}^{q}\zeta\left(n, \frac{\xi+j-1}{q}\right)\exp\left[\left(\frac{n}{2} - \frac{2(\xi+j-1)p}{q}\right)\pi i\right]\\ &+ \sum_{j=1}^{q}\zeta\left(n, \frac{j-\xi}{q}\right)\exp\left[\left(-\frac{n}{2} + \frac{2(j-\xi)p}{q}\right)\pi i\right]\Bigg\},\\ &(n \in \mathbb{N}\setminus\{1\};\ q \in \mathbb{N};\ p \in \mathbb{Z};\ \xi \in \mathbb{R}).\end{aligned} \tag{4.9}$$

Below we obtain a similar formula for the Apostol–Genocchi polynomials by using the Fourier expansions.

Theorem 4.1. *For* $n, q \in \mathbb{N};\ p \in \mathbb{Z};\ \xi \in \mathbb{R},\ |\xi| < 1$, *the following formula of Apostol–Genocchi polynomials at rational arguments*

$$\begin{aligned}\mathcal{G}_n\left(\frac{p}{q}; e^{2\pi i\xi}\right) = \frac{2\cdot n!}{(2q\pi)^n}\Bigg\{&\sum_{j=1}^{q}\zeta\left(n, \frac{2j+2\xi-1}{2q}\right)\exp\left[\left(\frac{n}{2} - \frac{(2j+2\xi-1)p}{q}\right)\pi i\right]\\ &+ \sum_{j=1}^{q}\zeta\left(n, \frac{2j-2\xi-1}{2q}\right)\exp\left[\left(-\frac{n}{2} + \frac{(2j-2\xi-1)p}{q}\right)\pi i\right]\Bigg\},\end{aligned} \tag{4.10}$$

holds true in terms of the Hurwitz zeta function.

Proof. Replacing k by $k-1$ in (2.3), we find

$$\mathcal{G}_n(x;\lambda) = \frac{2\cdot n! i^n}{\lambda^x}\left[\sum_{k=1}^{\infty}\frac{\exp[(n\pi/2-(2k-1)\pi x)i]}{[(2k-1)\pi i+\log\lambda]^n} + \sum_{k=1}^{\infty}\frac{\exp[(-n\pi/2+(2k-1)\pi x)i]}{[(2k-1)\pi i-\log\lambda]^n}\right]. \tag{4.11}$$

We employ the elementary series identity:

$$\sum_{k=1}^{\infty} f(k) = \sum_{j=1}^{l}\sum_{k=0}^{\infty} f(lk+j) \quad (l\in\mathbb{N}) \tag{4.12}$$

and the definition (4.1) to the formula (4.11), we obtain the following formula:

$$\mathcal{G}_n(x;\lambda)=\frac{2n! i^n\lambda^{-x}}{(2\pi i l)^n}\left[\sum_{j=1}^{l}\Phi\left(e^{-2\pi i l x},n,\frac{(2j-1)\pi i+\log\lambda}{2\pi i l}\right)\exp\left[\left(\frac{n\pi}{2}-(2j-1)\pi x\right)i\right] + \sum_{j=1}^{l}\Phi\left(e^{2\pi i l x},n,\frac{(2j-1)\pi i-\log\lambda}{2\pi i l}\right)\exp\left[\left((2j-1)\pi x-\frac{n\pi}{2}\right)i\right]\right]. \tag{4.13}$$

If setting $\lambda=\exp(2\pi i\xi)$, $x=p/q$, $l=q$ in (4.13), we then obtain the desired formula (4.10). This proof is complete. □

Taking $\xi=0$ in (4.10), we get the following corollary.

Corollary 4.1 ([7, Theorem 13]). *For $n,q\in\mathbb{N}$; $p\in\mathbb{Z}$. The following formula for the Genocchi polynomials at rational arguments*

$$G_n\left(\frac{p}{q}\right) = \frac{4\cdot n!}{(2q\pi)^n}\sum_{j=1}^{q}\zeta\left(n,\frac{2j-1}{2q}\right)\cos\left(\frac{(2j-1)p\pi}{q}-\frac{n\pi}{2}\right) \tag{4.14}$$

holds true.

REMARK 3. The same reason as Remark 1, we say here that the variable ξ take any real numbers in Theorem 4.1.

5. An explicit relationship between the Apostol–Genocchi polynomials and Gaussian hypergeometric function

Below we begin by stating and by proving the main result of this section.

Theorem 5.1. *For $n \in \mathbb{N}_0$ and $\lambda \in \mathbb{C} \setminus \{-1\}$, the following explicit series representation holds true*:

$$\begin{aligned}\mathcal{G}_n(x;\lambda) = 2n \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{\lambda^k}{(\lambda+1)^{k+1}} \sum_{j=0}^{k} (-1)^j \binom{k}{j} j^k (x+j)^{n-k-1} \\ \times {}_2F_1\left(k-n+1, k; k+1; \frac{j}{x+j}\right),\end{aligned} \tag{5.1}$$

where ${}_2F_1(a, b; c; z)$ denotes the Gaussian hypergeometric function defined by (*cf., e.g.,* [1, *p.* 556, *et seq.*])

$${}_2F_1(a, b; c; z) = {}_2F_1(b, a; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \tag{5.2}$$

$$(c \notin \mathbb{Z}_0^-;\ |z| < 1;\ z = 1 \text{ and } \Re(c-a-b) > 0;\ z = -1 \text{ and } \Re(c-a-b) > -1),$$

where

$$(\nu)_n = \nu(\nu+1)\cdots(\nu+n-1) = \frac{\Gamma(\nu+n)}{\Gamma(\nu)}, \quad \mathbb{Z}_0^- := \{0, -1, -2, \ldots\}.$$

Proof. Making use of Taylor's expansion and Leibniz's rule from (1.2), we have

$$\begin{aligned}\mathcal{G}_n(x;\lambda) &= D_z^n \left(\frac{2ze^{xz}}{\lambda e^z + 1} \right)\bigg|_{z=0} \quad (D_z = d/dz) \\ &= \frac{2}{\lambda+1} \sum_{k=1}^{n} \binom{n}{k} k x^{n-k} D_z^{k-1} \left\{ \left(1 + \frac{\lambda}{\lambda+1}(e^z - 1) \right)^{-1} \right\}\bigg|_{z=0}.\end{aligned} \tag{5.3}$$

By setting $\alpha = 1$ and $w = (\lambda/(\lambda+1))(e^z - 1)$ in the binomial expansion:

$$(1+w)^{-\alpha} = \sum_{r=0}^{\infty} \binom{\alpha+r-1}{r} (-w)^r \quad (|w| < 1), \tag{5.4}$$

and noting that (see [16, p. 58, Equation 1.5 (15)])

$$(e^z - 1)^k = k! \sum_{n=k}^{\infty} S(n, k) \frac{z^n}{n!}, \tag{5.5}$$

we obtain

$$\mathcal{G}_n(x;\lambda) = 2 \sum_{k=1}^{n} \binom{n}{k} k x^{n-k} \sum_{r=0}^{k-1} \frac{r!\,(-\lambda)^r}{(\lambda+1)^{r+1}} S(k-1, r) \tag{5.6}$$

$$= 2 \sum_{k=0}^{n} \binom{n}{k} k x^{n-k} \sum_{r=0}^{k} \frac{r!\,(-\lambda)^r}{(\lambda+1)^{r+1}} S(k-1, r). \tag{5.7}$$

Interchanging the order of summation in (5.7), we have

$$\mathcal{G}_n(x;\lambda) = 2\sum_{r=0}^{n} \frac{r!\,(-\lambda)^r}{(\lambda+1)^{r+1}} \sum_{k=r}^{n} \binom{n}{k} k x^{n-k} S(k-1,r) \tag{5.8}$$

$$= 2\sum_{r=0}^{n-1} \frac{r!\,(-\lambda)^r}{(\lambda+1)^{r+1}} \sum_{k=r+1}^{n} \binom{n}{k} k x^{n-k} S(k-1,r) \tag{5.9}$$

$$= 2\sum_{r=0}^{n-1} \frac{r!\,(-\lambda)^r}{(\lambda+1)^{r+1}} \sum_{k=0}^{n-r-1} \binom{n}{k+r+1} (k+r+1) x^{n-k-r-1} S(k+r,r). \tag{5.10}$$

Here we use the property of $S(n,k)$: when $n<k$, $S(n,k)=0$ in (5.7) and (5.9).

Applying

$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} j^n.$$

Replacing n and k by $k+r$ and r respectively, we get

$$\begin{aligned}\mathcal{G}_n(x;\lambda) = {} & 2\sum_{r=0}^{n-1} \frac{r!\,(-\lambda)^r}{(\lambda+1)^{r+1}} \sum_{k=0}^{n-r-1} \binom{n}{k+r+1} (k+r+1) x^{n-k-r-1} \frac{1}{r!} \\ & \times \sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j} j^{k+r},\end{aligned} \tag{5.11}$$

$$= 2\sum_{r=0}^{n-1} \frac{\lambda^r x^{n-r-1}}{(\lambda+1)^{r+1}} \sum_{j=0}^{r}(-1)^j\binom{r}{j} j^r \sum_{k=0}^{n-r-1} \binom{n}{k+r+1} (k+r+1)\left(\frac{j}{x}\right)^k. \tag{5.12}$$

Noting that (in view of $\binom{n}{k}=0$ when $k>n$)

$$\sum_{k=0}^{n-r-1} = \sum_{k=0}^{\infty},$$

and combining the definition of the Gaussian hypergeometric function

$${}_2F_1(a,b;c;z) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}, \tag{5.13}$$

after via some transmogrification and simplification, we give the evaluation of the third sum of (5.12) below.

$$\sum_{k=0}^{n-r-1} \binom{n}{k+r+1} (k+r+1)\left(\frac{j}{x}\right)^k = n\binom{n-1}{r}\,{}_2F_1\left(r-n+1,1;r+1;-\frac{j}{x}\right). \tag{5.14}$$

Substituting (5.14) into (5.12), and then letting $r \mapsto k$, we obtain

$$\begin{aligned}\mathcal{G}_n(x;\lambda) &= 2n \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{\lambda^k x^{n-k-1}}{(\lambda+1)^{k+1}} \\ &\quad \times \sum_{j=0}^{k} (-1)^j \binom{k}{j} j^k \, {}_2F_1\left(k-n+1, 1; k+1; -\frac{j}{x}\right).\end{aligned} \tag{5.15}$$

Finally, by applying the known Pfaff–Kummer hypergeometric transformation [1, p. 559, Equation (15.3.4)]:

$$\begin{aligned}{}_2F_1(a, b; c; z) &= (1-z)^{-a} \, {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right) \\ &\left(c \notin \mathbb{Z}_0^-; \ |\arg(1-z)| \leq \pi - \varepsilon \ (0 < \varepsilon < \pi)\right)\end{aligned} \tag{5.16}$$

to the equation (5.15), we arrive at the desired (5.1). This completes our proof. □

Setting $\lambda = 1$ in (5.1), we get the following corollary.

Corollary 5.1. *The following series representation for the Genocchi polynomials holds true*:

$$\begin{aligned}G_n(x) &= n \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{1}{2^k} \\ &\quad \times \sum_{j=0}^{k} (-1)^j \binom{k}{j} j^k (x+j)^{n-k-1} \, {}_2F_1\left(k-n+1, k; k+1; \frac{j}{x+j}\right).\end{aligned} \tag{5.17}$$

On the other hand, by using (1.2) in conjunction with the definition (1.1), it is easy to observe that

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{G}_n(x;\lambda) \frac{z^n}{n!} &= e^{-x\log\lambda} \frac{2(z+\log\lambda)}{e^{z+\log\lambda}+1} \frac{z}{z+\log\lambda} e^{x(z+\log\lambda)} \\ &= e^{-x\log\lambda} \sum_{k=0}^{\infty} G_k(x) \frac{(z+\log\lambda)^{k-1} z}{k!} \\ &= e^{-x\log\lambda} \sum_{k=0}^{\infty} G_k(x) \sum_{n=1}^{k} \binom{k-1}{n-1} \frac{z^n (\log\lambda)^{k-n}}{k!} \\ &= e^{-x\log\lambda} \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{k=0}^{\infty} \binom{n+k-1}{k} \binom{n+k}{k}^{-1} G_{n+k}(x) \frac{(\log\lambda)^k}{k!},\end{aligned}$$

which yields the following lemma:

Lemma 5.1. *For $n \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$, the Apostol–Genocchi polynomials is represented by*

$$\mathcal{G}_n(x;\lambda) = e^{-x\log\lambda}\sum_{k=0}^{\infty}\binom{n+k-1}{k}\binom{n+k}{k}^{-1}G_{n+k}(x)\frac{(\log\lambda)^k}{k!}, \tag{5.18}$$

in terms of the Genocchi polynomials.

Theorem 5.2. *For $n \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$, the following explicit series representation*

$$\begin{aligned}\mathcal{G}_n(x;\lambda) &= ne^{-x\log\lambda}\sum_{k=0}^{\infty}\frac{(\log\lambda)^k}{k!}\sum_{r=0}^{n+k-1}\frac{1}{2^r}\binom{n+k-1}{r}\\ &\quad\times\sum_{j=0}^{r}(-1)^j\binom{r}{j}j^r(x+j)^{n+k-r-1}\,{}_2F_1\left(r-n-k+1,r;r+1;\frac{j}{x+j}\right),\end{aligned} \tag{5.19}$$

holds true in terms of the Gaussian hypergeometric function.

Proof. By (5.17) and (5.18), we then obtain the assertion (5.19) immediately. □

REMARK 4. The proof of Theorem 5.1 can be applied *mutatis mutandis* in order to obtain an explicit formula for the Apostol–Genocchi polynomials $\mathcal{G}_n(x;\lambda)$ involving the Stirling numbers of the second kind as follows:

$$\mathcal{G}_n(x;\lambda) = 2\sum_{k=0}^{n}\binom{n}{k}k\sum_{j=0}^{k-1}\frac{j!(-\lambda)^j}{(\lambda+1)^{j+1}}S(k-1,j)x^{n-k} \quad (n \in \mathbb{N}_0;\ \lambda \in \mathbb{C}\setminus\{-1\}). \tag{5.20}$$

Further, setting $\lambda = 1$ in (5.20), we deduce the following formula for the Genocchi polynomials:

$$G_n(x) = \sum_{k=0}^{n}\binom{n}{k}k\sum_{j=0}^{k-1}\frac{j!\,(-1)^j}{2^j}S(k-1,j)x^{n-k} \quad (n \in \mathbb{N}_0). \tag{5.21}$$

REMARK 5. Applying the Gaussian summation theorem [1, p. 556, Equation (15.1.20)]:

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad (c \notin \mathbb{Z}_0^-;\ \Re(c-a-b) > 0)$$

for

$$a = k-n+1, \quad b = k, \quad \text{and} \quad c = k+1,$$

so that

$${}_2F_1(k-n+1,k;k+1;1) = \binom{n-1}{k}^{-1} \quad (k = 0,\ldots,n-1;\ n \in \mathbb{N}_0). \tag{5.22}$$

Therefore, by setting $x = 0$ in (5.1), we obtain the following explicit representation for the Apostol–Genocchi numbers involving the Stirling numbers of the second kind:

$$\mathcal{G}_n(\lambda) = 2n \sum_{k=0}^{n-1} \frac{k!\,(-\lambda)^k}{(\lambda+1)^{k+1}} S(n-1,k) \quad (n \in \mathbb{N}_0;\ \lambda \in \mathbb{C} \setminus \{-1\}). \tag{5.23}$$

Taking $\lambda = 1$ in (5.23), then we deduce the following formula for the Genocchi numbers:

$$G_n = n \sum_{k=0}^{n-1} \frac{(-1)^k k!}{2^k} S(n-1,k) \quad (n \in \mathbb{N}_0).$$

If $n = 0$, then the sum is considered to be null.

REMARK 6. Using the formula (5.23), we may calculate the first values of the Apostol–Genocchi numbers:

$$\mathcal{G}_0(\lambda) = 0, \quad \mathcal{G}_1(\lambda) = \frac{2}{\lambda+1}, \quad \mathcal{G}_2(\lambda) = -\frac{4\lambda}{(\lambda+1)^2},$$
$$\mathcal{G}_3(\lambda) = \frac{6\lambda(\lambda-1)}{(\lambda+1)^3}, \quad \mathcal{G}_4(\lambda) = -\frac{8\lambda(\lambda^2-4\lambda+1)}{(\lambda+1)^4},$$
$$\mathcal{G}_5(\lambda) = \frac{10\lambda(\lambda^3-11\lambda^2+11\lambda-1)}{(\lambda+1)^5}, \quad \mathcal{G}_6(\lambda) = -\frac{12\lambda(\lambda^4-26\lambda^3+66\lambda^2-26\lambda+1)}{(\lambda+1)^6}.$$

REMARK 7. The elementary properties of the Apostol–Genocchi polynomials can be readily derived from (1.2).

$$\lambda\mathcal{G}_n(x+1;\lambda) + \mathcal{G}_n(x;\lambda) = 2nx^{n-1} \ (n \geq 1), \quad \frac{\partial}{\partial x}\mathcal{G}_n(x;\lambda) = n\mathcal{G}_{n-1}(x;\lambda),$$
$$\int_a^b \mathcal{G}_n(x;\lambda)\,\mathrm{d}x = \frac{\mathcal{G}_{n+1}(b;\lambda) - \mathcal{G}_{n+1}(a;\lambda)}{n+1}, \quad \mathcal{G}_n(x+y;\lambda) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{G}_k(x;\lambda) y^{n-k},$$
$$\mathcal{G}_n(1-x;\lambda) = \frac{(-1)^{n+1}}{\lambda}\mathcal{G}_n(x;\lambda^{-1}).$$

6. Fourier expansions and integral representations for the Genocchi polynomials

In this section, we deduce the Fourier expansions and integral representations for the Genocchi polynomials.

Putting $\lambda = 1$ in Theorem 2.1, we then deduce the Fourier expansions for the Genocchi polynomials as follows:

Theorem 6.1 ([7, Theorem 1]). *For $n > 0$, $0 \le x \le 1$, we have*

$$G_n(x) = \frac{2 \cdot n!}{(\pi i)^n} \sum_{k \in \mathbb{Z}} \frac{e^{(2k-1)\pi i x}}{(2k-1)^n}, \tag{6.1}$$

$$= \frac{4 \cdot n!}{\pi^n} \sum_{k=0}^{\infty} \frac{\cos[(2k+1)\pi x - n\pi/2]}{(2k+1)^n}. \tag{6.2}$$

From Theorem 6.1, we have

Corollary 6.1 ([7, Corollary 2]). *For $n > 0$, $0 \le x \le 1$, we have*

$$G_{2n}(x) = (-1)^n \frac{4 \cdot (2n)!}{\pi^{2n}} \sum_{k=0}^{\infty} \frac{\cos[(2k+1)\pi x]}{(2k+1)^{2n}}, \tag{6.3}$$

$$G_{2n-1}(x) = (-1)^{n-1} \frac{4 \cdot (2n-1)!}{\pi^{2n-1}} \sum_{k=0}^{\infty} \frac{\sin[(2k+1)\pi x]}{(2k+1)^{2n-1}}. \tag{6.4}$$

By setting $\xi = 0$ in (3.1) and (3.4), we have the following integral representations for the Genocchi polynomials.

Theorem 6.2 ([7, Theorem 4 and Corollary 9]). *For $n = 1, 2, \ldots;$ $0 \le x \le 1$, we have*

$$G_n(x) = 2n \int_0^{\infty} \frac{e^{\pi t} \cos(\pi x - n\pi/2) - e^{-\pi t} \cos(\pi x + n\pi/2)}{\cosh 2\pi t - \cos 2\pi x} t^{n-1}\, \mathrm{d}t, \tag{6.5}$$

$$G_n(x) = (-1)^{n-1} \frac{4n}{\pi^n} \int_0^1 \frac{\cos(\pi x - n\pi/2) - t^2 \cos(\pi x + n\pi/2)}{t^4 - 2t^2 \cos 2\pi x + 1} (\log t)^{n-1}\, \mathrm{d}t. \tag{6.6}$$

Obviously, Theorem 6.2 implies the following corollary.

Corollary 6.2 ([7, Corollary 5 and 10]). *For $n = 1, 2, \ldots;$ $0 \le x \le 1$, we have*

$$G_{2n-1}(x) = 4(2n-1)(-1)^{n-1} \int_0^{\infty} \frac{\sin \pi x \cosh \pi t}{\cosh 2\pi t - \cos 2\pi x} t^{2n-2}\, \mathrm{d}t, \tag{6.7}$$

$$G_{2n}(x) = 8n(-1)^n \int_0^{\infty} \frac{\cos \pi x \sinh \pi t}{\cosh 2\pi t - \cos 2\pi x} t^{2n-1}\, \mathrm{d}t, \tag{6.8}$$

$$G_{2n-1}(x) = (-1)^{n-1} \frac{4(2n-1)}{\pi^{2n-1}} \int_0^1 \frac{(1+t^2) \sin \pi x}{t^4 - 2t^2 \cos 2\pi x + 1} (\log t)^{2n-2}\, \mathrm{d}t, \tag{6.9}$$

$$G_{2n}(x) = (-1)^{n-1} \frac{8n}{\pi^{2n}} \int_0^1 \frac{(1-t^2) \cos \pi x}{t^4 - 2t^2 \cos 2\pi x + 1} (\log t)^{2n-1}\, \mathrm{d}t. \tag{6.10}$$

Below we derive other integral representations for the Genocchi polynomials. We first need the following lemma.

Lemma 6.1 (see [13]).

$$\int \frac{2t(1-t^2)\cos x}{t^4-2t^2\cos 2x+1}\,\mathrm{d}x = \arctan\left(\frac{2t\sin x}{1-t^2}\right)+C, \tag{6.11}$$

$$\int \frac{4t(1+t^2)\sin x}{t^4-2t^2\cos 2x+1}\,\mathrm{d}x = \log\left(\frac{t^2-2t\cos x+1}{t^2+2t\cos x+1}\right)+C, \tag{6.12}$$

$$\int_0^1 \frac{\log(1+t)(\log t)^{n-1}}{t}\,\mathrm{d}t = (-1)^n(n-1)!\zeta(n+1)(2^{-n}-1), \tag{6.13}$$

$$\int_0^1 \frac{\log(1-t)(\log t)^{n-1}}{t}\,\mathrm{d}t = (-1)^n(n-1)!\zeta(n+1). \tag{6.14}$$

It follows that we give the following theorem.

Theorem 6.3 ([7, Corollary 11]). *For* $n=1,2,\ldots;\ 0\le x\le 1$, *we have*

$$G_{2n+1}(x) = (-1)^{n-1}\frac{4n(2n+1)}{\pi^{2n+1}}\int_0^1 \arctan\left(\frac{2t\sin\pi x}{1-t^2}\right)\frac{(\log t)^{2n-1}}{t}\,\mathrm{d}t, \tag{6.15}$$

$$G_{2n}(x) = (-1)^{n-1}\frac{2n(2n-1)}{\pi^{2n}}\int_0^1 \log\left(\frac{t^2-2t\cos\pi x+1}{t^2+2t\cos\pi x+1}\right)\frac{(\log t)^{2n-2}}{t}\,\mathrm{d}t. \tag{6.16}$$

Proof. Recalling the basic property of the Genocchi polynomials:

$$\int_a^x G_n(u)\,\mathrm{d}u = \frac{G_{n+1}(x)-G_{n+1}(a)}{n+1}. \tag{6.17}$$

From (6.17), we have

$$G_{2n}(x) = 2n\int_0^x G_{2n-1}(u)\,\mathrm{d}u + G_{2n}. \tag{6.18}$$

Letting $x\mapsto u$ in (6.9), and then substituting this into (6.18), we obtain

(6.19)

$$\begin{aligned} G_{2n}(x) &= 2n\int_0^x (-1)^{n-1}\frac{4(2n-1)}{\pi^{2n-1}}\int_0^1 \frac{(1+t^2)\sin\pi u}{t^4-2t^2\cos 2\pi u+1}(\log t)^{2n-2}\,\mathrm{d}t\,\mathrm{d}u + G_{2n} \\ &= (-1)^{n-1}\frac{2n(2n-1)}{\pi^{2n-1}}\int_0^1 \frac{(\log t)^{2n-2}}{t}\,\mathrm{d}t\int_0^x \frac{4t(1+t^2)\sin\pi u}{t^4-2t^2\cos 2\pi u+1}\,\mathrm{d}u + G_{2n}. \end{aligned}$$

Making the transformation $x=\pi u$ in (6.12), we have

(6.20)

$$\begin{aligned} \pi\int_0^x \frac{4t(1+t^2)\sin\pi u}{t^4-2t^2\cos 2\pi u+1}\,\mathrm{d}u &= \log\left(\frac{t^2-2t\cos\pi x+1}{t^2+2t\cos\pi x+1}\right)-\log\left(\frac{t^2-2t+1}{t^2+2t+1}\right) \\ &= \log\left(\frac{t^2-2t\cos\pi x+1}{t^2+2t\cos\pi x+1}\right)-2\log\left(\frac{1-t}{1+t}\right). \end{aligned}$$

Substituting (6.20) into (6.19), we get

$$
\begin{aligned}
G_{2n}(x) &= (-1)^{n-1}\frac{2n(2n-1)}{\pi^{2n}}\int_0^1 \log\left(\frac{t^2-2t\cos\pi x+1}{t^2+2t\cos\pi x+1}\right)\frac{(\log t)^{2n-2}}{t}\,\mathrm{d}t \\
&\quad + (-1)^n\frac{4n(2n-1)}{\pi^{2n}}\int_0^1 \log\left(\frac{1-t}{1+t}\right)\frac{(\log t)^{2n-2}}{t}\,\mathrm{d}t + G_{2n}.
\end{aligned}
\tag{6.21}
$$

Subtracting (6.13) from (6.14), we have

$$
\int_0^1 \log\left(\frac{1-t}{1+t}\right)\frac{(\log t)^{n-1}}{t}\,\mathrm{d}t = (-1)^n(n-1)!\,\zeta(n+1)(2-2^{-n}). \tag{6.22}
$$

From (6.22), it is easy to show that

$$
\int_0^1 \log\left(\frac{1-t}{1+t}\right)\frac{(\log t)^{2n-2}}{t}\,\mathrm{d}t = -(2n-2)!\,\zeta(2n)(2-2^{1-2n}). \tag{6.23}
$$

On the other hand, we recall the well-known formula (see [5, p. 35, (21)]):

$$
\zeta(2n) = \frac{(-1)^{n-1}(2\pi)^{2n}}{2(2n)!}B_{2n}.
$$

Using the known relation

$$
G_n = 2(1-2^n)B_n,
$$

we have

$$
G_{2n} = \frac{(-1)^{n-1}2^2(1-2^{2n})(2n)!}{(2\pi)^{2n}}\zeta(2n). \tag{6.24}
$$

Substituting (6.23) and (6.24) into (6.21), after a simplification, we obtain the formula (6.16) immediately.

Similarly, the formula (6.15) can also be proved. This proof is complete. □

By (6.8), (6.10) and (6.16), we have the integral representations for the Genocchi numbers.

Corollary 6.3 ([7, Corollary 12]). *For* $n = 0, 1, \ldots,$ *we have*

$$
\begin{aligned}
G_{2n} &= 4n(-1)^n\int_0^\infty \frac{t^{2n-1}}{\sinh(\pi t)}\,\mathrm{d}t && (6.25)\\
&= -\frac{8n(-1)^n}{\pi^{2n}}\int_0^1 \frac{(\log t)^{2n-1}}{1-t^2}\,\mathrm{d}t && (6.26)\\
&= (-1)^{n-1}\frac{4n(2n-1)}{\pi^{2n}}\int_0^1 \log\left(\frac{1-t}{1+t}\right)\frac{(\log t)^{2n-2}}{t}\,\mathrm{d}t. && (6.27)
\end{aligned}
$$

REMARK 8. The integral representations for Genocchi polynomials and numbers do not appear in the classical literatures, for example [1], [5] and [12]. Hence, the formulas of this section are presumably new.

7. Further observations and consequences

We define zeta functions as follows ($\xi \in \mathbb{R}$)

$$\begin{aligned}\lambda(n;\xi) &= \sum_{k=0}^{\infty} \frac{1}{(2k+2\xi+1)^n} = 2^{-n}\zeta\left(n, \frac{2\xi+1}{2}\right),\\ \lambda(n) &= \sum_{k=0}^{\infty} \frac{1}{(2k+1)^n} = 2^{-n}\zeta\left(n, \frac{1}{2}\right).\end{aligned} \tag{7.1}$$

Setting $\lambda = e^{2\pi i\xi}$ in (2.3) of Theorem 2.1, then we have

$$\begin{aligned}\mathcal{G}_n(x; e^{2\pi i\xi}) = \frac{2\cdot n!}{\pi^n e^{2\pi i\xi x}}\Bigg[&\sum_{k=0}^{\infty} \frac{\exp[(n\pi/2 - (2k+1)\pi x)i]}{(2k+2\xi+1)^n}\\ &+ \sum_{k=0}^{\infty} \frac{\exp[(-n\pi/2 + (2k+1)\pi x)i]}{(2k-2\xi+1)^n}\Bigg].\end{aligned} \tag{7.2}$$

Taking $x = 0$ in (7.2) and, noting that $\mathcal{G}_n(\lambda) = \mathcal{G}_n(0;\lambda)$ and the definition (7.1), we obtain the relationship between the Apostol–Genocchi numbers $\mathcal{G}_n(e^{2\pi i\xi})$ and the zeta function $\lambda(n;\xi)$:

$$\mathcal{G}_n(e^{2\pi i\xi}) = \frac{2\cdot n!}{\pi^n}\left[\exp\left(\frac{n\pi}{2}i\right)\lambda(n;\xi) + \exp\left(-\frac{n\pi}{2}i\right)\lambda(n;-\xi)\right]. \tag{7.3}$$

Replacing n by $2n$ in (7.3), we have

$$\mathcal{G}_{2n}(e^{2\pi i\xi}) = (-1)^n \frac{2\cdot(2n)!}{\pi^{2n}}[\lambda(2n;\xi) + \lambda(2n;-\xi)]. \tag{7.4}$$

If putting $\xi = 0$ in (7.4), we arrive directly at the following formula:

$$\lambda(2n) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{2n}} = \frac{(-1)^n\pi^{2n}}{4(2n)!}G_{2n}, \tag{7.5}$$

and noting that the formula $\lambda(n) = (1-2^{-n})\zeta(n)$ and $G_n = 2(1-2^n)B_n$ (see [1, p. 807, 23.2.20]), we derive the Euler formula:

$$\zeta(2n) = \frac{(-1)^{n-1}(2\pi)^{2n}}{2(2n)!}B_{2n}.$$

REMARK 9. We can also obtain the formulas (7.5) by (4.10).

REMARK 10. By taking $x = 0$ in Theorem 3.1 and Theorem 3.2 respectively, we deduce the integral representations for the Apostol–Genocchi polynomials numbers as follows:

$$\mathcal{G}_n(e^{2\pi i\xi}) = 2n\int_0^\infty \frac{\cos(n\pi/2)\cosh(2\pi\xi t) - i\,\sin(n\pi/2)\sinh(2\pi\xi t)}{\sinh(\pi t)} t^{n-1}\,\mathrm{d}t \tag{7.6}$$

$$= (-1)^{n-1}\frac{4n}{\pi^n}\int_0^1 \frac{\cos(n\pi/2)\cosh(2\xi\,\log t) + i\,\sin(n\pi/2)\sinh(2\xi\,\log t)}{1-t^2} \times (\log t)^{n-1}\,\mathrm{d}t. \tag{7.7}$$

Further setting $\xi = 0$ in (7.6) and (7.7) respectively, we deduce the integral representations for the Genocchi numbers in Corollary 6.3 once again.

REMARK 11. By (1.2) and the binomial theorem, yields that

$$\begin{aligned}\sum_{n=0}^\infty \mathcal{G}_n(a;\lambda)\frac{z^n}{n!} &= \frac{2ze^{az}}{\lambda e^z + 1} = 2z\sum_{k=0}^\infty (-\lambda)^k e^{(k+a)z} \\ &= \sum_{n=0}^\infty\left[2n\sum_{k=0}^\infty(-\lambda)^k(k+a)^{n-1}\right]\frac{z^n}{n!} = \sum_{n=0}^\infty\left[2n\sum_{k=0}^\infty\frac{(-\lambda)^k}{(k+a)^{1-n}}\right]\frac{z^n}{n!}.\end{aligned} \tag{7.8}$$

Therefore, we show an interesting relationship between the Apostol–Genocchi polynomials and Hurwitz–Lerch zeta function:

$$\mathcal{G}_n(a;\lambda) = 2n\Phi(-\lambda, 1-n, a) \quad (n\in\mathbb{N};\ \lambda\in\mathbb{C};\ |\lambda|\leqq 1;\ a\in\mathbb{C}\setminus\mathbb{Z}_0^-). \tag{7.9}$$

Therefore, we can also prove Theorem 2.1 by applying the relationship (7.9) in conjunction with Lerch's functional equation (4.8). Theorem 4.1 can also be proved with Lerch's functional equation (4.8), elementary series (4.12) and (7.9).

REMARK 12. Our methods in the present paper can be used to investigate the corresponding Apostol–Bernoulli and Apostol–Euler polynomials together with their classical cases, which have appeared in [8].

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