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CURVATURE PROPERTIES OF THE SLOWNESS SURFACE OF THE SYSTEM OF CRYSTAL ACOUSTICS FOR CUBIC CRYSTALS

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Abstract

In this paper we study geometric properties of the slowness surface of the system of crystal acoustics for cubic crystals. In particular we shall study curvature properties of the surface and the behaviour of the surface near singular points. The main result is that in the generic nearly isotropic case there are no planes which are tangent to the surface along entire curves. This is in contrast with what happens for the slowness surface of the system of crystal optics for bi-axial crystals. Geometric information of the type we shall obtain is needed to understand the long-time behaviour of global solutions of the system of crystal acoustics.

1. Introduction

The results in this paper are part of an attempt to understand the long time behavior of global solutions of the homogeneous system of crystal acoustics for cubic crystals. An essential part in this undertaking is to understand curvature properties of the slowness surface S associated with the system and to have information on the Gauss map (i.e., the map $\xi \to \vec{n}(\xi)$, where for $\xi \in S$, $\vec{n}(\xi)$ is the normal to S in ξ) defined on S. The main purpose of this paper is to provide this information. In principle our results also show that in the generic nearly isotropic case there is no internal conical refraction for acoustic waves in cubic crystals, but we shall not explain this statement any further. What we shall show is essentially (for more precise statements, see the theorems) that:

- when the crystal under consideration is not isotropic, then the total curvature of the slowness surface will always vanish along a number of curves,
- in the nearly isotropic case the mean curvature vanishes nowhere,
- and, still in the nearly isotropic case and generically, there are no planes tangent to S along entire curves.

The interest in results of this type in wave propagation for crystals has a long history. Indeed, in the case of crystal optics the surface corresponding to the slowness

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surface is Fresnel's surface, and R.S. Hamilton discovered in 1837 that for optically biaxial crystals there are planes which are tangent to Fresnel's surface along entire circles. (Cf. e.g., [1].) He inferred from this that for such crystals light rays which enter a crystal will not always propagate inside the crystal along one single straight ray, but could split up, at least in certain specific situations, into cones of light. When this phenomenon, which now goes under the name of "conical refraction", was confirmed experimentally a year later, it brought R.S. Hamilton instantaneous fame and led, still in the 19-th century, to a long series of papers on Fresnel's surface (both in the mathematical and in the physical literature.) A good reference for the history of the study of Fresnel's surface is [16]. Also see [9] for some more recent results in algebraic geometry which have their origin in this kind of problems. We are not aware of any comparable efforts made for the slowness surfaces which appear in crystal acoustics.

As for the present paper, the starting point has been to study decay estimates of global solutions of the system of crystal acoustics for cubic crystals. Cubic crystals are together with crystals from the hexagonal class the simplest non-isotropic crystals and seem interesting enough to merit an independent study. Consequently, we shall in fact only study the slowness surface for cubic crystals and we should say that we have not seriously tried to understand what happens for crystals in other classes. We regard our problem from a purely mathematical point of view, in the same vein in which decay estimates have been studied for the related case of the wave equation $\Box = \partial_t^2 - \sum_{j=1}^n \partial_{x_j}^2$. We also recall that decay estimates have been used to prove long-time existence for solutions of non-linear perturbations of the wave equation (cf. [6], [7] and many other papers) and a similar study has been undertaken for non-linear perturbations of Maxwell's system for optically biaxial crystals (cf. [11], [17]). Among the many papers on related arguments we only mention [21], [26] and [28] (which treats hexagonal crystals), respectively [3].

Rather than explaining the exact relation between decay estimates for solutions of the system of crystal acoustics and curvature properties for the associated slowness surface, we mention the following two results which are to some extend a preliminary step in this link and which have an independent interest.

Theorem 1.1 (Hlawka 1950, [4], [5]). *Let $S \subset \mathbb{R}^n$ be a smooth compact surface with nowhere vanishing total curvature. Also let $u\colon S \to \mathbb{C}$ be a smooth function on S. Then there is a constant $c > 0$ such that the Fourier transform $I(x)$ of $u\, d\sigma$, $d\sigma$ the surface element on S, defined by*

$$I(x) = \int_S \exp[i\langle x, \xi\rangle] u(\xi)\, d\sigma(\xi), \tag{1.1}$$

satisfies the estimate

$$|I(x)| \le c(1 + |x|)^{-(n-1)/2}. \tag{1.2}$$

Proposition 1.2. *Let $S \subset \mathbb{R}^3$ be a smooth algebraic surface given by a polynomial equation $S = \{\xi \in \mathbb{R}^3;\ p(\xi) = 0\}$, let $U' \Subset U \subset \mathbb{R}^3$ be open and bounded and assume that the following assumptions are satisfied*:

a) $\nabla_\xi p(\xi) \neq 0$ *for* $\xi \in S \cap U$ *and the mean curvature of* $S \cap U$ *does not vanish,*

b) *there is no plane tangent to* S *along an entire curve.*

Also consider some smooth function $u\colon S \to \mathbb{C}$ *such that* $u(\xi) = 0$ *if* $\xi \notin U'$. *Then there is a natural number* $k \geq 2$ *such that* $I(x) = \int_S \exp[i\langle x, \xi\rangle]u(\xi)\, d\sigma(\xi)$ ($d\sigma$ *the surface element on* S) *satisfies the estimate*

$$|I(x)| \leq c(1 + |x|)^{-1/2-1/k}. \tag{1.3}$$

("∇" denotes here and later on the gradient of some function f in the "natural variables" of the function.)

In order to give a flavor of why results on curvature and tangent planes along curves are related to decay estimates, we shall prove Proposition 1.2 in Section 2 below. Theorem 1.1 on the other hand is the first of a long list of results on estimates for Fourier (inverse) transforms of densities which live on surfaces in higher dimensions. (See [27].) In [4] it is used in a context of number theory (viz., the Gauss "Kreisproblem"), but it is clearly also linked to the counting function in eigenvalue problems for elliptic operators. Closer to the initial motivation for this paper is that it is underlying the classical estimates on long time behavior of solutions of the wave equation. The relation of Proposition 1.2 with decay estimates for solutions of the system of crystal acoustics will be explained in a forthcoming paper.

We should mention from the very beginning that most of the results in this paper shall be obtained by a perturbation argument starting from the isotropic case. It is for this reason that we can only obtain results for the nearly isotropic case and the case of general cubic crystals can probably only be treated using a different approach.

We now recall the system of crystal acoustics (or "crystal-elasticity") for cubic crystals in some detail. Additional information on crystal-acoustics for crystals, cubic or not, can be found e.g., in [2] and [19].

We are only interested in the homogeneous equation and in global solutions defined on $\mathbb{R}_t \times \mathbb{R}^3_x$ of the system. The system the solution will then satisfy has the following form:

$$P(D)u = \begin{pmatrix} \partial_t^2 - a\partial_{x_1}^2 - c\Delta & -b\partial_{x_1}\partial_{x_2} & -b\partial_{x_1}\partial_{x_3} \\ -b\partial_{x_2}\partial_{x_1} & \partial_t^2 - a\partial_{x_2}^2 - c\Delta & -b\partial_{x_2}\partial_{x_3} \\ -b\partial_{x_3}\partial_{x_1} & -b\partial_{x_3}\partial_{x_2} & \partial_t^2 - a\partial_{x_3}^2 - c\Delta \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 0 \tag{1.4}$$

with Δ denoting here the Laplacian in the variables $x = (x_1, x_2, x_3)$. (Later on, we shall use the letter "Δ" for "discriminants".)

Correspondingly the characteristic polynomial of the system is given by $\det P(\tau,\xi)$, $\tau \in \mathbb{R}$, $\xi \in \mathbb{R}^3$, where

$$(1.5)\qquad P(\tau,\xi) = -\begin{pmatrix} \tau^2 - a\xi_1^2 + c|\xi|^2 & -b\xi_1\xi_2 & -b\xi_1\xi_3 \\ -b\xi_2\xi_1 & \tau^2 - a\xi_2^2 + c|\xi|^2 & -b\xi_2\xi_3 \\ -b\xi_3\xi_1 & -b\xi_3\xi_2 & \tau^2 - a\xi_3^2 + c|\xi|^2 \end{pmatrix}.$$

The characteristic surface associated with the system is $\{(\tau,\xi) \in \mathbb{R}^4;\ \det P(\tau,\xi) = 0\}$. It is useful to write the characteristic polynomial in what is called "Kelvin's" form. We do so again only for the particular case of cubic crystals, when Kelvin's form is (cf. [2]):

$$(1.6)\qquad \frac{b\xi_1^2}{\tau^2 - c|\xi|^2 + (b-a)\xi_1^2} + \frac{b\xi_2^2}{\tau^2 - c|\xi|^2 + (b-a)\xi_2^2} + \frac{b\xi_3^2}{\tau^2 - c|\xi|^2 + (b-a)\xi_3^2} = 1.$$

In all this the a, b, c are real constants which can be calculated in terms of the 3 "essential" stiffness constants of a cubic crystal. (Cf. [2] and [13].) The fact that (1.6) defines the characteristic surface of a cubic crystal gives some restrictions on the a, b, c. Of these we mention that we must have $c > 0$, $a \neq 0$, $a+c > 0$, $3c-b+a > 0$ (see e.g., [13]). As in [2] we shall often assume that $b > 0$ (we shall justify this assumption in a moment) but additional restrictions shall be introduced later on in this introduction.

We also mention that the crystal is isotropic if and only if $a = b$. Recall that in the isotropic case, acoustic, respectively elastic, phenomena can be understood in terms of the Lamé constants "λ" and "μ". Expressed in terms of b and c they are $\lambda + 2\mu = c + b$, $\mu = c$. As to the physical interpretation, $\sqrt{\mu}$ is the velocity of the two shear waves, whereas $\sqrt{\lambda + 2\mu}$ is the velocity of the pressure wave. (Shear waves are also called transversal waves, whereas pressure waves are called longitudinal or compression waves.) Since the velocity of the shear waves is bigger than that of the pressure waves, in the isotropic case we must have $b > 0$. (Cf. [10], Section 22, where it is stated that we must have $\sqrt{\lambda + 2\mu} \geq \sqrt{4\mu/3}$. See also the footnote nr. 2 there.) We denote $b - a$ by d, so that d becomes a measure for the anisotropy of the crystal. While in (1.5) the "main" constants are visibly a, b, c, it may be argued that the primary constants in (1.6) are b, c, d and we shall write down calculations in terms of b, c, d henceforth.

The polynomial $\det P$ is immediately seen to be homogeneous in the variables (τ,ξ) and of degree six. The system (1.4) is a particular case of the system of elasticity for elastic media and as such it is known to be hyperbolic with respect to the time variable (when some conditions on the stiffness constants are satisfied). "Hyperbolicity" then implies that for every $\xi \in \mathbb{R}^3$ the equation $P(\tau,\xi) = 0$ has 6 real roots τ if multiplicities are counted, and it is obvious that for every fixed $\xi \neq 0$ three of them are positive and three negative. (Actually we shall always also assume that the roots are $\neq 0$

for $\xi \neq 0$.) Related to this is the fact that the surface defined by $S = \{\xi \in \mathbb{R}^3;\ q(\xi) = 0\}$,

$$q(\xi) = \prod_{j=1}^{3}(1 - c|\xi|^2 + d\xi_j^2) - \sum_{j=1}^{3} b\xi_j^2(1 - c|\xi|^2 + d\xi_{j+1}^2)(1 - c|\xi|^2 + d\xi_{j+2}^2) \tag{1.7}$$

(indices are counted modulo 3) is a closed bounded surface in $\mathbb{R}^3$.

DEFINITION 1.3. The surface S defined by the condition $q(\xi) = 0$ is called the "slowness surface" of the crystal. It is essentially the intersection of the characteristic surface $\{(\tau, \xi) \in \mathbb{R}^4;\ \det P(\tau, \xi) = 0\}$ with the plane $\tau = 1$.

When $d = 0$, the equation of the slowness surface reduces to $(1 - c|\xi|^2)^2(1 - (c + b)|\xi|^2) = 0$, and the slowness surface is the union of the "double sphere" $(1-c|\xi|^2)^2 = 0$ with the sphere $1-(b+c)|\xi|^2 = 0$. This is meaningful of course precisely when $c+b > 0$, so in particular we see that from a mathematical point of view there are cases with "$b < 0$" which make sense, although they may not have any physical interest.

REMARK 1.4. We say that some property holds in the nearly isotropic case, if for fixed b^0, c^0 there is $\varepsilon > 0$ such that the property holds when $|b - b^0| + |c - c^0| + |d| \leq \varepsilon$.

REMARK 1.5. As is already clear from the preceding remarks, almost all entities which we shall encounter later on depend on the value of the constants b, c, d. However, if we would make these dependencies explicit in the notations, then the notations would become rather heavy. If we consider some entity, as for example some function ρ, which depends on the variables ξ and also on the constants $b, c, d,$ we shall then write for example $\rho(\xi, b, c, d)$, $\rho(\xi, d)$ or $\rho(\xi)$, according to which, if any, of the constants b, c, d are relevant in the argument under consideration.

We now return to the discussion of some further restrictions on the constants a, b, c, d. One highly degenerate case is when $b = 0$. In this case the slowness surface has the form $\left\{\xi;\ \prod_{j=1}^{3}(1-c|\xi|^2+d\xi_j^2) = 0\right\}$, and the system is hyperbolic only when $d < c$.

More generally speaking, we shall call some triple of constants (b, c, d) "admissible", if the characteristic equation $\det P(\tau, \xi) = 0$ admits 3 strictly positive roots τ for every fixed $\xi \in \mathbb{R}^3 \setminus \{0\}$. (By continuity this implies that there is then a constant $\tilde{c} > 0$ such that $|\tau| \geq \tilde{c}|\xi|$ whenever $\det P(\tau, \xi) = 0$.) Thus "admissibility" refers in this paper strictly to the mathematical question of studying the characteristic surface associated with the system P and is not a question of physical relevance. Since our main interest in this paper is in the nearly isotropic case, we shall assume whenever convenient that $d < c$. (Note that this condition is also necessary if we want to work with a condition on d which is independent of the value of b.)

Kelvin's form of the characteristic equation is very useful when we want to obtain further information on the admissible region. This is based on the following remark (see [19] and also [13]): if a_i, b_i, $i = 1, 2, 3$ are strictly positive constants, $0 < a_1 < a_2 < a_3$, then the polynomial

$$g(t) = \prod_{i=1}^{3}(t - a_i) - \sum_{j=1}^{3} b_j(t - a_{j+1})(t - a_{j+2}), \tag{1.8}$$

is negative at $t = a_1$, $t = a_3$ and positive at $t = a_2$, $t = \infty$. It follows that it must have three positive roots, one in each of the intervals $(a_1, a_2), (a_2, a_3), (a_3, \infty)$. (Cf. [2] and also [13]. A sharper statement is (6.5) below.) On the other hand, when $b_j < 0$, but still $a_j \geq 0$, $j = 1, 2, 3$, then by a similar argument we shall have three strictly positive roots precisely when $g(0) < 0$. When we apply this for $a_j = c|\xi|^2 - d\xi_j^2$, $b_j = b\xi_j^2$ and assume $d < c$, we conclude the following, provided ξ does not lie on one of the axes and when we have $\prod_{i \neq j}(|\xi_i| - |\xi_j|) \neq 0$: in the case $b > 0$ there are always three positive roots; (we need no other condition, since the a_j are positive by our assumption $c - d > 0$) and in the case $b < 0$ we have three strictly positive roots precisely when $\prod_{j=1}^{3}(c|\xi|^2 - (b - a)\xi_j^2) - \sum_{j=1}^{3} b\xi_j^2(c|\xi|^2 - (b - a)\xi_{j+1}^2)(c|\xi|^2 - (b - a)\xi_{j+2}) > 0$ for every $\xi \in \mathbb{R}^3$, $\xi \neq 0$. (By continuity the results remain true then also when ξ lies on one of the axes or if $|\xi_i| = |\xi_j|$ for some i and j.) As a consequence, we conclude that if b, c, d are admissible and $b < 0$, then also every (b', c, d') with $d = d'$, $b < b'$ is admissible.

We now want to recall some known facts about the slowness surface for cubic crystals. To do so, we need two definitions from classical differential geometry.

DEFINITION 1.6 (Cf. [24]). Let S be a surface in $\mathbb{R}^3$ (definitions work equally well in $\mathbb{R}^n$) in which linear coordinates are denoted by $y = (y_1, y_2, y_3)$. We assume that $0 \in S$ and that in a neighborhood of 0, S is defined by an equation of form $\{y;\ f(y) = 0,\ y \in U\}$ for some function $f \in \mathcal{C}^\infty(U)$. Finally assume that $\nabla f(y) = 0$ precisely when $y = 0$ and denote by $J_k f(y) = \sum_{|\alpha|=k}(1/\alpha!)\, \partial_y^\alpha f(0) y^\alpha$ the homogeneous part of degree k in the Taylor expansion of f at 0.

a) We say that 0 is a conical singularity if for a suitable choice of linear coordinates $J_2 f$ has the form $J_2 f(y) = y_3^2 - y_1^2 - y_2^2$.

b) 0 is called a uniplanar singularity (or also a "uniplanar node") if the following happens: we can find linear coordinates in which $(\partial/\partial y_3)^2 f(0) \neq 0$ and such that $f = 0$ is locally equivalent to

$$y_3^2 + A(y_1, y_2)y_3 + B(y_1, y_2) = 0, \quad \text{with} \quad A(0) = 0,\ B(0) = 0,\ \nabla_y A(0) = 0,$$

for some $\mathcal{C}^\infty$-functions A, B. Moreover, we assume that if we denote by Δ the quantity $\Delta = A^2(y_1, y_2) - 4B(y_1, y_2)$, then we have $\Delta(y_1, y_2) \sim |(y_1, y_2)|^4$ for $(y_1, y_2) \to 0$. Geometrically speaking, S is thus near the origin the union of the two sheets $S^\pm =$

$\{y;\ y_3 = (1/2)(-A(y_1, y_2) \pm \sqrt{\Delta(y_1, y_2)})\}$ which have $y_3 = 0$ as a common tangent plane at 0.

We now return to our discussion of the slowness surface. In fact, S will consist of 3 "sheets", which we shall call the inner, middle and outer one. We can define them in the following way: if we fix $\omega \in \mathbb{R}^3$ of length 1, then the polynomial $\theta \to q(\theta\omega)$ is of degree 6 and is even in θ. "Hyperbolicity" of $P(D)$ gives that we shall have three positive roots $\theta_3(\omega) \le \theta_2(\omega) \le \theta_1(\omega)$ and three negative roots, which actually are $-\theta_i(\omega)$, $i = 1, 2, 3$. The sheets S^i are then parametrized by $S^i = \{\theta_i(\omega)\omega;\ \omega \in \mathbb{R}^3,\ |\omega| = 1\}$, $i = 1, 2, 3$. The following geometric information is well-established (cf. e.g., [2], [19], [13]):

- the inner sheet is strictly convex, (i.e., it is convex and no line intersects the inner sheet in more that 2 points),
- when $b \neq 0$, $d \neq 0$, then S has precisely 14 double points,
- 6 of these double points lie on the coordinate axes, exactly one on each semi-axis, and are of uniplanar type, the singular points on the ξ_3 axis being $\pm(0, 0, 1/\sqrt{c})$;
- the remaining 8 double points lie on the lines $|\xi_1| = |\xi_2| = |\xi_3|$, in each octant of $\mathbb{R}^3$ lying precisely one. The singularities of this type are conical and a point $\xi \in S$ with $|\xi_1| = |\xi_2| = |\xi_3|$ is singular precisely when $|\xi_j| = 1/\sqrt{3c - d}$;
- in the case $b = 0$, the slowness surface is $\prod_{j=1}^{3}(1 - c|\xi|^2 + d\xi_j^2) = 0$. The singular points are then more degenerate and we shall not consider this case, if not in some comment. Also the case $d = b$ (which means "$a = 0$") is somewhat more degenerate, and again, we shall not consider it in detail.

The plan of the paper is as follows. In the first part of the paper, we shall mainly study properties of the slowness surface which are related to curvature. It turns out that some of these properties are related to the structure of the singular points on S. We shall therefore also study these singularities in some detail. The main emphasis when doing so is to obtain estimates which are uniform for $d \to 0$, since we shall regard the nearly isotropic case as a small deformation of the isotropic situation. We also mention that the results on singularities which we obtain shall be useful in establishing decay properties for the solutions of the system of crystal acoustics in a forthcoming paper. In the last part of the paper we then turn to the study of the Gauss map.

Finally I would like to mention that a number of the arguments which shall be used in this paper (must) have been folklore some decades ago. The reason why this paper is written with many details is that part of them are not any more in the standard curriculum of people who work in PDE.

2. Proof of Proposition 1.2

In the argument we shall use the method of stationary phase and a lemma related to this method due to E. Stein, which, for the convenience of the reader, we now recall.

Lemma 2.1 (E. Stein. Cf. [27]). *Let φ be a real-valued function on the interval $[a, b]$ which is k times differentiable. Assume that $k \geq 2$ and that $|\varphi^{(k)}(x)| \geq 1$. Also consider $\psi \in \mathcal{C}^1[a, b]$. Then it follows that*

$$\left| \int_a^b e^{it\varphi(x)} \psi(x)\, dx \right| \leq c_k t^{-1/k} \left[|\psi(b)| + \int_a^b |\psi'(x)|\, dx \right], \quad \textit{for} \quad t > 0,$$

for some constant c_k which does not depend on φ, ψ, a and b.

Proof of Proposition 1.2. We may argue locally near some fixed point $\xi^0 \in S \cap U'$. We also fix x^0 and shall work for x in a small conic neighborhood of x^0. For simplicity of notation we assume that $\xi^0 = 0$, $x^0 = (0, 0, 1)$. Since we may restrict attention to a conic neighborhood of x^0, we may assume that $|x'| \leq c' x_3$ for some constant $c' > 0$, where $x' = (x_1, x_2)$. Correspondingly the estimate (1.3) can be written as $I(x) = O(|x_3|^{-1/2-1/k})$ for $x_3 \to \infty$, $|x'| \leq c' x_3$.

If x^0 is not normal to S at ξ^0, then the phase function $\xi \to \langle x, \xi \rangle$ is non-degenerate for ξ in a neighborhood of ξ^0 and x in a conic neighborhood of x^0 and decay estimates are easy to obtain by standard partial integration. We may therefore assume that x^0 is normal to S at ξ^0. We can then parametrize S in a neighborhood of 0 in the form $\{\xi \in \mathbb{R}^3;\ \xi_3 = g(\xi_1, \xi_2)\}$ for some analytic function g defined in a neighborhood of $0 \in \mathbb{R}^2$. It follows in particular that we must have $g(0) = 0$, $\nabla g(0) = 0$. The fact that the mean curvature of S near 0 does not vanish gives that $\sum_{|\alpha|=2} |\partial^\alpha_{\xi'} g(\xi_1, \xi_2)| \not\equiv 0$, where $\xi' = (\xi_1, \xi_2)$. It follows from this that we can find a direction v such that the second directional derivative $\partial^2_v g(0) \neq 0$ and it is no loss of generality to assume that $v = (0, 1)$. We can therefore find an analytic function $(x, \xi_1) \to h(x, \xi_1)$, defined for ξ_1 near $\xi_1^0 = 0$ and for x in a conic neighborhood G of x^0 such that $h(x^0, 0) = 0$, and

$$\frac{\partial}{\partial \xi_1}(x_1\xi_1 + x_2\xi_2 + x_3 g(\xi_1, h(x, \xi_1)) = x_2 + \frac{\partial}{\partial \xi_2} g(\xi_1, h(x, \xi_1)) x_3 \equiv 0.$$

We claim that $g(\xi_1, h(x^0, \xi_1))$ is not identically 0 near 0. Indeed, in the opposite case it would follow that $\partial_{\xi_1} g(\xi_1, h(x^0, \xi_1)) \equiv 0$, so we could conclude that $(\nabla_{\xi'} g)(\xi_1, h(x^0, \xi_1)) \equiv 0$ for every ξ_1 in a neighborhood of 0. Along the curve Γ given by $\xi_1 \to (\xi_1, h(x, g(\xi_1, h(x^0, \xi_1))$ the tangent planes to S would then all have normal $(0, 0, 1)$ and the entire curve Γ were contained in the plane $\xi_3 = g(\xi_1, h(x^0, \xi_1)) \equiv 0$. The existence of such curves is excluded by assumption, so our claim is established. We conclude then that we can find some positive natural integer $k \geq 2$ such that

$$\frac{\partial^k g(\xi_1, h(x^0, \xi_1))}{\partial \xi_1^k} \neq 0 \quad \text{at} \quad \xi_1 = 0 \tag{2.1}$$

and the same will then also hold when we let x run through a small conic neighborhood, which we shall again denote by G, of x^0.

The rest is now easy. We may in fact apply the method of stationary phase in the variable ξ_2. (Cf. any textbook on asymptotic methods.) This gives if the support of u lies in a sufficiently small neighborhood of 0

$$\begin{aligned} I(x) = c|x|^{-1/2} \int & \exp[i\xi_1 x_1 + x_2 h(x,\xi_1) + x_3 g(\xi_1, h(x,\xi_1))] \\ & \times u(\xi_1, h(x,\xi_1), g(\xi_1, h(x,\xi_1)))\chi(x,\xi_1)\, d\xi_1 + O(|x|^{-1}), \\ & \qquad \text{for} \quad x \to \infty,\ x \in G, \end{aligned} \tag{2.2}$$

for some function χ which comes from the surface element $d\sigma$. For the integral in ξ_1 we now apply Stein's lemma. In fact, when $|x_2| \le c_1 x_3$ with some sufficiently small constant c_1 then (2.1) implies

$$\left| \left(\frac{\partial}{\partial \xi_1} \right)^k \left[\frac{x_2}{x_3} h(x,\xi_1) + g(\xi_1, h(x,\xi_1)) \right] \right| \ge c_2.$$

We can therefore apply Stein's lemma with large parameter x_3 to estimate the integral in (2.2) in the form $O(|x_3|^{-1/k})$. It follows that $I(x) = O(|x|^{-1/2-1/k}) + O(|x|^{-1}) = O(|x|^{-1/2-1/k})$ for large x, $|x_2| \le c_1 x_3$, $x \in G$. □

3. Study of the discriminant: Preliminary remarks

Although in this paper we are foremost interested in geometric properties in the smooth part of the slowness surface, it will be useful to understand also the singularities of the surface in some detail. The reason is that near a singularity a surface is bent in a way which can to some extend already be understood in terms of some rough information concerning the singularity itself. A clear example of this principle is Proposition 7.3 below. In this section we shall start with a preliminary study of the discriminant and of the "local discriminant" of the polynomial q near a singular point of S. We shall work for the two singular points

$$\left(0, 0, \frac{1}{\sqrt{c}}\right), \quad \text{respectively} \quad \left(\frac{1}{\sqrt{3c-d}}, \frac{1}{\sqrt{3c-d}}, \frac{1}{\sqrt{3c-d}}\right), \tag{3.1}$$

(the first uniplanar, the second conical) for which the third component is strictly positive. The situation is similar for all the other singular points.

REMARK 3.1. For completeness, we also mention that the regular points on S on the semiaxis $(0, 0, \xi_3 > 0)$, respectively $\{t/\sqrt{3},\ t/\sqrt{3},\ t/\sqrt{3};\ t > 0\}$ are

$$\left(0, 0, \frac{1}{\sqrt{c-d+b}}\right) \quad \text{respectively} \quad \left(\frac{1}{\sqrt{3c-d+3b}}, \frac{1}{\sqrt{3c-d+3b}}, \frac{1}{\sqrt{3c-d+3b}}\right). \tag{3.2}$$

In particular, when $b > 0$, then the conically singular points lie all on $S^1 \cap S^2$, whereas when $b < 0$, they lie on $S^2 \cap S^3$. Moreover, when $b = 0$ or $d = b$ we have triple points for one or the other type of singular points. As already mentioned, we shall not discuss these cases.

Near the singularities which we have singled out, the slowness surface S can be parametrized by the variables $\xi' = (\xi_1, \xi_2)$. Let us for this purpose denote by $\xi' \to \rho_j(\xi')$, $j = 1, 2, 3, 4, 5, 6$, the roots of the polynomial $\rho \to q(\xi', \rho)$ labelled in such a way that

$$\rho_4(\xi') \le \rho_5(\xi') \le \rho_6(\xi') < 0 < \rho_3(\xi') \le \rho_2(\xi') \le \rho_1(\xi'). \tag{3.3}$$

In particular the functions ρ_j are continuous and we have

$$q(\xi', \rho_j(\xi')) \equiv 0. \tag{3.4}$$

We also observe that $\rho_j(\xi') = -\rho_{j+3}(\xi')$ when indices are calculated modulo 3 and that $\rho_j(\xi') \neq \rho_{j+1}(\xi')$ except when $(\xi', \rho_j(\xi'))$ and $(\xi', \rho_{j+1}(\xi'))$ are singular points.

Note that the functions ρ_j depend on the variables ξ' and the parameters b, c, d. As mentioned above in a more general context, we shall deliberately write them sometimes as functions of ξ' or (ξ', d) alone, to stress the fact that in some specific argument we are interested in the dependence on those variables, respectively parameters, and that the other parameters may be considered fixed. The surface S^j, $j = 1, 2, 3$, can then be represented locally ("locally" means here "in the region $\xi_3 > 0$") near the singularities mentioned above by the graph of the function ρ_j.

It is now useful to write q in a rather explicit form. In fact, q can be written as

$$q(\xi) = A_0\xi_3^6 + A_1(\xi')\xi_3^4 + A_2(\xi')\xi_3^2 + A_3(\xi'), \tag{3.5}$$

for some explicitly calculable coefficients A_j, which are polynomials of degree $2j$ in ξ' and also depend on the constants b, c, d. The A_j are easily calculated explicitly and are:

$$\begin{aligned}
A_0 &= bc^2 - c^2(-c + d),\\
A_1(\xi') &= -b\xi_1^2 c(-c + d) - b\xi_2^2(-c + d)c - bc(1 - c(\xi_1^2 + \xi_2^2) + d\xi_1^2)\\
&\quad - bc(1 - c(\xi_1^2 + \xi_2^2) + d\xi_2^2)\\
&\quad - (d - c)(-c(1 - c(\xi_1^2 + \xi_2^2) + d\xi_1^2) - c(1 - c(\xi_1^2 + \xi_2^2) + d\xi_2^2))\\
&\quad - c^2(1 - c(\xi_1^2 + \xi_2^2)),
\end{aligned}$$

$$
\begin{aligned}
A_2(\xi') = {}& b(d-c)\xi_1^2(1-c(\xi_1^2+\xi_2^2)+d\xi_2^2) - bc(\xi_1^2+\xi_2^2)(1-c(\xi_1^2+\xi_2^2)) \\
& + b\xi_2^2(-c+d)(1-c(\xi_1^2+\xi_2^2)+d\xi_1^2) \\
& + b(1-c(\xi_1^2+\xi_2^2)+d\xi_1^2)(1-c(\xi_1^2+\xi_2^2)+d\xi_2^2) \\
& - (1-c(x_1^2+\xi_2^2)+d\xi_1^2)(1-c(\xi_1^2+\xi_2^2)+d\xi_2^2)(-c+d) \\
& - (-c(1-c(\xi_1^2+\xi_2^2)+d\xi_1^2) - c(1-c(\xi_1^2+\xi_2^2)+d\xi_2^2))(1-c(\xi_1^2+\xi_2^2)), \\
A_3(\xi') = {}& b\xi_1^2(1-c(\xi_1^2+\xi_2^2)+d\xi_2^2)(1-c(\xi_1^2+\xi_2^2)) \\
& + b\xi_2^2(1-c(\xi_1^2+\xi_2^2))(1-c(\xi_1^2+\xi_2^2)+d\xi_1^2) \\
& - (1-c(\xi_1^2+\xi_2^2)+d\xi_1^2)(1-c(\xi_1^2+\xi_2^2)+d\xi_2^2)(1-c(\xi_1^2+\xi_2^2)).
\end{aligned}
$$

Calculations can now be simplified if we take into account the fact that q depends directly on ξ_j^2. If we introduce the notation $\xi_3^2 = \sigma$, the polynomial q can be written as

$$\tilde{q}(\xi', \sigma) = A_0\sigma^3 + A_1(\xi')\sigma^2 + A_2(\xi')\sigma + A_3(\xi'). \tag{3.6}$$

The most important instance of when this simplifies calculations is when we calculate the discriminant of q. We recall here that the discriminant of q, when regarded as a polynomial in ξ_3 is given by $A_0^4\prod_{i<j}(\rho_i(\xi')-\rho_j(\xi'))^2$. (The factor A_0 is not interesting from an analytic point of view. Note that anyway $A_0 \neq 0$.) We need some kind of discriminant since we want to understand the behavior of $\rho_1-\rho_2$, in the case when the singular sheet is S^1, respectively $\rho_2-\rho_3$, when the singular sheet is S^3. The reason why we resort to the discriminant, rather than directly to the expressions $(\rho_i-\rho_j)$, comes from the remark that in principle the discriminant can be calculated explicitly in terms of the coefficients of q. The expression of the discriminant of a polynomial of degree six is however rather complicated and it is here that we can use $\tilde{q}$. Indeed, if we denote $\rho_j^2(\xi')$ for $j = 1, 2, 3$, by $\sigma_j(\xi')$, the discriminant D of $\tilde{q}$ is on one hand equal to

$$D = A_0^4[(\sigma_1-\sigma_2)(\sigma_1-\sigma_3)(\sigma_2-\sigma_3)]^2 = A_0^4[(\rho_1^2-\rho_2^2)(\rho_1^2-\rho_3^2)(\rho_2^2-\rho_3^2)]^2, \tag{3.7}$$

and therefore already contains all information on $\rho_1-\rho_2$ (or $\rho_2-\rho_3$) which we need. (The factor A_0^4 is not interesting from an analytic point of view.) On the other hand, $\tilde{q}$ being a polynomial of degree three, it is standard (cf. e.g. [29]) that, calculated in terms of the coefficients A_i the discriminant D is also equal to

$$D = A_1^2A_2^2 - 4A_0A_2^3 - 4A_1^3A_3 - 27A_0^2A_3^2 + 18A_0A_1A_2A_3. \tag{3.8}$$

As for the "local discriminant" at T we define it when T is one of the two singular points in (3.1) by

$$\Delta(\xi') = (\rho_1(\xi') - \rho_2(\xi'))^2.$$

(More generally, if S is a surface defined near ξ^0 by an equation of form $\xi_3^2 + A(\xi')\xi_3 + B(\xi') = 0$, with ξ_3^0 a double root of $\xi_3 \to \xi_3^2 + A(\xi'^0)\xi_3 + B(\xi'^0)$, then the local discriminant of S is defined by $A^2(\xi') - 4B(\xi')$.)

Since $(\sigma_1 - \sigma_2)(\sigma_2 - \sigma_3)(\sigma_3 - \sigma_1)$ is a symmetric combination of the roots of the polynomial $\tilde{q}(\xi', \sigma)$, it must be an analytic function in d and ξ'. Moreover, D vanishes identically when $d = 0$, so it must be divisible by d. It is however a positive function, so it must actually vanish of order 2 at $d = 0$ and we have

$$D(\xi', d) = d^2 \tilde{D}(\xi', d), \quad \Delta(\xi', d) = d^2 \tilde{\Delta}(\xi', d), \tag{3.9}$$

for some analytic function $\tilde{D}$, $\tilde{\Delta}$.

We shall now continue our study separately for the case of uniplanar and of conical singularities.

4. Study of the discriminant near an uniplanar point

In this section we study the discriminant near the uniplanar point $(0, 0, 1/\sqrt{c})$. The notations are as in the preceding section. We know already that the local discriminant is of form $\Delta(\xi', d) = d^2 \tilde{\Delta}(\xi', d)$ for some analytic function $\tilde{\Delta}$ defined for small (ξ', d). The study of Δ is here simplified by the fact that evaluation of the derivatives of some polynomial at the origin is relatively easy. Moreover, it is known from uniplanarity that $\sum_{|\beta| \le 3} \left| \partial_{\xi'}^{\beta} \Delta(0, d) \right| = 0$ and therefore the first nontrivial term in the Taylor-expansion in ξ' at 0 is

$$J_4 \Delta(\xi', d) = \sum_{|\beta|=4} \partial_{\xi'}^{\beta} \Delta(0, d) \frac{\xi'^{\beta}}{\beta!}. \tag{4.1}$$

In order to calculate $J_4\Delta$, we now turn our attention to the "full" discriminant D of $\tilde{q}$, assuming, for the sake of the discussion, that $b > 0$. We notice that in D only the factor $(\rho_1 - \rho_2)^2$ vanishes at $\xi' = 0$. (See (3.7).) We conclude therefore that there is a constant $\tilde{\gamma}$ so that

$$J_4 \Delta = A_0^4 \tilde{\gamma} J_4 D. \tag{4.2}$$

Of course $\tilde{\gamma} = [(\rho_1 + \rho_2)(0)(\rho_1^2 - \rho_3^2)(0)(\rho_2^2 - \rho_3^2)(0)]^2$, so we obtain in view of (3.1), (3.2),

$$\tilde{\gamma} = \left[\frac{2}{\sqrt{c}} \left(\frac{1}{c} - \frac{1}{a + c} \right)^2 \right]^2. \tag{4.3}$$

$\tilde{\gamma} \neq 0$ in view of the condition $a \neq 0$. The expression of D can be calculated explicitly using (3.8). Here all the coefficients depend explicitly on ξ_1^2, ξ_2^2 and not directly on ξ_1, ξ_2. We conclude that $J_4 D$ is a polynomial in ξ_1^2, ξ_2^2 and we can calculate $J_4 D$.

What we get is (we have calculated it with St. Wolfram's "Mathematica" and double-checked with "Maple"):

$$J_4D = c^2d^2[4b^4 - 12b^3d + 13b^2d^2 - 6bd^3 + d^4](\xi_1^4 + \xi_2^4) \\ - c^2d^2[4b^4 + 16b^3d - 22b^2d^2 + 12bd^3 - 2d^4]\xi_1^2\xi_2^2.$$

(Note that for $d = 0$ J_4D vanishes identically, as it should, since $\rho_1 \equiv \rho_2$ then.) It is perhaps also worth noting that the main term for $d \to 0$ is $4b^4c^2d^2(\xi_1^4 + \xi_2^4 - \xi_1^2\xi_2^2)$, which has order of magnitude $b^4c^2d^2(|\xi_1|^4 + |\xi_2|^4)$. This is of course already an essential part of (3.9).

REMARK 4.1. We also conclude from our discussion that D must have the form $D = d^2[b^4c^2(\xi_1^4 + \xi_2^4 - \xi_1^2\xi_2^2) + dD_1(\xi', d) + D_2(\xi', d)]$, where D_1 and D_2 are polynomials in (ξ', d), D_1 with terms which are at least of degree 4 in ξ' and at least of degree 5 in ξ' in the case of D_2. For the local discriminant Δ this gives that $\Delta(\xi', d) = d^2[\tilde{\Delta}(\xi', d) + O(|\xi'|^5)]$ where $\tilde{\Delta}(\xi', d) \geq c_1|\xi'|^4$ for some constant $c_1 > 0$ and $\tilde{\Delta}$ is a polynomial of degree four in ξ'.

5. Study of the discriminant at a conical singularity

1. We denote by $\xi^0 = (1/\sqrt{3c-d}, 1/\sqrt{3c-d}, 1/\sqrt{3c-d})$ the conically singular point in the first octant on the slowness surface S. Our goal in this section is to evaluate the local discriminant in the variable ξ_3 of the defining equation $q(\xi) = 0$ at this point. Calculations are in principle similar to those in the preceding section, but technical details are more complicated since the fact that in Section 4 we had $\xi_1 = \xi_2 = 0$ simplified the situation there. We start with a comment on the Hessian of q. Since the term $\prod_{j=1}^{3}(1 - c|\xi|^2 + d\xi_j^2)$ vanishes of order 3 at ξ^0, the Hessian of q is equal b times the Hessian of

$$F(\xi) = \sum_{j=1}^{3} \xi_j^2(1 - c|\xi|^2 + d\xi_{j+1}^2)(1 - c|\xi|^2 + d\xi_{j+2}^2). \tag{5.1}$$

(Indices are counted modulo 3.) This suggests that in order to calculate the Hessian of the local discriminant of q, we may as well calculate the Hessian of the local discriminant of F. This is indeed the case and is based on some elementary remarks on the Weierstrass preparation theorem, which we have stated in the following lemma.

Lemma 5.1. *Let $\varphi(t,x) = t^2 + A(x)t + B(x)$ where A and B are analytic functions defined near $0 \in \mathbb{C}^2$ which vanish of order one, respectively of order two at 0 and consider some analytic function $\psi(t,x)$ defined for (t,x) near $(t,x) = (0,0)$, which vanishes of order three in (t,x) at $(0,0)$. Also consider*

$$\varphi(t,x) + \psi(t,x) = Q(t,x)(t^2 + \tilde{A}(x)t + \tilde{B}(x)), \tag{5.2}$$

the decomposition of $\varphi + \psi$ *given by the Weierstrass preparation theorem applied with respect to the variable* t. *Then* $\tilde{A}$ *and* $\tilde{B}$ *vanish of order one, respectively two at* 0 *and we have that* $H_{xx}(A^2 - 4B)(0) = H_{xx}(\tilde{A}^2 - 4\tilde{B})(0)$. ($H_{xx} f$ *stands for the Hessian of the function* f *in the variables* x.)

(Calculations shall be for real x. The reason why we work with "analytic functions" is that we did not want to invoke the Malgrange preparation theorem for a situation as simple as the one we really need.)

Proof of Lemma 5.1. We need to discuss some formal aspects of the Weierstrass preparation theorem. This is facilitated by the fact that we already know that Q, $\tilde{A}$, $\tilde{B}$ with (5.2) exist. We may assume that $n = 1$, since the Hessian of a function f is well-determined by the second directional derivatives of f. We also observe that if we can show that $\tilde{A}$ vanishes of order one at 0 and that $\tilde{B}$ vanishes of order two there, then $(d/dx)^2(\tilde{A}^2 - 4\tilde{B})(0) = 2[(d\tilde{A}/dx)(0)]^2 - 4(d/dx)^2\tilde{B}(0)$. To prove the lemma it suffices then to show that $\tilde{A}(0) = 0$, $(d/dx)\tilde{A}(0) = (d/dx)A(0)$, $\tilde{B}(0) = 0$, $(d/dx)\tilde{B}(0) = 0$, $(d/dx)^2\tilde{B}(0) = (d/dx)^2B(0)$.

The next remark is that (5.2) gives $Q(t, 0) \equiv 1 + O(t)$, so that $Q(0, 0) = 1$. It follows in particular that $0 = B(0) = Q(0, 0)\tilde{B}(0)$, whence $\tilde{B}(0) = 0$. We shall now calculate low order derivatives of $\tilde{B}$ and $\tilde{A}$ derivating the relation (5.2). This leads at first to $(d/dx)B(0) = (d/dx)Q(0, 0)\tilde{B}(0) + Q(0, 0)(d/dx)\tilde{B}(0)$ and gives (since we know already that $\tilde{B}(0) = 0$) $(d/dx)\tilde{B}(0) = 0$. A similar calculation shows that $(d/dx)^2\tilde{B}(0) = (d/dx)^2B(0) = 0$ (if we use that we already know that $\tilde{B}(0) = (d/dx)\tilde{B}(0) = 0$) and that $(d/dx)\tilde{A}(0) = (d/dx)A(0)$. (For the last equality we use again that $\tilde{B}(0) = (d/dx)\tilde{B}(0) = 0$. The derivatives of order less than two in x of ψ vanish at $(0, 0)$ and therefore have no bearing on our calculations.) This concludes the argument.

We now continue with the calculation of the Hessian of the local discriminant of q in the variable ξ_3. By the above (applied near ξ^0 rather than near 0, so the role of the variables (t, x) is played by the variables $\xi - \xi^0$), we may as well calculate the Hessian of the local discriminant of the function F. We apply the Weierstrass preparation theorem (the variables in which we apply the decomposition are (ξ, d); we need to consider d as a "variable" since we want to know that the various functions which are given by the decomposition depend in a smooth way on d) and can therefore write F locally near $(\xi^0, 0)$ in the form

$$F(\xi, d) = Q(\xi, d)[(\xi_3 - \xi_3^0)^2 + A(\xi', d)(\xi_3 - \xi_3^0) + B(\xi', d)], \tag{5.3}$$

for some Q, A, B, which are defined and analytic near $(\xi^0, 0)$, respectively $(\xi^{0\prime}, 0)$. We also have $A(\xi^{0\prime}, d) = B(\xi^{0\prime}, d) = 0$, $\nabla_{\xi'} B(\xi^{0\prime}, d) = 0$. The local discriminant is $\Delta(\xi', d) = A^2(\xi', d) - 4B(\xi', d)$.

Derivating (5.1) and (5.3) we now obtain

$$\frac{\partial^2 F(\xi^0, d)}{\partial \xi_j^2} = \frac{8c(3c-2d)}{(3c-d)^2}, \quad j = 1, 2, 3,$$

$$\frac{\partial^2 F(\xi^0, d)}{\partial \xi_i \partial \xi_j} = \frac{4(d^2 - 4cd + 6c^2)}{(3c-d)^2}, \quad \text{when} \quad i \neq j,$$

$$Q(\xi^0, d) = \frac{1}{2}\frac{\partial^2 F(\xi^0, d)}{\partial \xi_3^2} = \frac{4c(3c-2d)}{(3c-d)^2}, \quad \nabla_{\xi'} A(\xi^{0\prime}, d) = \frac{1}{Q(\xi^0, d)} \nabla_{\xi'} \frac{\partial F(\xi^0, d)}{\partial \xi_3},$$

$$\frac{\partial^2 B(\xi^0, d)}{\partial \xi_i \partial \xi_j} = \frac{1}{Q(\xi^0, d)} \frac{\partial^2 F(\xi^0)}{\partial \xi_i \partial \xi_j}, \quad i, j \in \{1, 2\}.$$

It follows that

$$\begin{aligned} \frac{\partial^2 \Delta(\xi^{0\prime}, d)}{\partial \xi_1^2} &= \frac{2}{Q^2(\xi^0, d)} \left(\left(\frac{\partial^2 F(\xi^0, d)}{\partial \xi_1 \partial \xi_3} \right)^2 - \left(\frac{\partial^2 F(\xi^0, d)}{\partial \xi_1^2} \right)^2 \right) \\ &= \frac{2}{Q^2(\xi^0, d)} \frac{16d^2(d^2 - 8cd + 12c^2)}{(3c-d)^4} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2 \Delta(\xi^{0\prime}, d)}{\partial \xi_1 \partial \xi_2} &= 2\frac{\partial A(\xi^{0\prime}, d)}{\partial \xi_1} \frac{\partial A(\xi^{0\prime}, d)}{\partial \xi_2} - 4\frac{\partial^2 B(\xi^{0\prime}, d)}{\partial \xi_1 \partial \xi_2} \\ &= \frac{2}{Q^2(\xi^0, d)} \frac{\partial^2 F(\xi^0, d)}{\partial \xi_1 \partial \xi_3} \left(\frac{\partial^2 F(\xi^0)}{\partial \xi_2 \partial \xi_3} - 2Q(\xi^0, d) \right) \\ &= \frac{2}{Q^2(\xi^0, d)} \frac{\partial^2 F(\xi^0, d)}{\partial \xi_1 \partial \xi_3} \frac{4d^2}{(3c-d)^2} \\ &= \frac{2}{Q^2(\xi^0, d)} \frac{4(d^2 - 4cd + 6c^2)}{(3c-d)^2} \frac{4d^2}{(3c-d)^2}. \end{aligned}$$

It is visible from this that the Hessian of the local discriminant is divisible by d^2 in a smooth way. Actually, we know already from Section 3 that the local discriminant itself is divisible by d^2, but we claim that from the explicit expressions we can see that $H_{\xi'\xi'}\Delta(\xi^{0\prime}, d)/d^2$ is strictly positive definite with an estimate from below which does not depend on d when d is small. Indeed, it will suffice to check this with $d = 0$ when we have with the notation $E(\xi^0, d) = H_{\xi'\xi'}\Delta(\xi^0, d)/(d^2)$ that

$$E(\xi^0, 0) = \frac{2}{Q^2(\xi^0, 0)(3c)^4} \begin{pmatrix} 192c^2 & 96c^2 \\ 96c^2 & 192c^2 \end{pmatrix} = \frac{64}{Q^2(\xi^0, 0)3^3c^2} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

In analogy with Section 4 we may thus write

$$\Delta(\xi', d) = d^2(\tilde{\Delta}(\xi', d) + \Delta_1(\xi', d)) \tag{5.4}$$

where $\tilde{\Delta}(\xi', d)$ is a positively definite quadratic form in $\xi' - \xi^{0\prime}$ with $\tilde{\Delta}(\xi', d) \geq \tilde{c}|\xi'|^2$, $\tilde{c} > 0$, and $\Delta_1(\xi', d)$ vanishes of order three at $\xi' = \xi^{0\prime}$. □

6. Study of the discriminant and of curvature in the regular region

In this section we consider a regular direction ξ^0 and a small open convex neighborhood K of ξ^0. In particular we may assume that K contains no singular direction and (after a renotation for the variables, and replacing, if necessary, ξ^0 by $-\xi^0$) that $\xi_3 \geq c'|\xi|$ for $\xi \in K$. Our final goal is to show that we have

Proposition 6.1. *Consider $b^0 \neq 0$, $c^0 > 0$, and let (ξ^0, K) be as before. Then there is a constant c' so that if $|b - b^0| + |c - c^0| + |d| < c'$, then the total curvature of the slowness surface $S(b, c, d)$ associated with (b, c, d) is strictly positive for all points $\xi \in S(b, c, d) \cap K$.*

In the argument we shall assume $b > 0$, the case $b < 0$ being similar. For $b > 0$ and d small compared with b the sheet S^3 will stay away from the other two sheets and its defining equation will depend (in view of the implicit function theorem) in an analytic way on the parameters. Since for $d = 0$, S^3 is a sphere, total curvature will be strictly positive on S^3 for small d. In the sequel we may therefore concentrate our attention on $S^1 \cup S^2$. The proof of Proposition 6.1 will be by a perturbation argument in which we shall start from the isotropic case $d = 0$. It seems convenient to calculate curvature in polar coordinates, so we shall basically study the derivatives of the functions which correspond to the functions ρ_j when in $\mathbb{R}^3$ we work with such coordinates. Let us then denote by $\omega = \xi/|\xi|$ and by $\theta = |\xi|$. Further, denote by $\theta_j(\omega)$, $\omega \in \mathbb{R}^3$, $|\omega| = 1$, $j = 1, 2, 3$, the positive roots of the polynomial $\theta \to q(\theta\omega)$ labelled in such a way that $\theta_3(\xi/|\xi|) \leq \theta_2(\xi/|\xi|) \leq \theta_1(\xi/|\xi|)$. It follows in particular that $\theta_j(\xi/|\xi|) = 1/\tau_j(\xi/|\xi|)$, where the τ_j are the positive roots of the characteristic equation of the system of crystal optics labelled in such a way that $0 < \tau_1(\xi) \leq \tau_2(\xi) \leq \tau_3(\xi)$. We shall mainly be interested in the case $j = 1, 2$. The main step in the argument is the following proposition, in which we write $\theta_j(\omega, d)$ for the roots (rather than $\theta_j(\omega)$) in order to make the dependence of the θ_j on d explicit.

Proposition 6.2. *Under the assumptions of* Proposition 6.1 *there are constants c_i, $i = 1, 2, 3$, such that*

$$(6.1)\quad c_1|d| \leq |\theta_1(\omega, d) - \theta_2(\omega, d)| \leq c_2|d|, \quad \textit{if} \quad |c - c^0| + |b - b^0| + |d| < c_3, \quad \omega \in K.$$

Note that $\theta_1 - \theta_2$ is an essential factor in the full discriminant D associated with the polynomial defining the slowness surface, when the latter is calculated with respect to the variable θ and is expressed in the coordinates ω. (For fixed $d \neq 0$, $(\theta_1 - \theta_2)^2$ is not strictly speaking a "local discriminant" in terms of the terminology introduced above, but it is roughly speaking an object of this type when we consider also d as a "variable" and work in a neighborhood of $d = 0$.)

It seems difficult to understand the analytic expression of the full discriminant D given by the formula (3.8) in a generic regular direction. We shall therefore use an indirect argument. As a preparation we prove

Proposition 6.3. *Let $c_1 > 0$, $c_2 > 0$, $a_i \in \mathbb{R}$, $b_i \in \mathbb{R}$, $i = 1, 2, 3$, be given such that*

$$a_1 < a_2 < a_3, \quad 0 < b_3 < b_2 < b_1, \quad |a_i| \le c_2, \quad c_1 \le b_2, \quad b_i \le c_2. \tag{6.2}$$

Then we can find $c_3 > 0$, c', which depend only on c_1, c_2 and not on the a_i, b_i, such that if $|d| \le c'$ is fixed and if $t_1 < t_2 < t_3$ are the three roots of the polynomial $\tilde{g}$ defined (*in analogy with* (1.8)) *by*

$$\tilde{g}(t) = \prod_{i=1}^{3}(t - da_i) - \sum_{j=1}^{3} b_j(t - da_{j+1})(t - da_{j+2}), \tag{6.3}$$

then it follows that $|t_i - t_j| \ge c_3|d|\,|a_3 - a_1|$, if $i \ne j$.

Conversely, there is also a constant $\tilde{c}$ such that

$$\min_{i \ne j}|t_i - t_j| \le \tilde{c}|d|. \tag{6.4}$$

Proof. We shall show that the assumption $\min(t_2 - t_1, t_3 - t_2) < c_3|d|\,|a_3 - a_1|$ leads to a contradiction if c_3 is small enough. We may assume that $d > 0$. It follows (see the introduction for a similar situation) that

$$da_1 < t_1 < da_2 < t_2 < da_3 < t_3. \tag{6.5}$$

This already proves (6.4). To prove the first statement, we have to examine four cases, in which c_4 stands for $|a_3 - a_1|/2$ (and is thus fixed throughout the argument):

a) $a_2 - a_1 \ge c_4,\ t_2 - t_1 < 2c_3c_4d$,
b) $a_2 - a_1 \ge c_4,\ t_3 - t_2 < 2c_3c_4d$,
c) $a_3 - a_2 \ge c_4,\ t_2 - t_1 < 2c_3c_4d$,
d) $a_3 - a_2 \ge c_4,\ t_3 - t_2 < 2c_3c_4d$,

and show that none of these cases can in fact hold if d and c_3 are small. □

Incompatibility with the situation described respectively in a), b), c), d), will be reached by looking at the condition $\tilde{g}(t_i) = 0$ for some suitable i. To do so, we shall look at the expressions $I_1(t) = -b_1(t - da_2)(t - da_3)$, $I_2(t) = -b_2(t - da_3)(t - da_1)$, $I_3(t) = -b_3(t - da_1)(t - da_2)$, $I_4(t) = (t - da_1)(t - da_2)(t - da_3)$, in terms of which $\tilde{g}(t)$ is build and shall show that in each of the cases a), b), c), d), one of the expressions $I_j(t_i)$, $j \le 3$, will either dominate the others (if i is suitable) or else will have the same sign with those I_k which are not dominated by it. The mechanism by which this will happen will be the following:

– one of the three terms $(t_i - da_1)$, $(t_i - da_2)$, $(t_i - da_3)$, can be estimated by $2c_3c_4d$ and is therefore relatively small (when we say in this proof that some quantity is "relatively small", or that it is "relatively large", then we mean that it is small, respectively large, when compared with "d"; in particular relatively small terms can be made small with respect to b_2 and b_1) when c_3 is small,

– one of the terms can be estimated by $2c_4(c_3+1)d$ and is small for small d, when compared with b_2, or b_1,

– another term will in absolute value be larger than $(c_4 - 2c_3c_4)d$ and is therefore relatively large when c_3 is small,

– the remaining term cannot always be estimated appropriately, but it will have the "correct" sign.

CASE a). The relatively small factor in this case is $da_2 - t_1 \le t_2 - t_1 \le 2c_3c_4d$ (cf. (6.5)), $t_1 - da_1 = da_2 - da_1 + t_1 - da_2 \ge d(c_4 - 2c_3c_4)$ is relatively large, and we use that $\tilde{g}(t_1) = 0$. We have that

$\alpha)$ $|b_1(t_1 - da_2)(t_1 - da_3)| \le 2c_2c_3c_4d|t_1 - da_3|$,

$\beta)$ $b_2(t_1 - da_1)(t_1 - da_3) \le -c_1d(c_4 - 2c_3c_4)|t_1 - da_3|$,

$\gamma)$ $b_3(t_1 - da_1)(t_1 - da_2) \le 0$.

The term in $\beta)$ dominates the term in $\alpha)$ for small c_3 and also the product $\prod_{j=1}^3(t_1 - da_j)$. Since it has the same sign with the term in $\gamma)$, we get $\tilde{g}(t_1) < 0$ when c_3 is small, a contradiction.

CASE b). The factor $t_3 - da_3 \le t_3 - t_2$ is relatively small whereas $t_3 - da_1$ is relatively large, and we shall argue on $\tilde{g}(t_3) = 0$. In view of what we just said, $\prod_{j=1}^3(t_3 - da_j)$ is dominated for small c_3 by $b_2(t_3 - da_1)(t_3 - da_3)$. The other terms in $\tilde{g}(t_3)$ are harmless, since they have the same sign with $b_2(t_3 - da_1)(t_3 - da_3)$ (in that $0 \le t_3 - da_i$, $\forall i$).

CASE c). The relatively large factor now is $t_2 - da_3$, in that $da_3 - t_2 = da_3 - da_2 + da_2 - t_2 \ge (c_4 - 2c_3c_4)d$ and $|t_2 - da_2| \le t_2 - t_1 \le 2c_3c_4d$ is relatively small with c_3. We shall argue on $\tilde{g}(t_2) = 0$. Observe then that $b_2(t_2 - da_1)(t_2 - da_3)$ dominates $\prod_{j=1}^3(t_2 - da_j)$ for small c_3 and also $b_3(t_2 - da_2)(t_2 - da_1)$. The term $b_1(t_2 - da_2)(t_2 - da_3)$ gives no problems since it has the correct sign.

CASE d). In the present situation $t_3 - da_1 \le c_4d + t_3 - t_2 \le (c_4 + 2c_3c_4)d$. In particular, $t_3 - da_1$ is dominated by b_1 if d is small. We shall use $\tilde{g}(t_3) = 0$. By the preceding, $\prod_{j=1}^3(t_3 - da_j)$ is dominated by $b_1(t_3 - da_2)(t_3 - da_3)$ for d small and the other terms have the correct sign (again in view of $0 \le t_3 - da_i$, $\forall i$).

REMARK 6.4. A similar result is valid if $a_1 \le a_2 \le a_3$, with only one of the inequalities strict. If, e.g., $a_1 = a_2$, then $t_1 = a_1$, and a discussion similar to Proposition 6.3 applies. It is also clear that the case considered in this remark is somehow a limit case for the case considered in the proposition. (Cf. [13].) It also follows if we take into account symmetries that without any restriction for the relative position of the a_j we have

$$|t_i - t_j| \geq c_3 d\left(\sum_{r \neq s} |a_r - a_s|\right).$$

We now turn to the proof of Proposition 6.2. We shall first study the positive roots τ_j of the characteristic equation of the system of crystal acoustics, which, as we may recall, are labelled in such a way that $0 < \tau_1(\xi) \leq \tau_2(\xi) \leq \tau_3(\xi)$. In particular, the τ_j thus satisfy Kelvin's equation (1.6). We shall now apply Proposition 6.3. We may assume (after a renotation of the variables, if necessary) that $0 \leq \xi_1 \leq \xi_2 \leq \xi_3$ and since we are not close to one of the axes, that $\xi_2/|\xi| > c_1$ for some $c_1 > 0$. We next denote by $b_j = b\xi_j^2/|\xi|^2$ and by $a_j = -\xi_j^2/|\xi|^2$. Also consider the auxiliary variable $t = \tau^2 - c$. If $\tau_j(\xi/|\xi|, d)$ is a solution of Kelvin's equation, $t_j = \tau_j^2(\xi/|\xi|, d)$ satisfies $\tilde{g}(t_j) = 0$, where $\tilde{g}$ is the function associated in (6.3) with the choices of a_j, b_j just mentioned. Proposition 6.3 implies then at first that $|\tau_1^2(\xi/|\xi|, d) - \tau_2^2(\xi/|\xi|, d)| \geq \tilde{c}d$ and therefore (since $\tau_1(\omega, d) + \tau_2(\omega, d) \leq \tilde{c}_1$) that

$$\left|\tau_1\left(\frac{\xi}{|\xi|}, d\right) - \tau_2\left(\frac{\xi}{|\xi|}, d\right)\right| \geq \tilde{c}_2 d, \quad \text{if} \quad \xi \in K.$$

We shall now use the relation $\theta_j(\omega, d) = 1/\tau_j(\xi/|\xi|, d)$ to rewrite this in order to prove (6.1). We obtain at first $|1/\theta_1(\omega, d) - 1/\theta_2(\omega, d)| \geq \tilde{c}_2 d$ if $\omega \in K$ and then also (since $\theta_1(\omega, d) \geq \tilde{c}_3$, $\theta_2(\omega, d) \geq \tilde{c}_3$) that

$$|\theta_1(\omega, d) - \theta_2(\omega, d)| \geq \tilde{c}_4 d \quad \text{if} \quad \omega \in K.$$

In the following corollary and in the remainder of this section we shall work with the "local discriminant" defined with respect to polar coordinates. Thus we shall put (assuming, to make a choice, $b > 0$) $\Delta(\omega, d) = (\theta_1(\omega, d) - \theta_2(\omega, d))^2$.

Corollary 6.5. *There is a constant c_1 such that for $j = 1, 2$, $|\nabla_\omega \theta_j(\omega, d)| \leq c_1$ for $\omega \in K$.*

Indeed, we can write for example $\theta_1(\omega, d) = [\theta_1(\omega, d) + \theta_2(\omega, d) + \sqrt{\Delta(\omega, d)}]/2$ and clearly $|\nabla_\omega[\theta_1(\omega, d) + \theta_2(\omega, d)]| \leq c_2$. In addition, we have $\nabla_\omega \sqrt{\Delta(\omega, d)} = (1/2)(\nabla_\omega \Delta(\omega, d))/\sqrt{\Delta}(\omega, d)$. We can therefore conclude the argument which estimates $|\nabla_\omega \theta_j(\omega, d)|$ by observing that $|\nabla_\omega \Delta(\omega, d)/\sqrt{\Delta}(\omega, d)| \leq c_3 d$.

A similar argument gives:

Corollary 6.6. *$|H_{\omega\omega}\sqrt{\Delta}(\omega, d)| \leq c_1 d$ and also $H_{\omega\omega}(\theta_1 + \theta_2)(\omega, d) = H_{\omega\omega}(\theta_1 + \theta_2)(\omega, 0) + dO(1)$.*

It follows that $H_{\omega\omega}(\theta_j(\omega, d)) = (1/2)H_{\omega\omega}[(\theta_1+\theta_2)(\omega, d) \pm \sqrt{\Delta(\omega, d)}] = (1/2)H_{\omega\omega}(\theta_1 + \theta_2)(\omega, d) \pm H_{\omega\omega}\sqrt{\Delta(\omega, d)}/2 = (1/2)H_{\omega\omega}(\theta_1 + \theta_2)(\omega, d) \pm d\,O(1)$.

This now implies that the second derivatives in ω of $\theta_j(\omega, d)$ are for d small close to those for the case $d = 0$. Since the total curvature is not vanishing for $d = 0$, it will be non-vanishing in the nearly isotropic case. This concludes the proof of Proposition 6.1.

7. Curvature properties near the singular points

1. The curvature properties of the slowness surface of the system of crystal optics (at smooth points of the surface) are well-established: cf. e.g., [1]. We have not found information of a similar quality for the case of the system of elasticity for crystals in the literature. Cf. anyway [19] for some partial results. In this paper we are interested mainly in the case of cubic crystals in the nearly isotropic case. The principal result which we shall obtain in this section (also see the beginning of this paper) is the following:

Theorem 7.1. *Assume $b > 0$.*
a) *When $d \neq 0$, the total curvature will always vanish on entire curves in the smooth part of $S^1 \cup S^2$. It does not vanish however in the nearly isotropic case on S^3.* (*Note that when d is small compared with b, the sheet S^3 will be smooth.*)
b) *The mean curvature will vanish nowhere in the smooth part of S, at least if we are close to the isotropic case.*

Similar results are true when $b < 0$, only that then the smooth sheet is S^1 and the conically singular points lie in $S^2 \cap S^3$.

2. We shall assume $b > 0$. The proof of Theorem 7.1. a) will be based on two statements which are perhaps of independent interest:

Theorem 7.2. *In the nearly isotropic case, the total curvature of S^1 is*
i) *negative near conical points,*
ii) *positive near uniplanar points.*

Once i) and ii) are established, we will of course also have proved part a) of Theorem 7.1, since the regions where the total curvature is strictly positive must be separated by non-trivial curves of vanishing total curvature from the regions where the total curvature is negative. (The set where the total curvature vanishes must be algebraic and is a subset of the slowness surface. Since the single sheets of the slowness surface are not reducible, this set can not have geometric dimension two and must therefore consists of curves and possibly, some additional isolated points. We do not know if such points are present.)

That i) is true is a consequence of a simple remark on surfaces which have defining equations of the form considered in the following proposition. To increase readability in calculations, we shall temporarily denote the coordinates in $\mathbb{R}^3$ by (x, y, z).

Proposition 7.3. *Let $Q_1 = Q_1(x, y, d)$, $Q_2 = Q_2(x, y, d)$ be positive definite quadratic forms in the variables (x, y) with coefficients which depend in a $\mathcal{C}^\infty$ way on d for small d and assume that there are constants $c_1 > 0$, $c_2 > 0$ such that*

$$Q_1(x, y, d) \geq c_1(x^2 + y^2), \quad Q_2(x, y, d) \geq c_2(x^2 + y^2). \tag{7.1}$$

Also assume that f_1, f_2 are $\mathcal{C}^\infty$-functions of (x, y, d), defined, say, for $|x| + |y| < 1$, $|d| < 1$, such that

$$\partial_x^k \partial_y^l f_i(0) = 0 \quad \text{for} \quad k + l \leq 2,\ i = 1, 2, \tag{7.2}$$

and denote by $\tilde{S}$ the surface

$$\tilde{S} = \{(x, y, z);\ z = -Q_1(x, y, d) + f_1(x, y, d) + |d|\sqrt{Q_2(x, y, d) + f_2(x, y, d)}\}.$$

(Thus $\tilde{S}$ depends on d.) Then there is $c > 0$, which depends only on c_1, c_2, such that the total curvature $K(P)$ at any point $P \in \tilde{S}$ is strictly negative when $|P| + |d| < c$.

Proof. We shall use (7.2) by writing that $|\partial_x^k \partial_y^l f_i(x, y, d)| \leq c_3(|x| + |y|)^{3-k-l}$ for $k+l \leq 2$. The proposition is intuitively clear and the proof is by direct calculation. Recall that for a surface represented as the graph of a function of form $(x, y) \to z(x, y)$, the total curvature K at the point $(x, y, z(x, y))$ is given by $K = (z_{xx}z_{yy} - z_{xy}^2)(x, y)/[(1 + z_x^2 + z_y^2)(x, y)]^2$. In the present situation, we take $z(x, y) = -Q_1(x, y, d) + f_1(x, y, d) + |d|\sqrt{Q_2(x, y, d) + f_2(x, y, d)}$ and have to show that $(z_{xx}z_{yy} - z_{xy}^2)(x, y) < 0$ for small $(x, y) \neq 0$ and $|d|$. After an orthogonal (but not necessarily orthonormal) change of coordinates, we may assume that $Q_1(x, y, d) = c'(x^2 + y^2)$, $Q_2(x, y, d) = \alpha(d)x^2 + 2\beta(d)xy + \gamma(d)y^2$ with $c' \geq c_1/2$, $\alpha(d) > c_4 > 0$, $\alpha(d)\gamma(d) - \beta^2(d) \geq c_5 > 0$ if d is small. Then we have

$$z_x = -2c'x + f_{1,x} + \frac{|d|}{2}\frac{Q_{2,x} + f_{2,x}}{\sqrt{Q_2 + f_2}},$$

$$z_{xx} = -2c' + f_{1,xx} + \frac{|d|}{2}\frac{Q_{2,xx} + f_{2,xx}}{\sqrt{Q_2 + f_2}} - \frac{|d|}{4}\frac{(Q_{2,x} + f_{2,x})^2}{(Q_2 + f_2)^{3/2}},$$

$$z_{xy} = f_{1,xy} + \frac{|d|}{2}\frac{Q_{2,xy} + f_{2,xy}}{\sqrt{Q_2 + f_2}} - \frac{|d|}{4}\frac{(Q_{2,x} + f_{2,x})(Q_{2,y} + f_{2,y})}{(Q_2 + f_2)^{3/2}}.$$

It follows that

$$z_{xx}z_{yy}-z_{xy}^2=\left[-2c'+f_{1,xx}+\frac{|d|}{2}\frac{Q_{2,xx}+f_{2,xx}}{\sqrt{Q_2+f_2}}-\frac{|d|}{4}\frac{(Q_{2,x}+f_{2,x})^2}{(Q_2+f_2)^{3/2}}\right]$$
$$\times\left[-2c'+f_{1,yy}+\frac{|d|}{2}\frac{Q_{2,yy}+f_{2,yy}}{\sqrt{Q_2+f_2}}-\frac{|d|}{4}\frac{(Q_{2,y}+f_{2,y})^2}{(Q_2+f_2)^{3/2}}\right]$$
$$-\left[f_{1,xy}+\frac{|d|}{2}\frac{Q_{2,xy}+f_{2,xy}}{\sqrt{Q_2}+f_2}-\frac{|d|}{4}\frac{(Q_{2,x}+f_{2,x})(Q_{2,y}+f_{2,y})}{(Q_2+f_2)^{3/2}}\right]^2.$$

We next write

$$\frac{1}{(Q_2+f_2)^{1/2}}=\frac{1}{Q_2^{1/2}}\left(\frac{1}{1+f_2/Q_2}\right)^{1/2}=\frac{1}{Q_2^{1/2}}(1+f_3),$$

where $|f_3(x,y,d)|\le c_5|(x,y)|$. This gives that

$$z_{xx}z_{yy}-z_{xy}^2=\left[-2c'+\frac{|d|}{2}\frac{Q_{2,xx}}{\sqrt{Q_2}}-\frac{|d|}{4}\frac{Q_{2,x}^2}{Q_2^{3/2}}+df_4+g_4\right]$$
$$\times\left[-2c'+\frac{|d|}{2}\frac{Q_{2,yy}}{\sqrt{Q_2}}-\frac{|d|}{4}\frac{Q_{2,y}^2}{Q_2^{3/2}}+df_5+g_5\right]$$
$$-\left[f_{1,xy}+\frac{|d|}{2}\frac{Q_{2,xy}}{\sqrt{Q_2}}-\frac{|d|}{4}\frac{Q_{2,x}Q_{2,y}}{Q_2^{3/2}}+df_6+g_6\right]^2,$$

where the f_4, f_5, f_6, respectively g_4, g_5, g_6, are functions of (x,y,d) which satisfy

$$|f_i(x,y,d)|\le c,\quad |g_i(x,y,d)|\le c|(x,y)|,\quad i=4,5,6.$$

The most singular part for $(x,y)\to 0$ in this expression is apparently

$$L(x,y,d)=d^2\left\{\left[\frac{1}{2}\frac{Q_{2,xx}}{\sqrt{Q_2}}-\frac{1}{4}\frac{Q_{2,x}^2}{Q_2^{3/2}}\right]\left[\frac{1}{2}\frac{Q_{2,yy}}{\sqrt{Q_2}}-\frac{1}{4}\frac{Q_{2,y}^2}{Q_2^{3/2}}\right]-\left[\frac{1}{2}\frac{Q_{2,xy}}{\sqrt{Q_2}}-\frac{1}{4}\frac{Q_{2,x}Q_{2,y}}{Q_2^{3/2}}\right]^2\right\}.$$

However, a direct calculation shows that $L(x,y,d)\equiv 0$.

(Since Q_2 is a quadratic form we have the trivial relation $[Q_{2,xx}Q_2/2-Q_{2,x}^2/4]\times[Q_{2,yy}Q_2/2-Q_{2,y}^2/4]-[Q_{2,xy}Q_2/2-Q_{2,x}Q_{2,y}/4]^2\equiv 0$.)

The “next-most” singular part is

$$(-2c'+df_4)\left[\frac{|d|}{2}\frac{Q_{2,yy}}{\sqrt{Q_2}}-\frac{|d|}{4}\frac{Q_{2,y}^2}{Q_2^{3/2}}\right]+(-2c'+df_5)\left[\frac{|d|}{2}\frac{Q_{2,xx}}{\sqrt{Q_2}}-\frac{|d|}{4}\frac{Q_{2,x}^2}{Q_2^{3/2}}\right]$$
$$+df_6\left[\frac{|d|}{2}\frac{Q_{2,xy}}{\sqrt{Q_2}}-\frac{|d|}{4}\frac{Q_{2,x}Q_{2,y}}{Q_2^{3/2}}\right].$$

For small d, the dominant term is here

$$-2c'\frac{|d|}{Q_2^{3/2}}[(\alpha(d)+\gamma(d))Q_2-(\alpha(d)x+\beta(d)y)^2-(\beta(d)x+\gamma(d)y)^2]$$
$$=-2c'\frac{|d|}{Q_2^{3/2}}(\alpha(d)\gamma(d)-\beta^2(d))(x^2+y^2).$$

The assumption had given that $\alpha(d)\gamma(d)-\beta^2(d)>c_5$, so the curvature will be negative for small (x,y) away from 0. This concludes the proof of Proposition 7.3 and therefore also the proof of part i) in Theorem 7.2 above. □

A similar argument shows that the curvature of $z=-Q_1+f_1-|d|\sqrt{Q_2+f_2}$ is positive for (x,y,d) small.

The argument to prove that total curvature is positive near uniplanar points (i.e., statement ii) in Theorem 7.2 above), e.g. on S^1, is even simpler. We work near the uniplanar point $\xi^0=(0,0,1/\sqrt{c})$ and parametrize S^1 near ξ^0 by (ξ_1,ξ_2). It follows from our study of the local discriminant that the defining equation for S^1 can be written (locally) in the form

$$\xi_3=f_1(\xi_1,\xi_2,d)+|d|\sqrt{Q_4(\xi_1,\xi_2,d)+Q_5(\xi_1,\xi_2,d)}$$

where Q_4 is homogeneous of fourth order in (ξ_1,ξ_2), $|Q_5(\xi_1,\xi_2,d)|\leq c|(\xi_1,\xi_2)|^5$ and $Q_4(\xi_1,\xi_2,d)\geq c|(\xi_1,\xi_2)|^4$ for some $c>0$ and small (ξ_1,ξ_2,d). Here the functions f_1, Q_4 and Q_5 depend smoothly on ξ_1,ξ_2 and d. Moreover, we know that the function Q_4+Q_5 is strictly positive when $(\xi_1,\xi_2)\neq 0$. We also know that when $d=0$, then the defining equation of S^1 near P is simply $\xi_3=\sqrt{1/c-\xi_1^2-\xi_2^2}$. It follows in particular from what we have said that second order derivatives of $|d|\sqrt{Q_4+Q_5}$, calculated at points $(\xi_1,\xi_2)\neq 0$, close to $(0,0)$ can be estimated by $\tilde{c}|d|$. This shows that for $(\xi_1,\xi_2)\neq 0$ small and for d small, the total curvature of S^1 is close to the total curvature in the isotropic case, which is of course positive.

REMARK 7.4. It is quite trivial to show that on S^1 there are points of positive total curvature when we make the additional assumption that $d>0$. In fact, it is obvious that then the distance from the origin will increase near conical points to values bigger than $1/\sqrt{c-d/3}$ (which is the distance from the conically singular points to the origin). Since the distance from the uniplanarly singular points to the origin is $1/\sqrt{c}$, we conclude that the points $\tilde{P}\in S^1$ farthest away from the origin must lie in the smooth part of S^1. At such a point $\tilde{P}$ the total curvature must be positive. (Consider in fact such a point $\tilde{P}$ and consider the sphere with center at the origin which is tangent to S^1 at this point. The total curvature of S^1 at $\tilde{P}$ is then bigger than that of the sphere and is therefore strictly positive. This type of reasoning is standard in classical differential geometry. Cf., e.g., [25].)

Statement a) in Theorem 7.1 above is now proved, and we may turn to statement b).

Theorem 7.5. *Let G be a small convex open conic neighborhood of the conically singular direction* $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ *which stays away from the uniplanarly singular direction* $(0, 0, 1)$. *Then the mean curvature of points in the smooth part of $S \cap G$ does not vanish in the nearly isotropic case.*

(We should say that we have not checked what happens for general, not necessarily small, d.)

The following remark is central in our argument:

REMARK 7.6. Let $H(\xi', d)$ be a function which depends analytically on (ξ', d) and such that $H(\xi', d)$ is of form $H(\xi', d) = d^2[Q_2(\xi', d) + O(|\xi'|^3)]$ for $\xi' \to 0$, where Q_2 is a positive definite quadratic form in ξ' with coefficients which may depend analytically on d. Also fix (α, β) with $\alpha^2 + \beta^2 = 1$ and consider the function $t \to f(\alpha, \beta, d, t) = \sqrt{H(\alpha t, \beta t, d)}$ defined for $|t|$ small, the square roots being taken positive. Then the function

$$\tilde{f}(\alpha, \beta, d, t) = \begin{cases} \sqrt{H(\alpha t, \beta t, d)} & \text{for} \quad t \leq 0, \\ -\sqrt{H(\alpha t, \beta t, d)} & \text{for} \quad t > 0, \end{cases}$$

depends analytically on (t, d). This is a consequence of the fact that $H(\alpha t, \beta t, d)$ must be of form $g(\alpha, \beta, d)t^2 + O(t^3)$ for some positive function $g(\alpha, \beta, d)$.

Proof of Theorem 7.5. The singular point in S on the half-ray with direction $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ is $\xi^0 = (1/\sqrt{3c - d}, 1/\sqrt{3c - d}, 1/\sqrt{3c - d})$. We shall assume that $b > 0$, so the sheets which contain the singular point ξ^0 are S^1 and S^2. We denote by $\rho_1(\xi', d)$, $\rho_2(\xi', d)$, $\rho_3(\xi', d)$, the functions considered in (3.3), respectively the functions $\sigma_i = \rho_i^2$. The sheets S^i, $i = 1, 2, 3$, are locally the graphs of the functions ρ_i.

The σ_i are thus the roots of the polynomial $\sigma \to A_0\sigma^3 + A_1(\xi', d)\sigma^2 + A_2(\xi', d)\sigma + A_3(\xi', d)$ introduced in (3.5).

The main steps in the argument are described in the following remarks α), β), γ) and δ).

α) $\rho_3(\xi', d)$ is analytic (when we say in this section "analytic", we always mean "real-analytic") in (ξ', d). This follows in fact from the implicit function theorem applied in the variables (ξ, d) near the point $(\tilde{\xi} = (1/\sqrt{3c + 3b}, 1/\sqrt{3c + 3b}, 1/\sqrt{3c + 3b}), d = 0)$. Application of the implicit function theorem is possible since for $d = 0$ the defining equation of S is $q(\xi) = (1 - c|\xi|^2)^2(1 - (c + b)|\xi|^2) = 0$ and therefore the derivative $(\partial/\partial\xi_3)q$ is not vanishing at $(\tilde{\xi}, 0)$. This also shows that the derivatives in the variables ξ' of ρ_3 for $d \neq 0$ are close to those of $\rho_3(\xi', 0)$ when d is small. We next mention that for $d = 0$ we have $\sigma_3(\xi', 0) = (c+b)^{-1}(1-(c+b)(\xi_1^2+\xi_2^2))$. Since the total curvature of S^3

is non-vanishing when $d = 0$, we conclude that the total curvature of S^3 near the point on the conically singular direction is non-vanishing. In the sequel of the argument we may therefore concentrate on what happens on S^1 and S^2.

β) $\sigma_1(\xi', d) + \sigma_2(\xi', d)$, respectively $\sigma_1(\xi', d)\sigma_2(\xi', d)$, is analytic for (ξ', d) close to $(\tilde{\xi}', 0)$. This follows from α) and the Vietè relations for $\sigma_1(\xi', d) + \sigma_2(\xi', d) + \sigma_3(\xi', d)$, respectively $\sigma_1(\xi', d)\sigma_2(\xi', d) + \sigma_2(\xi', d)\sigma_3(\xi', d) + \sigma_3(\xi', d)\sigma_1(\xi', d)$. For $d = 0$ we obtain $\sigma_1(\xi', 0) + \sigma_2(\xi', 0) = c^{-1}(2 - 2c(\xi_1^2 + \xi_2^2))$, $\sigma_1(\xi', 0)\sigma_2(\xi', 0) = c^{-2}(1 - c(\xi_1^2 + \xi_2^2))^2$. Note incidentally that both quantities are strictly positive for $\xi = \tilde{\xi}$ and will therefore also be strictly positive in the nearly isotropic case when $\xi' - \tilde{\xi}'$ is small.

γ) $\rho_1(\xi', d) + \rho_2(\xi', d)$ is analytic in (ξ', d) in the nearly isotropic case if (ξ', d) is close to $(\tilde{\xi}', 0)$. This is clear from the fact that $\sigma_1 + \sigma_2 > 0$, respectively $\sigma_1\sigma_2 > 0$ and $\rho_1 + \rho_2 = (\sigma_1 + \sigma_2 + 2\sqrt{\sigma_1\sigma_2})^{1/2}$, if square roots are taken positive.

δ) The discriminant $D(\xi', d)$ in σ of the polynomial $\sigma \to A_0(d)\sigma^3 + A_1(\xi', d)\sigma^2 + A_2(\xi', d)\sigma + A_3(\xi', d)$ is a positive function in (ξ', d) of form $d^2\tilde{D}(\xi', d)$. We denote the local discriminant $(\rho_1(\xi', d) - \rho_2(\xi', d))^2$ by $\Delta(\xi', d)$ and conclude that we must have $\Delta(\xi', d) = d^2\tilde{\Delta}(\xi', d)$, where $\tilde{\Delta}$ is analytic in (ξ', d) for small $(\xi' - \tilde{\xi}', d)$ and is positive. If we fix a line L in the (ξ_1, ξ_2)-plane of form $\{(\alpha t, \beta t);\ t \in \mathbb{R}\}$, α, β, arbitrary constants with $\alpha^2 + \beta^2 = 1$, then we can apply Remark 7.6 and find a function $F(t, d, \alpha, \beta)$, which depends continuously on (α, β), such that

$$F(t, d, \alpha, \beta) = \sqrt{\tilde{\Delta}(\alpha t, \beta t, d)} \quad \text{for} \quad t \leq 0,$$

and which is analytic in t and continuous in (t, d) for (t, d) small. We conclude from all this that for every fixed (α, β) the functions $\tilde{\rho}_1(\alpha t, \beta t, d) = [\rho_1(\alpha t, \beta t, d) + \rho_2(\alpha t, \beta t, d) + F(t, d, \alpha, \beta)]/2$ and $\tilde{\rho}_2(\alpha t, \beta t, d) = \tilde{\rho}_1(\alpha t, \beta t, d) - F(t, d, \alpha, \beta)$ are analytic in t and continuous in (t, d) for (t, d) small and depend continuously on (α, β). The functions $\tilde{\rho}_j$ are thus analytic extensions to the lines L of the functions ρ_j considered as functions on the half-lines $L_- = \{(\alpha t, \beta t);\ t < 0\}$ and the graphs $\{(\alpha t, \beta t, \rho_j(\alpha t, \beta t, d);\ |t| \text{ small}\}$ define analytic curves in S which depend in a continuous way on d. Clearly, when $d = 0$ we are in the isotropic case and the curves which we obtain are portions of circles. □

The statement from Theorem 7.5 will now follow, if we observe that the plane curvature of these curves above the lines L is non-vanishing in the isotropic case and approximates the one for fixed d when d is small.

8. Study of the Gauss map: Preliminaries

In this section we shall repeatedly use the following

DEFINITION 8.1. Let S be a surface in $\mathbb{R}^3$ of form $S = \{(\xi', h(\xi'));\ \xi' \in U\}$, $h \in \mathcal{C}^\infty(U)$ and let L be a line in $\mathbb{R}^3$ in some plane $\{(\xi', \xi_3);\ \xi_3 = \text{const}\}$, which has some point ξ^0 in common with S. We say that L has a contact of order k with S if when L is written as $L = \{\xi^0 + tv;\ t \in R\}$ for some suitable $v \in \mathbb{R}^3 \setminus \{0\}$, $v_3 = 0$, then we have $(d/dt)^j h(\xi^{0\prime} + tv')_{|t=0} = 0$ for $j < k$, $(d/dt)^k h(\xi^{0\prime} + tv')_{|t=0} \neq 0$. (For $v = (v_1, v_2, v_3) \in \mathbb{R}^3$ we denote by $v' = (v_1, v_2)$. Note that we must have $v_3 = 0$.)

We now turn our attention to the last statement made in the beginning of this paper. In this section we shall collect some preliminary material. The first relates curves on which the Gauss map degenerates, to curves of vanishing curvature.

Proposition 8.2. *Let T be a smooth surface in $\mathbb{R}^3$ and assume that there is a piecewise smooth curve Γ in T and a plane Σ which is tangent along Γ to T. Then Γ is a curve of vanishing total curvature of T.*

Proof. This seems standard, but we shall give a complete proof since it is easy to check. We may assume that the tangent plane is $\xi_3 = 0$. Since Γ is piecewise smooth it is (by definition) smooth except for a finite number of points. We now take a point P in the smooth portion of the curve and consider a local, smooth, parametrization of the curve $t \to \gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$, $\gamma_3(t) \equiv 0$ near P. We may also assume that $P = 0$ and that the tangent line Λ to Γ at 0 is $s \to (s, 0, 0)$. Finally, we assume that T is given locally near 0 as the graph of some function $\xi' \to h(\xi')$, which will satisfy $\nabla_{\xi'} h(\gamma_1(t), \gamma_2(t)) \equiv 0$. It follows that $\dot{\gamma}_1(0) \neq 0$, $\dot{\gamma}_2(0) = 0$, $h(\gamma_1(t), \gamma_2(t)) \equiv 0$. We conclude from this that

$$0 = \frac{d}{dt}[\nabla_{\xi'} h(\gamma_1(t), \gamma_2(t))]_{t=0} = H_{\xi',\xi'} h(0) \begin{pmatrix} \dot{\gamma}_1(0) \\ \dot{\gamma}_2(0) \end{pmatrix}.$$

This gives $(\partial^2/\partial\xi_1^2)h(0, 0) = 0$, $(\partial^2/\partial\xi_1\partial\xi_2)h(0) = 0$. It follows that the total curvature of S is zero at 0. Total curvature is therefore vanishing in the smooth part of Γ. It must then vanish by continuity also in the singular points of Γ. □

Our next concern is to understand what kind of curves can appear, if the curve is to be such that there is a plane tangent to a given sextic along it. We start with the following simple (and classical) remark:

REMARK 8.3. If Q_1 and Q_2 are two polynomials in two variables ξ_1, ξ_2 which have no common factors, then the set $\{(\xi_1, \xi_2) \in \mathbb{C}^2;\ Q_1(\xi_1, \xi_2) = Q_2(\xi_1, \xi_2) = 0\}$ is finite. (Indeed, after a linear change of variables we may assume that $Q_1 = a_0\xi_1^\sigma + \tilde{Q}_1$, where $a_0 \in \mathbb{C}$, $a_0 \neq 0$, σ is the degree of Q_1, and $\tilde{Q}_1$ is a polynomial in (ξ_1, ξ_2) which, as a polynomial in the variable ξ_1, has degree strictly less than σ. The resultant R in the variable ξ_1 of the two polynomials is a polynomial in ξ_2, which is not identically zero, since Q_1 is irreducible. Common zeros of Q_1 and Q_2 can only occur if $R(\xi_2) = 0$, so we obtain a finite number of values ξ_2 for which we can have common zeros. Actually, for every ξ_2 with $R(\xi_2) = 0$, we obtain σ values of ξ_1 (when multiplicities are counted) such that $Q_1(\xi_1, \xi_2) = 0$ and conversely, the pairs (ξ_1, ξ_2) which appear in this way, are precisely the common zeros of Q_1 and Q_2. For the theory of resultants of two polynomials see almost any textbook in algebra, e.g., [29].)

The following is an immediate consequence

REMARK 8.4. Let $\tilde{g}$ be an irreducible polynomial in two variables with real coefficients. Then the set of common real zeros of $\tilde{g}$ and of $(\partial/\partial\xi_1)\tilde{g}$ is finite if $(\partial/\partial\xi_1)\tilde{g}$ is not identically vanishing.

For reference reasons, we mention the following trivial:

REMARK 8.5. Let f be a real-valued polynomial in $\mathbb{R}^3$ and denote $S = \{\xi \in \mathbb{R}^3;\ f(\xi) = 0\}$. Consider $\xi^0 \in S$ and assume that $f(\xi^0) = 0$, $\partial_{\xi_3} f(\xi^0) \neq 0$. Also denote $g(\xi') = f(\xi', \xi_3^0)$. Then $\Sigma = \{\xi, \xi_3 = \xi_3^0\}$ is tangent to S at ξ^0 precisely when $\nabla_{\xi'} g(\xi^{0\prime}) = 0$.

Proposition 8.6. *Let f be a real-valued polynomial on $\mathbb{R}^3$ such that except for a finite number of points $P^1, \ldots, P^s \in S = \{\xi \in \mathbb{R}^3;\ f(\xi) = 0\}$ we have that $f(\xi) = 0$ implies $\nabla_\xi f(\xi) \neq 0$. Assume that f is of degree* 6. *We also assume that S is bounded and that there is a plane Σ which is tangent to S along a smooth curve Γ. Then Σ is tangent to S along an ellipse which contains Γ. In addition to this ellipse of tangency, there can at most be finitely additional points at which Σ is tangent to S.*

Proof. We may assume that Σ is given by $\xi_3 = 0$ and denote by $g(\xi') = f(\xi_1, \xi_2, 0)$. In particular, $g(\xi') = 0$ implies that $(\xi', 0) \in S$. Moreover, g is a polynomial of degree at most 6. Also consider $P \in \mathbb{R}^2$. Σ is tangent to $(P, 0) \in S$ precisely when $g(P) = 0$, $\nabla_{\xi'} g(P) = 0$. (See Remark 8.5.) We denote by $\prod_{j=1}^{s} q_j$, the decomposition of g into irreducible, possibly multiple, real factors. In particular, $s \leq 6$. Since S, and therefore also Γ, is bounded, no factor can be first or third order and therefore in particular the number of factors is actually at most 3. (When q' is a polynomial of odd degree, with real coefficients then its set of zeros is unbounded.)

We now also claim that the number of factors (i.e., s) must at least be two. Indeed, if there were only one irreducible factor, i.e., if g itself were irreducible, then it could have only finitely many zeros in common with $\nabla_{\xi'} g$ (see Remark 8.4), so we could not have an entire curve of tangency. Actually, by this same argument it is also clear that, except for a finite number of points in Γ, the points in Γ must be zeros of at least two factors q_i, q_j, $i \neq j$, since only in this way can we make sure that generically simultaneously $q_i(\xi')q_j(\xi') = 0$ and $\nabla_{\xi'}[q_i(\xi')q_j(\xi')] = 0$. This excludes the possibility that we have an irreducible factor of degree 4. Indeed, in view of Remark 8.4, the set of common zeros of such a factor with the remaining factor of degree 2 would have to be finite and could not be an entire curve. We are finally left with the case of two or three factors of degree two. In the first case, the sets of zeros of q_1 and of q_2 must be ellipses and these ellipses must both contain Γ as a subset. It follows that actually the two ellipses coincide and the set of tangency of Σ with S must be this common ellipse. Moreover, in this case the two factors q_1, q_2 will have to be proportional. The other case is when we have three factors of degree two. Again we obtain two factors which must vanish on a common ellipse, which contains Γ as a subset and along which Σ is tangent to S. As for the third factor, it may give rise to a finite number of additional points along which Σ is tangent to S. □

Let us also give an example of a situation when Σ is tangent along a circle and is, in addition, also tangent at a point in Σ which lies outside this circle. The example thus shows that the last case considered in the proof of Proposition 8.6 can effectively occur.

REMARK 8.7. Let $\tilde{P}(\tau, \xi)$ be the characteristic polynomial of the system of crystal optics for some fixed biaxial crystal and let $Q_1(\xi) = \tilde{P}(1, \xi)$ be the polynomial defining the corresponding slowness surface. Then it is known that Q_1 is fourth order and that there are 4 circles imbedded in $S = \{\xi \in \mathbb{R}^3,\ Q_1(\xi) = 0\}$ such that for each of these circles there is a tangent plane which is tangent to S along the respective circle. (For the explicit form of $\tilde{P}$, of S, and information about the four circles see, e.g., [1].) Let Σ be one of these planes, denote by $\mathcal{C}$ the corresponding circle and let Q_2 a polynomial of form $|\xi - \xi^0|^2 - 1$ such that the sphere $S' = \{\xi;\ |\xi - \xi^0|^2 - 1 = 0\}$ is also tangent to Σ in a point $\tilde{\xi}$ which does not lie on $\mathcal{C}$. Then the polynomial $Q_1(\xi)Q_2(\xi)$ is a polynomial of degree six such that Σ is tangent to the surface $\tilde{S} = \{\xi;\ Q_1(\xi)Q_2(\xi) = 0\}$ for all points in $\mathcal{C} \cup \{\tilde{\xi}\}$.

9. Study of the Gauss map

In this section we shall use the term "curve" in a somewhat non-orthodox way: a curve shall be a finite union of otherwise piecewise smooth standard curves. Parametrizations shall be defined for the single smooth pieces of which our curves are made. We shall denote by ξ^0 the conically singular point $\xi_1^0 = \xi_2^0 = \xi_3^0 = 1/\sqrt{3c - d}$. In particular, ξ^0 depends on the parameters c and d.

In some arguments it will be necessary to work simultaneously with more than one set of parameters b, c, d. To distinguish between the various situations, we shall denote the corresponding slowness surfaces often (but not always) by $S(b, c, d)$, rather than by S. Accordingly, $S^1(b, c, d)$, $S^2(b, c, d)$, $S^3(b, c, d)$ are then the outer, middle and inner sheet of the slowness surface for some given b, c, d. We shall say that some property $\mathcal{P}$ holds "generically for (b, c, d) near some (b^0, c^0, d^0)" if we can find a neighborhood U of (b^0, c^0, d^0), a non-vanishing algebraic function φ: $U \to \mathbb{R}$, and a constant $\tilde{c}$ such that the property $\mathcal{P}$ holds for $(b, c, d) \in U$ when $\varphi(b, c, d) \neq \tilde{c}$. We shall say, more generally, that the property $\mathcal{P}$ holds generically if it holds generically near (b^0, c^0, d^0) for any (b^0, c^0, d^0) under consideration. The "property" which we have in mind is that for given (b, c, d) there are no planes Σ which are tangent to $S(b, c, d)$ along entire curves.

Our main result is

Theorem 9.1. *If we are sufficiently close to the isotropic case, then for generic values of b, c, d no plane Σ which is tangent to S along an entire curve can exist.*

(We have no idea if the result remains true for arbitrary b, c, d, but clearly the methods of proof used in this paper do not suffice to study the general case.)

REMARK 9.2. Let K be an open convex cone in the first octant in $\mathbb{R}^3$ which contains the singular direction $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$. In the nearly isotropic case the total curvature of the surfaces $S^i(b, c, d)$ is strictly positive in the first octant as long as we remain in the complement of K. (See Proposition 6.1 and Theorem 7.2.) In fact, we have seen that both in the regular region and close to the uniplanarly singular directions, $S(b,c,d)$ has non-vanishing total curvature. It follows in the nearly isotropic case, that if a plane Σ is for some value of b, c, d, tangent to $S(b, c, d)$ along an entire curve Γ with points in the first octant then every connected component of Γ with points in the first octant must lie in K. Actually we also know that Γ must be an ellipse, so it can have only one connected component.

Proof of Theorem 9.1. We shall only discuss the physically interesting case $b > 0$, the case $b < 0$ being similar. Since we shall work in the nearly isotropic case, we may assume then that S^3 is smooth and strictly convex. In follows from the remark that for suitable (α, β), $\alpha^2 + \beta^2 = 1$, Γ must intersect the curve $\{\xi \in S^1(b, c, d) \cup S^2(b, c, d);\ \alpha(\xi_1 - \xi_1^0) = \beta(\xi_2 - \xi_2^0)\}$. (Note that ξ^0 will depend on (b, c, d), and when we need to specify this, we shall write $\xi^0(b, c, d)$ rather than ξ^0.) To continue our argument, it will now be necessary to obtain some minimal information about such intersections. □

To begin this study, we look at the family of curves

$$\mathcal{L}(\alpha, \beta) = \{\xi \in S^1(b, c, d) \cup S^2(b, c, d);\ \alpha(\xi_1 - \xi_1^0) = \beta(\xi_2 - \xi_2^0),\ \xi_i > 0,\ i = 1, 2, 3\}$$

with $\alpha, \beta \in \mathbb{R}$, $\alpha^2 + \beta^2 = 1$.

(Note that the pairs (α, β) and $(-\alpha, -\beta)$ define the same curve in this family. Since the argument is always about some fixed (α, β) this does not lead to ambiguities.) The curves $\mathcal{L}(\alpha, \beta)$ are singular at $\xi = \xi^0$. We also mention that $\mathcal{L}(\alpha, \beta)$ depends on (b, c, d), but since we shall work for a fixed (b, c, d), we do not make this explicit in the notation.

Theorem 9.3. *Fix b^0, c^0. If $|b-b^0|+|c-c^0|+|d|$ is sufficiently small, the curves $\mathcal{L}(\alpha, \beta)$ admit no inflection points in their smooth portions.*

Proof. We denote by $\xi^0(b, c, d)$ the conically singular point for some given (b, c, d) for which $|b-b^0|+|c-c^0|+|d|$ is sufficiently small.

When $|\alpha| \geq \sqrt{1/2}$, we parametrize the curves $\mathcal{L}(\alpha, \beta)$ by ξ_2, otherwise by ξ_1. Indeed, working e.g., for the case $|\alpha| \geq \sqrt{1/2}$ it is reasonable to write ξ_1 in terms of ξ_2: $\xi_1 = \xi_1^0(b, c, d) + (\beta/\alpha)(\xi_2 - \xi_2^0(b, c, d))$. ξ_3, on the other hand, can be calculated from the defining equation of $S^1(b, c, d) \cup S^2(b, c, d)$. Since we want to use a perturbation argument in the parameters b, c, d, we study at first the defining equations of $S^1(b, c, d) \cup S^2(b, c, d)$ in a neighborhood of the singular point $\xi^0(b, c, d)$. We shall assume that we are working in the first octant in $\mathbb{R}^3$. Applying the Weierstrass preparation theorem to the six-th order polynomial with parameters (b, c, d), which defines the $S(b, c, d)$ in a neighborhood of (ξ^0, b^0, c^0, d^0) we see that $S^1(b, c, d) \cup S^2(b, c, d)$ can be defined locally by an equation of form $\tilde{q}(\xi, b, c, d) = 0$,

$$\tilde{q}(\xi, b, c, d) = \xi_3^2 + g_1(\xi_1, \xi_2, b, c, d)\xi_3 + g_2(\xi_1, \xi_2, b, c, d), \tag{9.1}$$

where the g_i are holomorphic in ξ_1, ξ_2, b, c, d, and $\Delta = g_1^2 - 4g_2$ is the local discriminant. In terms of the roots $\xi_{3,1}, \xi_{3,2}$ of the polynomial $\xi_3 \to \tilde{q}(\xi, b, c, d)$ the local discriminant is $(\xi_{3,1}(\xi_1, \xi_2, b, c, d) - \xi_{3,2}(\xi_1, \xi_2, b, c, d))^2$. It is important here that the functions g_1 and g_2 are defined on a neighborhood of (ξ_1^0, ξ_2^0) which is independent of (b, d) if (b, d) is close to $(b^0, 0)$ for some previously fixed b^0. (Note that $\xi^0(b, c, d)$ depends on (b, c, d), but we have applied the Weierstrass preparation theorem at $(\xi^0, b^0, c^0, 0)$, so the functions g_1, g_2 have the indicated domains of definition.) This gives for every α, β and every b, c, d with $|b-b^0|+|c-c^0|+|d|$ small, two solutions $\xi_{3,i}$, $i = 1, 2$, which depend on (ξ_1, ξ_2, b, c, d). As a function of the two variables ξ_1, ξ_2 these functions have to be singular at $\xi^0(b, c, d)$, but along each of the lines $\xi_1 - \xi_1^0(b, c, d) = (\beta/\alpha)(\xi_2 - \xi_2^0(b, c, d))$ we can define solution-functions in such a way that they depend analytically on $\xi_2, \alpha, \beta, b, c, d$. Indeed, if

$$\begin{aligned} \xi_{3,\pm} = {} & \frac{-g_1(\xi_1^0(b, c, d) + (\alpha/\beta)(\xi_2 - \xi_2^0(b, c, d)), \xi_2, b, c, d)}{2} \\ & \pm \frac{\sqrt{\Delta(\xi_1^0(b, c, d) + (\alpha/\beta)(\xi_2 - \xi_2^0(b, c, d)), \xi_2, b, c, d)}}{2}, \end{aligned} \tag{9.2}$$

are the two standard roots of the polynomial $\xi_3 \to \tilde{q}(\xi, b, c, d)$ (square roots are to be taken positive), then we set $\xi_{3,1} = \xi_{3,+}$ when $\xi_2 < \xi_2^0(b, c, d)$ and $\xi_{3,1} = \xi_{3,-}$ when $\xi_2 \geq \xi_2^0(b, c, d)$. (Also cf. here Remark 7.6.) The function $\xi_{3,2}$ is then defined similarly. We know from Section 5 that $\Delta = d^2(\tilde{\Delta}(\xi', d) + \Delta_1(\xi', d))$ where $\tilde{\Delta}(\xi', d)$ is a positively definite quadratic form in ξ' with $\tilde{\Delta}(\xi', d) \geq \tilde{c}|\xi'|^2$, $\tilde{c} > 0$ and $\Delta_1(\xi', d)$ vanishes of order three at $\xi' = \xi^{0\prime}$.

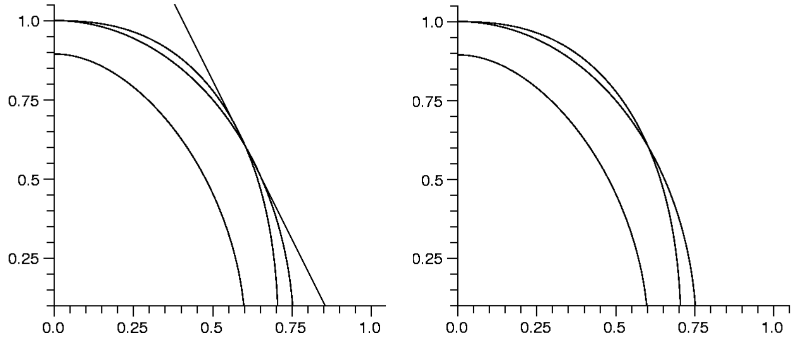

Fig. 1. Part of the section of $S \cap \{\xi_1 = \xi_2\}$ with and without common tangent in the case $c = 1$, $b = 0.5$, $d = 0.25$. The two points of tangency must be separated by the double point on the curve.

Since our curves have no inflection points in the isotropic case, we can conclude that in the nearly isotropic case there are no inflection points in the smooth portions of the curves $\mathcal{L}(\alpha, \beta)$ near the singular direction. Away from the singular directions, we again obtain that there are no inflection points in the nearly isotropic case, since the curves $\mathcal{L}(\alpha, \beta)$ are analytic functions which depend analytically on the parameters b, c, d.

Now assume that for some (b, c, d) there is a plane Σ which is tangent to $S(b, c, d)$ along an entire curve Γ of which some portion is contained in the first octant. We know from Remark 9.2 that if we fix some open convex cone $K \subset \mathbb{R}^3$ containing $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ and if we are sufficiently close to the isotropic case, then Γ must lie completely in K. It will then also intersect for some suitable α, β the curves $\mathcal{L}(\alpha, \beta)$ considered above. Actually, since the curves $\mathcal{L}(\alpha, \beta)$ have no inflection points, then generically any curve which intersects Γ must have at least two distinct points on Γ and the line L which is determined by these two points will be tangent to the curve $\mathcal{L}(\alpha, \beta)$. This is only possible if the singular point ξ^0 separates the two points of $\mathcal{L}(\alpha, \beta) \cap \Sigma$ on $\mathcal{L}(\alpha, \beta)$, i.e., these points lie on different connected components of the curve $\mathcal{L}(\alpha, \beta) \setminus \{\xi^0(b, c, d)\}$. This shows that the only way to have a plane Σ which is tangent along an entire curve Γ on $S(b, c, d)$ and has points in the first octant, is when Γ contains the conically singular direction $\xi^0(b, c, d)$ in its "interior". It is then also clear that Γ must intersect the curves $\mathcal{L}(\alpha, \beta)$ for every α, β in two points which lie on $S^1(b, c, d)$. (For part of this, see Fig. 1.)

To see that planes Σ tangent to $S^1(b, c, d)$ along curves can not exist generically (for the exact statement, see Theorem 9.1), we argue now by considering the intersection of $S^1(b, c, d) \cup S^2(b, c, d)$ with $\xi_1 = \xi_2$. This is a curve of type $\mathcal{L}(\alpha, \beta)$ with $\alpha = \beta = 1/\sqrt{2}$, but the structure of $\mathcal{L}(1/\sqrt{2}, 1/\sqrt{2})$ is quite easy to understand, so it may be worthwhile to say a few things about it. The sextic defining $T = \{\xi \in$

$S(b, c, d)$; $\xi_1 = \xi_2\}$ can be factored in the form

$$(1 - 2c\xi_1^2 - c\xi_3^2 + d\xi_1^2)[2b\xi_1^2(1 - 2c\xi_1^2 - c\xi_3^2 + d\xi_3^2) + b\xi_3^2(1 - 2c\xi_1^2 - c\xi_3^2 + d\xi_1^2) - (1 - 2c\xi_1^2 - c\xi_3^2 + d\xi_1^2)(1 - 2c\xi_1^2 - c\xi_3^2 + d\xi_3^2)]. \tag{9.3}$$

(This is checked by direct verification.) The double points on this sextic clearly satisfy the equation of the ellipse associated with the first factor in (9.3), so that the quartic defined by the second factor has no double points. It is therefore a smooth curve and the double points of the sextic T corresponding to the conically singular points on S come from the intersection of this curve with the ellipse $1 - 2c\xi_1^2 - c\xi_3^2 + d\xi_1^2 = 0$. □

The idea to conclude the proof of Theorem 9.1 is now as follows. We know that if we assume that Σ were for some fixed values $b = b^0$, $c = c^0$, $d = d^0$, d^0 sufficiently small, a tangent plane to $S^1(b^0, c^0, d^0)$ along a curve Γ in the first octant, then it must intersect the curves $\{\xi \in S^1(b^0, c^0, d^0);\ \xi_j = \xi_k,\ \xi_i > 0\}$, $k \neq j$, in two points for every $j \neq k$. Since the situation is symmetric with respect to permutations of the variables ξ_i, we see that the normal to the plane Σ must be $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$. When we restrict to the plane $\xi_1 = \xi_2$ we conclude that the line $L = \{(\xi_1, \xi_3);\ \xi \in \Sigma,\ \xi_1 = \xi_2\}$ must have normal $\vec{n}^0 = (2, 1)/\|(2, 1)\|$. We call this $\vec{n}^0$ the "correct normal". Let us now denote by P^+ and P^- the points of tangency of L and $\{(\xi_1, \xi_3);\ (\xi_1, \xi_1, \xi_3) \in S(b^0, c^0, d^0)\}$.

To bring in the perturbation argument, we now change point of view slightly and look at the intersection of $S^1(b, c, d)$ with the plane $\xi_1 = \xi_2$ for generic values of (b, c, d). We parametrize the plane $\xi_1 = \xi_2$ by its natural coordinates (ξ_1, ξ_3). We may assume that in the first quadrant the pieces of algebraic curves $\Lambda_+ = \{(\xi_1, \xi_3) \in \mathbb{R}^2;\ (\xi_1, \xi_1, \xi_3) \in S^1(b, c, d),\ \xi_3 > \xi_3^0(b, c, d)\}$ and $\Lambda_- = \{(\eta_1, \eta_3) \in \mathbb{R}^2;\ (\eta_1, \eta_1, \eta_3) \in S^1(b, c, d),\ \eta_3 < \xi_3^0(b, c, d)\}$ are defined respectively by $q_+(\xi_1, \xi_3, b, c, d) = 0$ and by $q_-(\eta_1, \eta_3, b, c, d) = 0$. We changed the notations for the variables (from ξ to η) in the region $\{\xi \in \mathbb{R}^3;\ \xi_3 < \xi_3^0(b, c, d)\}$ in order to make it easier to distinguish the contributions of this region from the contributions of its complement. $S^1(b, c, d)$ is here calculated for the values (b, c, d) under consideration. Arguing as above (or using the explicit form of the factors in (9.3)) we can show that in the nearly isotropic case the curvature of the curves $\Lambda_\pm$ may be assumed non-vanishing. The two curves have exactly one common tangent line with points of tangency $P^+ = (\tilde{\xi}_1, \tilde{\xi}_3)$ and $P^- = (\tilde{\eta}_1, \tilde{\eta}_3)$ in the first quadrant, where P^+ is chosen in the region $\xi_3 > \xi_3^0(b, c, d)$ and P^- in $\xi_3 < \xi_3^0(b, c, d)$. (See again Fig. 1.)

We claim that P^+ and P^- are algebraic functions of the b, c, d. Indeed, in terms of the defining equations $q_+ = 0$ and $q_- = 0$, considered above, the conditions for P^+ and P^- are

$$\begin{aligned} &q_+(P^+, b, c, d) = 0, \quad q_-(P^-, b, c, d) = 0, \\ &\langle \nabla q_+(P^+, b, c, d),\ P^+ - P^- \rangle = 0, \quad \langle \nabla q_-(P^-, b, c, d),\ P^+ - P^- \rangle = 0, \end{aligned} \tag{9.4}$$

and all these equations are polynomial. We have here four equations for the four coordinates (in the (ξ_1, ξ_3)-plane) of P^+, P^-. Since in the nearly isotropic case the solutions

P^+ and P^- are locally unique, we can solve these equations to obtain P^+ and P^- as algebraic functions of (b, c, d). This is indeed a consequence of the implicit function theorem. To see this, it is convenient to make a linear change of coordinates in which $\nabla q_+(P^+, b, c, d)$ and $\nabla q_-(P^-, b, c, d)$ are parallel to $(1, 0)$. It also follows that in these coordinates we must have $\tilde{\xi}_1 = \tilde{\eta}_1$ and therefore also that $\tilde{\xi}_3 \neq \tilde{\eta}_3$. Since the curvatures of Λ_+, Λ_- were assumed non-vanishing, we must have $(\partial/\partial\xi_3)^2 q_+(P^+, b, c, d) \neq 0$, $(\partial/\partial\eta_3)^2 q_-(P^-, b, c, d) \neq 0$. To see that the implicit function theorem is applicable it remains then to calculate the Jacobian (in the variables $(\xi_1, \xi_3, \eta_1, \eta_3)$) of the map

$$(\xi_1, \xi_3, \eta_1, \eta_3) \to \begin{pmatrix} q_+(\xi_1, \xi_3, b, c, d) \\ \langle \nabla q_+(\xi_1, \xi_3, b, c, d), (\xi_1, \xi_3) - (\eta_1, \eta_3) \rangle \\ q_-(\eta_1, \eta_3, b, c, d) \\ \langle \nabla q_-(\eta_1, \eta_3, b, c, d), (\xi_1, \xi_3) - (\eta_1, \eta_3) \rangle \end{pmatrix}$$

and show that it is nonsingular at (P^+, P^-). This is immediate in our special coordinates.

The normal to the line in the plane $\xi_1 = \xi_2$ which is tangent to the curve $\{(\xi_1, \xi_1, \xi_3) \in S^1(b, c, d);\ \xi_1 = \xi_2\}$ in two points P^+, P^- in the first quadrant as above, is an algebraic function of the parameters (since it is determined by the two points P^+, P^- which depend algebraically on the parameters). To conclude the argument, it will then suffice to show that for generic choices of the constants b, c, d, we do not get the "correct normal". Many ways to show this are available. For example, it is already clear from graphical evidence that the directions of the normals in question are not constant, and therefore there must also be instances where the direction of the normal is not the correct one. Although we have explored this (using "Maple"), we think that this approach is not necessarily convincing enough for everybody. Another possibility is to study what happens when b, c, d move in the admissible region towards the boundary of the admissible region. Finally, we can try to calculate the normals for special values of the parameters chosen in such a way that calculations become simple. The simplest such choice is when $b = 0$, when the equation of the slowness surface is the product of three second order polynomials. Since we have generally sticked in this paper to values $b \neq 0$ we shall here argue for the case $a = -2b$, which is closer to the physically interesting situations and where moreover results are somewhat more precise than in the general generic case. The first remark is that when $a = -2b$, then the condition $a + c > 0$ reduces to $c > 2b$. We shall assume that $b > 0$, so the only remaining restriction on parameters is (see the introduction) $c - d + b > 0$. In particular, we may also consider the nearly isotropic case for $a = -2b$, although it has perhaps no physical interest. The main remark is now that for the case $\xi_1 = \xi_2$, the polynomial q factors into the product of three polynomials. This is based on the fact that for $a = -2b$ the characteristic surface can be written as

$$(\tau^2 - c|\xi|^2)[(\tau^2 + (b - c)|\xi|^2)^2 - b^2(\xi_1^4 + \xi_2^4 + \xi_3^4 - \xi_1^2\xi_2^2 - \xi_1^2\xi_3^2 - \xi_2^2\xi_3^2)] = 0. \tag{9.5}$$

(See e.g., [20].) The most interesting part of the slowness surface is then in some sense

$$f(\xi, b, c) = (1 + (b - c)|\xi|^2)^2 - b^2(\xi_1^4 + \xi_2^4 + \xi_3^4 - \xi_1^2\xi_2^2 - \xi_1^2\xi_3^2 - \xi_2^2\xi_3^2) = 0. \tag{9.6}$$

When e.g. $\xi_1 = \xi_2$, the equation (9.6) reduces to

$$(1 + (b - c)(2\xi_1^2 + \xi_3^2))^2 - b^2(\xi_1^2 - \xi_3^2)^2 = 0. \tag{9.7}$$

This can also be written as

$$[(1 + (b - c)(2\xi_1^2 + \xi_3^2)) - b(\xi_1^2 - \xi_3^2)] \times [(1 + (b - c)(2\xi_1^2 + \xi_3^2)) + b(\xi_1^2 - \xi_3^2)] = 0, \tag{9.8}$$

so all in all, we obtain

$$\begin{aligned} q(\xi_1, \xi_1, \xi_3) = (1 - 2c\xi_1^2 - c\xi_3^2)[(1 + (b - c)(2\xi_1^2 + \xi_3^2)) - b(\xi_1^2 - \xi_3^2)] \\ \times [(1 + (b - c)(2\xi_1^2 + \xi_3^2)) + b(\xi_1^2 - \xi_3^2)]. \end{aligned} \tag{9.9}$$

(Notice that $q(\xi_1, \xi_1, \xi_3)$ is a polynomial in the two variables (ξ_1, ξ_3).) The problem is then reduced to see whether the common tangent in the first quadrant to the two ellipses Λ', Λ'' in the (ξ_1, ξ_3) plane

$$\begin{aligned} \Lambda' &= \{(\xi_1, \xi_3);\ (b - 2c)\xi_1^2 + (2b - c)\xi_3^2 + 1 = 0\}, \\ \Lambda'' &= \{(\xi_1, \xi_3);\ (3b - 2c)\xi_1^2 - c\xi_3^2 + 1 = 0\}, \end{aligned} \tag{9.10}$$

has the correct normal. It is Λ' which has intersection point with the ξ_3-axis farther away from the origin when $b > 0$.

Since we are dealing with ellipses, this common tangent can be calculated effectively. Let us briefly describe how this is done.

In fact, we may search the common tangent to the ellipses Λ', Λ'', in the form $\xi_1 = m\xi_3 + n$ where m and n have to be determined. We denote by $P^+ = (\tilde{\xi}_1, \tilde{\xi}_3)$, respectively $P^- = (\tilde{\eta}_1, \tilde{\eta}_3)$, the points of tangency of the tangent with Λ', respectively Λ'', in the quadrant $\xi_1 \geq 0$, $\xi_3 \geq 0$. We have in particular that

$$\begin{aligned} &\tilde{\xi}_1 = m\tilde{\xi}_3 + n, \quad \tilde{\eta}_1 = m\tilde{\eta}_3 + n, \quad \text{(and therefore } \tilde{\xi}_1 - m\tilde{\xi}_3 = \tilde{\eta}_1 - m\tilde{\eta}_3) \\ &(b - 2c)\tilde{\xi}^2 + (2b - c)\tilde{\xi}_3{}^2 + 1 = 0, \quad (3b - 2c)\tilde{\eta}_1{}^2 - c\tilde{\eta}_3{}^2 + 1 = 0. \end{aligned} \tag{9.11}$$

The notations are here chosen as above, in that P^+ will have to lie in the region $\xi_3 > \xi_1$ and P^- in the region $\xi_3 < \xi_1$. (Recall that Λ' intersects the ξ_3-axis in a point farther away from the origin than the point of intersection of Λ'' with the ξ_3-axis and therefore P^+ will have to lie in $\xi_3 > \xi_1$.) We still have to add the condition that $\xi_1 = m\xi_3 + n$ is tangent to Λ', respectively Λ''. To make "tangency" explicit, we choose two locally defined functions φ_1 and φ_2 which parametrize Λ', respectively Λ'', near P^+, respectively P^-. Thus

$$(b-2c)\varphi_1^2(\xi_3)+(2b-c)\xi_3^2+1=0, \quad (3b-2c)\varphi_2^2(\xi_3)-c\xi_3^2+1=0, \tag{9.12}$$

and $\tilde{\xi}_1=\varphi_1(\tilde{\xi}_3)$, $\tilde{\eta}_1=\varphi_2(\tilde{\eta}_3)$. By derivating (9.12) we obtain (with "dots" indicating derivatives) for $\dot{\varphi}_1(\tilde{\xi}_3)$, $\dot{\varphi}_2(\tilde{\eta}_3)$ the expressions:

$$\dot{\varphi}_1(\tilde{\xi}_3)=\frac{(c-2b)\tilde{\xi}_3}{(b-2c)\tilde{\xi}_1}, \quad \dot{\varphi}_2(\tilde{\eta}_3)=\frac{c\tilde{\eta}_3}{(3b-2c)\tilde{\eta}_1}. \tag{9.13}$$

We could now continue to calculate $\tilde{\xi}_1, \tilde{\xi}_3, \tilde{\eta}_1, \tilde{\eta}_3$ from these conditions, but actually we are only interested in showing that when we assume that $m=-1/2$, (which is the value corresponding to the correct normal) then we cannot find $\tilde{\xi}_1, \tilde{\xi}_3, \tilde{\eta}_1, \tilde{\eta}_3$ which satisfy all the conditions.

Indeed, with this value for m the $\tilde{\xi}_1, \tilde{\xi}_3, \tilde{\eta}_1, \tilde{y}_3$ must satisfy the following conditions:

$$\tilde{\xi}_1+\frac{1}{2}\tilde{\xi}_3=\tilde{\eta}_1+\frac{1}{2}\tilde{\eta}_3, \tag{9.14}$$

$$-\frac{1}{2}=\frac{(c-2b)\tilde{\xi}_3}{(b-2c)\tilde{\xi}_1}=\frac{c\tilde{\eta}_3}{(3b-2c)\tilde{\eta}_1}, \tag{9.15}$$

$$(b-2c)\tilde{\xi}_1^2+(2b-c)\tilde{\xi}_3^2+1=0, \quad (3b-2c)\tilde{\eta}_1^2-c\tilde{\eta}_3^2+1=0. \tag{9.16}$$

We have here five equations for four unkowns, and we shall see that these relations are not compatible.

The equations (9.15) give in fact

$$\tilde{\xi}_3=\frac{1}{2}\frac{(b-2c)\tilde{\xi}_1}{2b-c}, \quad \tilde{\eta}_3=\frac{1}{2}\frac{(2c-3b)\tilde{\eta}_1}{c}.$$

When we insert this into the equations of Λ' and Λ'', we obtain the two equations

$$\left(b-2c+\frac{1}{4}\frac{(b-2c)^2}{2b-c}\right)\tilde{\xi}_1^2+1=0, \quad \left(3b-2c-\frac{1}{4}\frac{(3b-2c)^2}{c}\right)\tilde{\eta}_1^2+1=0.$$

We solve these equations explicitly to get (remember that we are looking for solutions in the first quadrant)

$$\tilde{\xi}_1=\frac{2}{3}\frac{\sqrt{-(-24bc+9b^2+12c^2)(-c+2b)}}{-8bc+3b^2+4c^2}=\frac{2}{\sqrt{3}}\frac{\sqrt{c-2b}}{\sqrt{4c^2-8bc+3b^2}}$$

$$\tilde{\eta}_1=\frac{2}{3}\frac{\sqrt{3}\sqrt{(-8bc+3b^2+4c^2)c}}{-8bc+3b^2+4c^2}=\frac{2}{\sqrt{3}}\frac{\sqrt{c}}{\sqrt{4c^2-8bc+3b^2}}.$$

We now insert this in the first relation in (9.14) to obtain (after dividing both sides by the common factor $2/\sqrt{3(-8bc+3b^2+4c^2)}$):

$$\sqrt{c-2b} = \sqrt{c}.$$

This shows that the system (9.14), (9.15), (9.16) is compatible only when $b = 0$. Since we must also have that $a = 0$ in this case, we are in the completely degenerate situation when S, the full slowness surface, is a triple sphere. Not even in this case there are planes which are tangent to S on complete curves however. This completes the proof of Theorem 9.1.

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