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K-THEORY OF QUASI-TORIC MANIFOLDS

PARAMESWARAN SANKARAN and V. UMA*

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Abstract

In this note we shall give a description of the K-ring of a quasi-toric manifolds in terms of generators and relations. We apply our results to describe the K-ring of Bott-Samelson varieties.

1. Introduction

The notion of quasi-toric manifolds is due to M. Davis and T. Januszkiewicz [9] who called them 'toric manifolds.' The quasi-toric manifolds are a natural topological generalization of the algebraic geometric notion of non-singular projective toric varieties. However there are compact complex non-projective non-singular toric varieties which are quasi-toric manifolds. See [3]. Recently Civan [6] has constructed an example of a compact complex non-singular toric variety which is *not* a quasi-toric manifold.

In [17], we obtained, among other things, a description of the K-ring of projective non-singular toric varieties in terms of generators and relations. (In fact our result is applicable to slightly more general class of varieties.) The purpose of this note is to extend the K-theoretic results of [17] to the context of quasi-toric manifolds. As an application we obtain a description of the K-ring of Bott-Samelson varieties.

We remark that the K-ring of Bott towers, special cases of which are deformations of Bott-Samelson varieties, have been obtained recently by Willems [20] and Civan and Ray [7]. Willems [19] also has computed the torus-equivariant K-ring of Bott towers. A very general description of the K-ring for any finite CW complex X has been obtained by Conner and Floyd [8]. It is shown in [8] that $\Omega_U^*(X)\otimes_{\Omega_U^*}\mathbb{Z}\cong K^*(X)$ where $\Omega_U^*(X)$ is the complex cobordism ring of X. Here the ring of integers is being regarded as a Ω_U^*-module via the map that sends the class $[M]\in\Omega_U^n$ of a weakly almost complex manifold M to $(-1)^n Td(M)$ where $Td(M)$ is the Todd genus of M. The complex cobordism ring of a quasi-toric manifold has been computed explicitly by Buchstaber and Ray [5] as $\Omega_U^*(M)\cong\Omega_U^*[x_F; F\in\mathcal{F}]/(I+J)$ where the generators of the ideal I, J are exactly as in the description of the singular cohomology of M, given in [9].

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As for the *KO*-theory, Bahri and Bendersky [1] have shown that the Adams spectral sequence for the real connective *KO*-theory of any quasi-toric manfold collapses. They also show that the *KO*-ring is completely determined by the mod 2 cohomology of the quasi-toric manifold. Explicit description of the *KO*-ring as has been obtained by Civan and Ray for Bott towers which are 'totally even' or 'terminally odd' (see §6, [7]).

Let $G = (\mathbb{S}^1)^n$ be an n-dimensional *compact* torus and let $P \subset \mathbb{R}^n$ be a simple convex polytope of dimension n. That is, P is a convex polytope in which exactly n facets—codimension 1 faces of P—meet at each vertex of P. A G-quasi-toric manifold over P is a (smooth) G-manifold M where the G-action is locally standard with projection $\pi\colon M \to M/G \cong P$. Here 'local standardness' means that every point of M has an equivariant neighbourhood U such that there exists an automorphism $\theta\colon G \to G$, an equivariant open subset $U' \subset \mathbb{C}^n$ where G action on $\mathbb{C}^n$ is given by the standard inclusion of $G \subset U(n)$, and a diffeomorphism $f: U \to U'$ where $f(t \cdot x) = \theta(t)f(x)$ for all $x \in U$, $t \in G$. Any two points of $\pi^{-1}(p)$ have the same isotropy group its dimension being codimension of the face of P which contains p in its relative interior. It is known that M admits a CW-structure with only even dimensional cells. In particular M is simply connected and hence orientable.

Let $\mathcal{F}_P$ (or simply $\mathcal{F}$) denote the set of facets of P and let $|\mathcal{F}| = d$. For each $F_j \in \mathcal{F}$, let $M_j = \pi^{-1}(F_j)$ and G_j be the (1-dimensional) isotropy subgroup at any 'generic' point of M_j. Then M_j is orientable for each j. The subgroup G_j determines a primitive vector v_j in $\mathbb{Z}^n = \mathrm{Hom}(\mathbb{S}^1, G)$ which is unique upto sign. The sign is determined by choosing an omni-orientation on M, i.e. orientations on M as well as one on each M_j, $1 \le j \le d$. Choosing such a v_j for $1 \le j \le d$ defines the 'characteristic map' $\lambda\colon \mathcal{F} \to \mathbb{Z}^n \cong \mathrm{Hom}(\mathbb{S}^1, G)$ where $F_j \mapsto v_j$. Suppose that $F_1, \ldots, F_d$ are the facets of P, then writing $v_j = \lambda(F_j)$, the primitive vectors $v_1, \ldots, v_d$ are such that

$$\text{if} \quad \bigcap_{1\le r\le k} F_{j_r} \subset P \quad \text{is of codimension } k \text{ then} \tag{1.1}$$
$$v_{j_1}, \ldots, v_{j_k} \quad \text{extends to a } \mathbb{Z}\text{-basis} \quad v_{j_1}, \ldots, v_{j_k}, w_1, \cdots, w_{n-k} \quad \text{of} \quad \mathbb{Z}^n.$$

Fix an orientation for M. The omni-orientation on M determined by λ is obtained by orienting M_j so that the oriented normal bundle corresponds to the 1-parameter subgroup given by v_j.

We shall call any map $\lambda\colon \mathcal{F}_P \to \mathbb{Z}^n$ that satisfies (1.1) a characteristic map.

Conversely, starting with a pair (P, λ) where P is any simple convex polytope and a characteristic map $\lambda\colon \mathcal{F} \to \mathbb{Z}^n$ there exists a quasi-toric manifold M over P whose characteristic map is λ. The data (P, λ) determines the G-manifold M and an omni-orientation on it. We refer the reader to [9] and [3] for basic facts concerning quasi-toric manifolds.

Suppose that P is a simple convex polytope of dimension n and that $F_j \mapsto v_j$, $1 \le j \le d$ is a characteristic map $\lambda\colon \mathcal{F} \to \mathbb{Z}^n$. Assume that $F_1 \cap \cdots \cap F_n$ is a vertex of P so that $v_1, \ldots, v_n$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^n$. Let S be any commutative ring with identity and let $r_1, \ldots, r_n$ be invertible in S.

DEFINITION 1.1. Consider the ideal $\mathcal{I}$ of the polynomial algebra $S[x_1, \ldots, x_d]$ generated by the following two types of elements:

$$x_{j_1} \cdots x_{j_k} \tag{1.2}$$

whenever $F_{j_1} \cap \cdots \cap F_{j_k} = \emptyset$, and the elements

$$z_u := \prod_{j, u(v_j) > 0} (1 - x_j)^{u(v_j)} - r_u \prod_{j, u(v_j) < 0} (1 - x_j)^{-u(v_j)} \tag{1.3}$$

where $u \in \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z}) \cong \mathrm{Hom}(G, \mathbb{S}^1)$ and $r_u := \prod_{1 \le i \le n} r_i^{u(v_i)}$. We denote the quotient $S[x_1, \ldots, x_d]/\mathcal{I}$ by $\mathcal{R}(S; \lambda)$ or simply by $\mathcal{R}$. Note that $\mathcal{R}(S; \lambda)$ depends not only on λ but also on the choice of the sequence $r_1, \ldots, r_n$ of invertible elements of S.

Let $E \to B$ be a principal G-bundle with base space B a compact Hausdorff space. Denote by $E(M)$ the associated M-bundle with projection map $p\colon E \times_G M \to B$. The choice of the basis $v_1, \ldots, v_n$ for $\mathbb{Z}^n = \mathrm{Hom}(\mathbb{S}^1, G)$ yields a product decomposition $G = \prod_{1 \le i \le n} G_i \cong (\mathbb{S}^1)^n$. Also one obtains principal $\mathbb{S}^1$-bundles ξ_i, $1 \le i \le n$, over B associated to the i-th projection $G = (\mathbb{S}^1)^n \to \mathbb{S}^1$. The projection $E \to B$ is then the projection of the bundle $\xi_1 \times \cdots \times \xi_n$ over B. For any G-equivariant vector bundle V over M denote by $\mathcal{V}$ the bundle over $E(M)$ with projection $E(V) \to E(M)$. We shall often denote the complex line bundle associated to a principal $\mathbb{S}^1$ bundle ξ also by the same symbol ξ.

Suppose that V is the product bundle $M \times \mathbb{C}_\chi$, where $\mathbb{C}_\chi$ is the 1-dimensional G-representation given by the character $\chi : G \to \mathbb{S}^1$. Then $\mathcal{V}$ is isomorphic to the pull-back of the bundle $p^*(E_\chi)$, where E_χ is obtained from $E \to B$ by 'extending' the structure group to $\mathbb{S}^1$ via the character χ. Writing $\chi = \sum_{1 \le i \le n} a_i \rho_i$, where $\rho_i\colon G \to \mathbb{S}^1$ is the i-th projection, one has $\mathcal{V} \cong p^*(\xi_1^{a_1} \cdots \xi_n^{a_n})$, where $\xi^a = (\xi^*)^{-a}$ when $a < 0$.

Theorem 1.2. *Let M be a quasi-toric manifold over a simple convex polytope $P \subset \mathbb{R}^n$ and characteristic map $\lambda\colon \mathcal{F} \to \mathbb{Z}^n$. Let $E \to B$ be a principal $G = (\mathbb{S}^1)^n$ bundle over a compact Hausdorff space. With the above notations, there exist equivariant line bundles L_j over M such that, setting $r_i = [\xi_i] \in K(B)$, $1 \le i \le n$, one has an isomorphism of $K(B)$-algebras $\varphi\colon \mathcal{R}(K(B); \lambda) \to K(E(M))$ defined by $x_j \mapsto (1 - [\mathcal{L}_j])$.*

The proof is given in §3. A technical result, namely Proposition 2.1, needed in the proof is established in §2.

In §4 we apply our results to obtain explicit description of K-rings of connected sum $M \# \mathbb{P}^n_{\mathbb{C}}$ where M is a quasi-toric manifold of dimension $2n$, and a few other examples.

In §5 we obtain the K-ring of Bott-Samelson varieties. Indeed we consider the more general class of spaces, namely, Bott towers. We apply our main theorem to obtain K-ring of Bott-Samelson varieties, although their description as iterated 2-sphere bundles allows one to use well-known facts (cf. Proposition 4.2, §4) to obtained the same result.

2. Generators of $\mathcal{R}$

We keep the notations of the previous section. In this section we give a convenient generating set for S-module $\mathcal{R}(S; \lambda)$ where S is any commutative ring with identity and $\lambda\colon \mathcal{F}_P \to \mathbb{Z}^n$ a characteristic map, $P \subset \mathbb{R}^n$ being a simple convex n dimensional polytope.

Let $h\colon \mathbb{R}^n \to \mathbb{R}$ be a linear map which is injective when restricted to the set P_0 of vertices of P. Then h is a generic "height function" with respect to the polytope P. That is, h is injective when restricted to any edge of P. The map h induces an ordering on the set P_0 of vertices of P, where $w < w'$ in P_0 if $h(w) < h(w')$. We call $h(w)$ the *height* of w. The ordering on P_0 induces an orientation on the edges P_1 of P in the obvious fashion. Since P is simple, there are exactly n edges which meet at each vertex of P. Given any $w \in P_0$, denote by T_w the face of P spanned by all those edges incident at w which point *away* from w. Then the following property holds:

$$\text{if} \quad w' \in P_0 \quad \text{belongs to} \quad T_w, \quad \text{then} \quad w \leq w'. \tag{2.1}$$

This is a consequence of the assumption that P is simple and can be proved easily using for example Lemma 3.6 of [21]. Property (2.1) is 'dual' to property ($*$) of §5.1, [11]. Note that when $w \in P$ is the lowest vertex (i.e., $h(w)$ is the least,) then $T_w = P$; when w is the highest vertex, T_w is the vertex w.

It is shown in [9] that M has a perfect cell decomposition with respect to which the submanifolds $M_w := \pi^{-1}(T_w)$ are closed cells. (Cf. [2], and [15].)

Any proper face Q of P is the intersection of facets of P which contain Q, the number of distinct facets which contain Q being equal to the codimension of Q in P as P is simple. For each vertex w of P, denote by V_w the collection of all facets of P which contain w and by $U_w \subset \mathcal{F}$ the collection of facets which contains T_w. When $T_w = P$, we convene that $U_w = \emptyset$. Note that $U_w \subset V_{w'} \Rightarrow w' \in T_w$.

If $I \subset \mathcal{F}$ is non-empty, we denote by P_I the face of P obtained as the intersection of all facets in I. We shall write $x(I)$ to denote the product $\prod_{F_i \in I} x_i$. Note that $x(I) = 0$ in $\mathcal{R}$ if $P_I = \emptyset$ by (1.2). On the other hand, if I is empty, we set $P_I = P$ and $x(I) = 1$, the unit element of S. We say that $I \subset \mathcal{F}$ is *facial* if $F_I = \bigcap_{F_i \in I} F_i$ is non-empty. Note that I is facial if and only if $x(I) \neq 0$. Property (1.2) implies that if

I is any facial subset of $\mathcal{F}$, then there exists a unique $w \in P_0$ such that

$$U_w \subset I \subset V_w. \tag{2.2}$$

Indeed, the vertex w is just the least vertex in P_I.

The main result of this section is

Proposition 2.1. *With the above notations, the monomials $x(U_w)$, $w \in P_0$, form a generating set for the S-module $\mathcal{R}$.*

The proof will depend on the following lemma, which is the K-theoretic analogue of the 'moving lemma' proved in the context cohomology of nonsingular toric varieties in §5.2 [11]. The above proposition and the lemma were stated in the language of fans in Lemma 2.2, [17]. We include here the complete proof.

Lemma 2.2. *We keep the above notations.*
(i) *The set of monomials $\{x(I) \mid I \subset \mathcal{F}$ facial$\}$, spans $\mathcal{R}$ as an S-module. More precisely, let $J \subset I \subset L \subset \mathcal{F}$ be facial subsets. Suppose that $F_j \in I$, then*

$$x_j x(I) = (1 - r_u)x(I) + \sum_{\widetilde{I}} a_{j,\widetilde{I}}\, x\big(\widetilde{I}\big) \tag{2.3}$$

for suitable elements $u \in \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z})$, $a_{j,\widetilde{I}} \in S$, where $\widetilde{I}$ varies over those facial subsets of $\mathcal{F}$ not contained in L with $\big|\widetilde{I}\big| > |I|$, and such that $J \subset \widetilde{I}$. In particular square-free monomials $x(I)$, $I \subset \mathcal{F}$ facial, span $\mathcal{R}$ as an S-module.
(ii) *Let $J \subsetneq I \subset L$ in $\mathcal{F}$. Then*

$$x(I) = bx(J) + \sum b_{\widetilde{I}}\, x\big(\widetilde{I}\big) \tag{2.4}$$

for suitable elements $b, b_{\widetilde{I}} \in S$ where the sum varies over the set of all facial subsets $\widetilde{I} \subset \mathcal{F}$ which are not contained in L and $J \subsetneq \widetilde{I}$.
(iii) *Suppose that $(1 - r_i)$ is nilpotent for $1 \le i \le n$. Then $x_j \in \mathcal{R}$ is nilpotent for $1 \le j \le d$.*

Proof. (i) Without loss of generality, we may assume that $J = I$ and that $|L| = n$. Since P is simple, F_L is a vertex of P. We prove the statement by descending induction on the $|I|$.

Fix $F_j \in I$ and let u be the dual basis element of $\mathbb{Z}^n$ for the basis $\{\lambda(F_i) =: v_i \mid F_i \in L\}$ such that $\langle u, v_i\rangle = \delta_{i,j}$. Multiplying the relation $z_u = 0$ on both sides by $x(I)$, we obtain

$$(x(I) - x_j x(I)) \prod_p (1 - x_p)^{\langle u, v_p\rangle} = r_u x(I) \prod_q (1 - x_q)^{-\langle u, v_q\rangle} \tag{2.5}$$

where the product is over those p, (resp. q) such that the facets $F_p \notin L$, (resp. $F_q \notin L$), such that $\langle u, v_p\rangle > 0$ (resp. $\langle u, v_q\rangle < 0$.)

Note that when $I = L$, $x_p x(I) = 0$ for $F_p \notin L$. Therefore the above equation reduces to $x_j x_L = (1 - r_u)x_L$ in this case. Assume, by induction hypothesis, that any monomial divisible by $x\big(\widetilde{I}\big)$ with $I \subsetneq \widetilde{I}$ can be expressed as a S-linear combination of (square-free) monomials $x\big(\widetilde{J}\big)$ where $I \subsetneq \widetilde{J}$ with $\widetilde{J} \subset \mathcal{F}$ facial as in (2.1). Expanding each factor in (2.5) and multiplying out we obtain, using induction hypothesis, the equation

$$x(I) - x_j x(I) + \sum_{\widetilde{J}'} c_{\widetilde{J}'} x\big(\widetilde{J}'\big) = r_u \left(x(I) + \sum_{\widetilde{J}''} d_{\widetilde{J}''} x\big(\widetilde{J}''\big) \right)$$

where, in view of (1.1), only those $x\big(\widetilde{J}'\big), x\big(\widetilde{J}''\big)$ with $\widetilde{J}', \widetilde{J}''$ non-empty, facial and not contained in L occur in the above equation. This proves (i).

(ii) Again, without loss of generality, we assume that $|L| = n$. Fix a $j \in I \setminus J$, and set $I' = I \setminus \{j\}$. Recall that $x(I') = 1$ if $I' = \emptyset$. We choose $u \in \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z})$ as in the proof of (i) above and multiply both sides of the relation $z_u = 0$ by $x(I')$. Observe that $x_j x(I') = x(I)$ to obtain

$$x(I) = (1 - r_u)x(I') + \sum_{\widetilde{I}'} a_{\widetilde{I}'} x\big(\widetilde{I}'\big)$$

for suitable elements $a_{\widetilde{I}'}$ where the sum is over nonempty facial sets $\widetilde{I}'$ such that $J \subset I' \subset \widetilde{I}'$ and $\widetilde{I}'$ not contained in L. If $I' = J$ we are done. Otherwise, $J \subset I'$. By induction on $|I \setminus J|$, one can express $x(I')$ in the form (2.4), which can be substituted in the above expression for $x(I)$.

(iii) Observe that $(1 - ab) = a(1 - b) + (1 - a)$ is nilpotent if $(1 - a), (1 - b)$ are nilpotent. It follows from this observation and our hypothesis that $(1 - r_u)$ is nilpotent for all $u \in \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z})$.

Fix $j \le d$. Let $\mathsf{p}_1, \ldots, \mathsf{p}_t$ be the vertices of F_j. For $1 \le i \le t$, let $L_i \subset \mathcal{F}$ be the collection of all those facets which are incident at the vertex p_i. Suppose that $F_{k_i} \notin L_i$, $1 \le i \le t$, then we claim that the monomial $x_j x_{k_1} \cdots x_{k_t} = 0$, or, equivalently, that $F_j \cap F_{k_1} \cap \cdots \cap F_{k_t} = \emptyset$. Since any non-empty face of F_j must contain a vertex of F_j, in order to establish the claim it suffices to show that $F_{k_1} \cap \cdots \cap F_{k_t}$ does not contain any p_i, $1 \le i \le t$. Since $F_{k_i} \notin L_i$, it is clear that $\mathsf{p}_i \notin F_{k_i}$, $1 \le i \le t$. Hence *none* of the vertices p_i belong to the $F_{k_1} \cap \cdots \cap F_{k_t}$ and consequently $x_j x_{k_1} \cdots x_{k_t} = 0$.

To complete the proof, we express x_j in different ways using (2.4) each time taking the L to be L_i, $1 \le i \le t$. This leads to

$$x_j^t = \prod_{1 \le i \le t} \left(b_j + \sum_{\widetilde{I}'_i} b_{\widetilde{I}'_i} x\big(\widetilde{I}'_i\big) \right)$$

with $\widetilde{I}'_i$ not contained in L_i and $b_i \in S$ nilpotent. On multiplying out, we see that each term on the right hand side appears with coefficient a nilpotent or is divisible by a monomial $x_{k_1} \cdots x_{k_t}$ with $x_{k_i} \notin L_i$. From what has been established above, it follows that x_j^{t+1} is nilpotent. Hence x_j is also nilpotent. □

Proof of Proposition 2.1. In view of Lemma 2.2 (i), it suffices to show that $x(I)$ is in the S-span of $x(U_w)$, $w \in P_0$, for any facial subset $I \subset \mathcal{F}$. Assume that $I \subset \mathcal{F}$ is a nonempty (facial) subset of $\mathcal{F}$. By (2.2), there exists a unique $w \in P_0$ such that $U_w \subset I \subset V_w$.

The proof will be by downward induction on the height of w. If w is the highest vertex, then $U_w = I = V_w$ and hence $x(I) = x(U_w)$. Let $U_w \subsetneq I \subset V_w$ and assume that for any $I' \subset \mathcal{F}$ facial with $U_{w'} \subset I' \subset V_{w'}$ where $w' > w$, the monomial $x(I')$ is in the S-span of $x(U_{w''})$, $w'' \geq w'$.

We apply (2.4) with $J := U_w \subset I$ and $L := V_w \subset \mathcal{F}$. Applying (2.4), we see that $x(I)$ can be expressed as an S-linear combination of $x(U_w)$ and $x(I')$ with $U_w \subset I'$ and I' not contained in V_w. We claim that if $U_{w'} \subset I' \subset V_{w'}$, then $w' > w$. Indeed, since $U(w) \subset I'$, we have $U(w) \subset V(w')$, which implies that $w' \in T_w$. In view of (2.1) we conclude that $w' \geq w$. Since I' is not contained in V_w, we must have $w' > w$. □

REMARK 2.3. (i) Under the hypotheses of Lemma 2.2 (iii), $(1 - x_j)$, $1 \leq d$, are invertible in $\mathcal{R}$. Hence the relation $z_u = 0$ in $\mathcal{R}$ can be rewritten as

$$\zeta_u := \prod_{1 \leq j \leq d} (1 - x_j)^{u(v_j)} = r_u.$$

Furthermore, the relations $\zeta_u = r_u$ for u varying in some $\mathbb{Z}$-basis of $\mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z})$, implies that $\zeta_u = r_u$ for *all* $u \in \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z})$ since $\zeta_{au-bu'} = \zeta_u^a / \zeta_{u'}^b$ and $r_{au-bu'} = r_u^a / r_{u'}^b$, for $a, b \geq 0$ positive integers.
(ii) In case $r_i = 1$ for all $i \leq n$, then $r_u = 1$ for all u and our proof actually shows that $x^{n+1} = 0$.

3. Proof of Theorem 1.2

We keep the notations of previous sections. Let M be the quasi-toric manifold over a simple convex polytope $P \subset \mathbb{R}^n$ with characteristic map $\lambda \colon \mathcal{F} \to \mathbb{Z}^n$. Let $|\mathcal{F}| = d$. As in the previous section we shall assume that $\bigcap_{1 \leq i \leq n} F_i$ is a vertex of P and $G = \prod_{1 \leq i \leq n} G_i \cong (\mathbb{S}^1)^n$ the corresponding product decomposition.

Set $\widetilde{G} := (\mathbb{S}^1)^d$ and let $\theta \colon \mathcal{F} \to \mathbb{Z}^d$ be defined by $F_j \mapsto e_j$, the standard basis vector, for each $j \leq d$. For any face $F = F_{j_1} \cap \cdots \cap F_{j_k}$, set $\widetilde{G}_F$ denote the subgroup $\{(t_1, \ldots, t_d) \in \widetilde{G} \mid t_i = 1,\ i \neq j_1, \ldots, j_k\}$. One has a $\widetilde{G}$-manifold $\mathcal{Z}_P := \widetilde{G} \times P/\!\sim$ where $(g, \mathsf{p}) \sim (g', \mathsf{p}')$ if and only if $\mathsf{p} = \mathsf{p}'$ and $g^{-1}g' \in \widetilde{G}_F$ where p is in the relative interior of the face $F \subset P$. The action of $\widetilde{G}$ on $\mathcal{Z}_P$ is given by $g.[g', \mathsf{p}] = [gg', \mathsf{p}]$ for

$g \in \widetilde{G}$ and $[g', \mathsf{p}] \in \mathcal{Z}_P$. One has the projection map $\mathbf{p}\colon \mathcal{Z}_P \to P$. However $\mathcal{Z}_P$ is not a quasi-toric manifold over P since $d > n$. When P is clear from the context we shall denote $\mathcal{Z}_P$ simply by $\mathcal{Z}$.

Let $\widetilde{\lambda}\colon \mathbb{Z}^d \to \mathbb{Z}^n$ be defined as $e_j \mapsto v_j := \lambda(F_j)$, $1 \le j \le d$. This corresponds to a surjective homomorphism of groups $\Lambda\colon \widetilde{G} \to G$ with kernel $H \subset \widetilde{G}$ for the subgroup corresponding to $\ker\bigl(\widetilde{\lambda}\bigr)$. One has a splitting $\mathbb{Z}^d = \ker\bigl(\widetilde{\lambda}\bigr) \oplus \mathbb{Z}^n$ induced by the injection $\mathbb{Z}^n \rightarrowtail \mathbb{Z}^d$ defined as $v_i \mapsto e_i$, $1 \le i \le n$. This injection corresponds to an imbedding $\Gamma\colon G \rightarrowtail \widetilde{G}$. Identifying $\mathbb{Z}^d$ with $\mathrm{Hom}\bigl(\mathbb{S}^1, \widetilde{G}\bigr)$, the splitting yields an identification $\widetilde{G} \cong G \times H$, $\widetilde{g} = g.h$ where $g = \Gamma \circ \Lambda\bigl(\widetilde{g}\bigr) \in G$ and $h = g^{-1}\widetilde{g}$. The group H is the subgroup of $\widetilde{G}$ with $\ker\bigl(\widetilde{\lambda}\bigr) = \mathrm{Hom}(\mathbb{S}^1, H) \subset \mathrm{Hom}\bigl(\mathbb{S}^1, \widetilde{G}\bigr)$. We let group H act on the *right* of $\mathcal{Z}$ where $x.h = h^{-1}x \in \mathcal{Z}$ for $x \in \mathcal{Z}$, $h \in H \subset \widetilde{G}$. In view of our hypothesis (1.1), we note that the intersection of H with $\widetilde{G}_F$ is trivial for any proper face F of P. Hence the action of H on $\mathcal{Z}$ is free. The quotient of $\mathcal{Z}$ by H is the quasi-toric manifold M. (cf. §4, [9].)

Let $\chi\colon H \to \mathbb{S}^1$ be the restriction to H of any character $\widetilde{\chi}\colon \widetilde{G} \to \mathbb{S}^1$. One obtains a G-equivariant complex line bundle $L_{\widetilde{\chi}}$ over M with projection $\mathcal{Z} \times_H \mathbb{C}_\chi \to M$ where $\mathbb{C}_\chi$ denotes the 1-dimensional complex representation space corresponding to χ. Here the Borel construction $\mathcal{Z} \times_H \mathbb{C}_\chi$ is obtained by the identification

$$(xh, z) \sim (x, \chi(h)z), \quad h \in H, \quad x \in \mathcal{Z}, \quad z \in \mathbb{C}. \tag{3.1}$$

Equivalently $\mathcal{Z} \times_H \mathbb{C}_\chi$ is the quotient of the diagonal action by H on the left on $\mathcal{Z} \times \mathbb{C}_\chi$. The equivalence class of (x, z) is denoted by $[x, z]$. The G-action on $L_{\widetilde{\chi}}$ is given by $g.[x, z] := [gx, \widetilde{\chi}(g)z]$ for $x \in \mathcal{Z}$, $z \in \mathbb{C}_\chi$.

When $\widetilde{\chi} = \widetilde{\rho}_j$ is the j-th projection $\widetilde{G} = (\mathbb{S}^1)^d \to \mathbb{S}^1$, the corresponding G-line bundle on M will be denoted L_j. Denote by $\pi_j\colon L_j \to M$ the projection of the bundle L_j.

Henceforth we shall identify the character group of $\widetilde{G}$ with $\mathrm{Hom}(\mathbb{Z}^d, \mathbb{Z})$ etc. If $\widetilde{u} \in \mathrm{Hom}(\mathbb{Z}^d, \mathbb{Z})$ vanishes on $\ker\bigl(\widetilde{\lambda}\bigr)$, then the line bundle $L_{\widetilde{u}}$ is isomorphic to the product bundle. However the G action on it is given by the character $\widetilde{u}|G$.

Given $u \in \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z}) = \mathrm{Hom}(G, \mathbb{S}^1)$, composing with surjection $\widetilde{G} \to G$, we obtain a character of $\widetilde{G}$ which is trivial on H. As an element of $\mathrm{Hom}(\mathbb{Z}^d, \mathbb{Z})$, this is just the composition $\widetilde{u} := u \circ \widetilde{\lambda}$. Let $e_1^*, \ldots, e_d^*$ be the dual of the standard basis for $\mathbb{Z}^d$. Note that the character $\widetilde{G} \to \mathbb{S}^1$ corresponding to e_j^* is just the j-th projection ρ_j. Clearly,

$$\widetilde{u} = u \circ \widetilde{\lambda} = \sum_{1 \le j \le d} u\bigl(\widetilde{\lambda}(e_j)\bigr)e_j^* = \sum_{1 \le j \le d} u(v_j)e_j^*.$$

Hence we obtain the following isomorphism of G-bundles:

$$L_{\widetilde{u}} = \prod_{1 \le j \le d} L_j^{u(v_j)}. \tag{3.2}$$

Note that since $\widetilde{u}|H$ is trivial, $\mathcal{Z} \times_H \mathbb{C}_{\widetilde{u}} = M \times \mathbb{C}$ and so $L_{\widetilde{u}}$ is isomorphic to the product bundle.

Let $1 \leq j \leq d$. Choose an affine map $h_j \colon \mathbb{R}^n \to \mathbb{R}$ such that h_j vanishes on F_j and $h_j(\mathsf{p}) > 0$ for $\mathsf{p} \in P \setminus F_j$. Since $\widetilde{G}_{F_j} := \widetilde{G}_j$ acts freely on $\mathcal{Z} - p^{-1}(F_j)$, one has a well-defined trivialization $\sigma_j \colon \pi_j^{-1}(M - M_j) \to (M - M_j) \times \mathbb{C}_j$ given by $\sigma_j([x, z]) = ([x], \widetilde{\rho}_j(\widetilde{g}^{-1})z)$ where $x = [\widetilde{g}, \mathsf{p}] \in \mathcal{Z}$, $z \in \mathbb{C}_j$.

Using σ_j and h_j one obtains a well-defined section $s_j \colon M \to L_j$ by setting $s_j([x]) = [x, h_j(\mathsf{p})\widetilde{\rho}_j(\widetilde{g})]$ where $x = [\widetilde{g}, \mathsf{p}] \in \mathcal{Z}$. Note that the section s_j vanishes precisely on M_j. It is straightforward to verify that s_j is G equivariant.

Now let $1 \leq j_1, \ldots, j_k \leq d$ be such that $F_{j_1} \cap \cdots \cap F_{j_k} = \emptyset$. Thus $M_{j_1} \cap \cdots \cap M_{j_k} = \emptyset$. Consider the section $s \colon M \to V$ defined as $s(m) = (s_{j_1}(m), \ldots, s_{j_k}(m))$ where V is (the total space of) the vector bundle $L_{j_1} \oplus \cdots \oplus L_{j_k}$. The section s is nowhere vanishing: indeed $s(m) = 0 \iff s_{j_r}(m) = 0\ \forall r \iff m \in M_{j_r}\ \forall r$. Since $\bigcap_{1 \leq r \leq k} M_{j_r} = \emptyset$, we see that s is nowhere vanishing. Applying the γ^k-operation to $[V] - k$ we obtain $\gamma^k([V] - k) = \gamma^k\big(\bigoplus_{1 \leq r \leq k}([L_{j_r}] - 1)\big) = \prod_{1 \leq r \leq k}([L_{j_r}] - 1)$. Since V has geometric dimension at most $k - 1$,

$$\prod_{1 \leq r \leq k} (1 - [L_{j_r}]) = 0 \tag{3.3}$$

whenever $\bigcap_{1 \leq r \leq k} F_{j_r} = \emptyset$.

REMARK 3.1. Let $\widetilde{L}_j$ denote the pull-back of L_j by the quotient map $\mathcal{Z} \to M$. Since H acts freely on $\mathcal{Z}$, $\widetilde{L}_j$ is isomorphic to the product bundle. This is the same as *dual* of the bundle $\widetilde{L}_j$ considered in §6.1 of [9]. A description of the stable tangent bundle of M was obtained in Theorem 6.6 of [9]. It follows from their proof that the $L_j|M_j$ is isomorphic to the normal bundle to the imbedding $M_j \subset M$. Therefore we have $c_1(L_j) = e(L_j) = \pm[M_j] \in H^2(M; \mathbb{Z})$ where $e(L_j)$ denotes the Euler class of L_j (see [16].) The omni-orientation corresponding to λ is so chosen as to have $c_1(L_j) = +[M_j]$.

Proposition 3.2. *With notations as above, let* $\mathcal{R} = \mathcal{R}(\mathbb{Z}; \lambda)$ *where* $r_i = 1\ \forall i \leq n$. *One has a well-defined homomorphism* $\psi \colon \mathcal{R} \to K(M)$ *of rings which is in fact an isomorphism. Explicitly,* $K(M) \cong \mathbb{Z}[x_1, \ldots, x_d]/\!\sim$ *where the generating relations are*
(i) $x_{j_1} \cdots x_{j_k} = 0$ *whenever* $F_{j_1} \cap \cdots \cap F_{j_k} = \emptyset$, $x_i^{n+1} = 0$ *for* $1 \leq i \leq d$,
(ii) $\prod_{1 \leq j \leq d}(1 - x_j)^{u(v_j)} = 1$ *for any* $u \in \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z})$.
Indeed it suffices to let u vary over a basis of $\mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z})$.

Proof. Relations (3.3) and (3.3) above clearly imply that ψ is a well-defined algebra homomorphism.

From Theorem 4.14 [9], the integral cohomology of M is generated by degree 2 elements. Indeed these can be taken to be dual cohomology classes $[M_j], 1 \leq j \leq d$, where $M_j = \pi^{-1}(F_j)$, F_j being the facets of P. As noted in Remark 3.1, $c_1(L_j) = [M_j]$, $1 \leq j \leq d$.

Now Lemma 4.1, [17], implies that $K(M)$ is generated by the line elements $[L_j]$, $1 \leq j \leq d$. This shows that ψ is surjective. To show that it is injective, we observe that since M is a CW complex with cells only in even dimensions, $K(M)$ is a free abelian group of rank $\chi(M)$ the Euler characteristic of M. But $\chi(M) = m$, the number of vertices of P. (In fact a $\mathbb{Z}$-basis for the integral cohomology of M is the set of dual cohomology classes $[M_w]$, $w \in P_0$.) Since by Prop. 2.1 the rank of $\mathcal{R}$, as an abelian group, is *at most* m, it follows that ψ is in fact an isomorphism of rings.

The remaining parts of the proposition follow from Lemma 2.2 (iii) and Remark 2.3. □

REMARK 3.3. Since L_j restricted to $M - M_j$ is trivial, it follows that $L_j|M_i$ is trivial if $M_i \cap M_j = \emptyset$. On the other hand, if $M_i \cap M_j \neq \emptyset$, then $L_j|M_i$ can be described as follows: choose a vertex in F_j *not* in F_i and let $u \in \operatorname{Hom}(\mathbb{Z}^n, \mathbb{Z})$ be such that $u(\lambda(F_j)) = 1$ and $u(F) = 0$ for any other F that contains v. Then $L_j|M_i = \prod_k L_k^{a_k}|M_i$ where $a_k = -\langle u, \lambda(F_k)\rangle$ and the product is over all those facets $F_k \neq F_j$ which meet F_i.

We shall now prove the main theorem.

Proof of Theorem 1.2. Note that the complex line bundles L_j are G-equivariant. Denote by $\mathcal{L}_j$ the bundle $E(L_j) := E \times_G L_j$ over $E(M) = E \times_G M$. Since the sections s_j are equivariant, so is the section $s = (s_{j_1}, \ldots, s_{j_k})$ of $V = L_{j_1} \oplus \cdots \oplus L_{j_k}$. Hence we obtain a section $E(s)\colon E(M) \to E(V)$. If $\bigcap_{1\leq r\leq k} F_{j_r} = \emptyset$, then s and hence $E(s)$ is nowhere vanishing. Again by applying the γ^k-operation to $[E(V)] - k \in K(E(M))$, we conclude that

$$\prod_{1\leq r\leq k} ([\mathcal{L}_{j_r}] - 1) = 0 \tag{3.4}$$

whenever $\bigcap_{1\leq r\leq k} F_{j_r} = \emptyset$.

Since the isomorphism in equation (3.2) is G-equivariant, one obtains an isomorphism $\mathcal{L}_{\tilde{u}} = \prod_{1\leq j\leq d} \mathcal{L}_j^{u(v_j)}$. Since $L_{\tilde{u}}$ is the product bundle $M \times \mathbb{C}_{\tilde{u}} \to M$, the bundle $\mathcal{L}_{\tilde{u}} \cong p^*(\xi_1^{a_1} \cdots \xi_n^{a_n})$ where $a_i = u(v_i)$. (See §1.) It follows that, in the $K(B)$-algebra $K(E(M))$ one has

$$\prod_{1\leq j\leq d} [\mathcal{L}_j]^{u(v_j)} = [\xi_1]^{a_1} \cdots [\xi_n]^{a_n}. \tag{3.5}$$

Setting $r_i = [\xi_i]$ for $1 \leq i \leq n$, in view of (3.4) and (3.5) we see that there is a well-defined homomorphism of $K(B)$-algebras $\psi\colon \mathcal{R}(K(B); \lambda) \to K(E(M))$ which maps x_j to $(1 - [\mathcal{L}_j])$ for $1 \leq j \leq d$.

It follows from Prop. 3.2 that, the monomials in the L_j generate $K(M)$. Hence the fibre-inclusion $M \to E(M)$ is totally non-cohomologous to zero in K-theory as the bundles $\mathcal{L}_j$ restrict to L_j. As B is compact Hausdorff and $K(M)$ is free Abelian, we observe that the hypotheses of Theorem 1.3, Ch. IV, [14], are satisfied. It follows that $K(E(M)) \cong K(B) \otimes K(M)$ as a $K(B)$-module. In particular, we conclude that ψ is surjective. To see that ψ is a monomorphism, note that $K(E(M))$ is a free module over $K(B)$ of rank $\chi(M) = m$, the number of vertices in P. Since, by Prop. 2.1, $\mathcal{R}(K(B), \lambda)$ is generated by m elements, it follows that ψ is an isomorphism. $\square$

4. Examples

In this section we illustrate our results. Our first example is a well-known result concerning the K ring of a bundle where the fibre is the n-dimensional complex projective space $\mathbb{P}^n$; see Theorem 2.16, Ch. VI of [14]. We include the proof here for the sake of completeness. Our next example is that of a 'quasi-toric bundle' which parallels the notion of toric bundle in toric geometry [11]. We end this section with the example of K-ring of a connected sum $M \# \mathbb{P}^n_{\mathbb{C}}$ where M is any quasi-toric manifold of dimension $2n$.

EXAMPLE 4.1 (Projective space bundle). The complex projective n-space $\mathbb{P}^n$ is a quasi-toric manifold over the standard n-simplex $\Delta^n = \left\{x = \sum_{1 \leq i \leq n} x_i e_i \in \mathbb{R}^n \;\middle|\; \sum_i x_i \leq 1,\ 0 \leq x_i \leq 1 \quad \forall i \geq 1\right\}$. The characteristic map λ sends the i-th facet—the face opposite the vertex e_i—to the standard basis element $v_i := e_i \in \mathbb{Z}^n$ for $i \geq 1$ and sends the 0-th facet which is opposite the vertex 0 to $v_0 := -(e_1 + \cdots + e_n) \in \mathbb{Z}^n$. The space $E(\mathbb{P}^n)$ is just the projective space bundle $\mathbb{P}(\mathbf{1} \oplus \xi_1 \oplus \cdots \oplus \xi_n)$ over B. Here $\mathbf{1}$ denotes the trivial complex line bundle $B \times \mathbb{C}e_0 \to B$ over B. Indeed the map which sends $[e, x] = [(w_1, \ldots, w_n), [x_0, \ldots, x_n]]$ to the complex line spanned by the vector $x_0 e_0 + x_1 w_1 + \cdots + x_n w_n$ in the fibre $\mathbb{P}(\mathbb{C}e_0 + \mathbb{C}w_1 + \cdots + \mathbb{C}w_n)$ over $\pi(e) \in B$, where $e = (w_1, \ldots, w_n)$, is a well defined bundle isomorphism.

Proposition 4.2 (cf. Theorem 2.16, Chapter IV, [14]).

$$K(E(\mathbb{P}^n)) \cong K(B)[y] \Big/ \Big\langle \prod_{0 \leq i \leq n} (y - [\xi_i]) \Big\rangle$$

where $\xi_0 := 1$ and y is the class of the canonical bundle over $E(\mathbb{P}^n)$ which restricts to the tautological bundle on each fibre of $E(\mathbb{P}^n) \to B$.

Proof. By choosing $u \in \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z})$ to be the dual basis element e_i^*, $i \geq 1$, relation (1.3) gives $[\mathcal{L}_i] = [\xi_i][\mathcal{L}_0]$ in $\mathcal{R}(K(B); \lambda)$ since $\xi_0 = 1$. It can be seen easily that other choices of u in relation (1.3) do not lead to any new relation. Substituting for $[\mathcal{L}_i]$ in relation (1.2), we obtain $\prod_{0 \leq j \leq n}(1 - [\mathcal{L}_0][\xi_i]) = 0$. Setting $y = [\mathcal{L}_0]^{-1}$ we obtain

$\prod([\xi_i] - y) = 0$. By Theorem 1.2 it follows that $K(E(\mathbb{P}^n)) \cong K(B)[y]/\langle\prod([\xi_i] - y)\rangle$. Since $y = [\mathcal{L}_0^*]$, the proof is completed by observing that $\mathcal{L}_0^*$ restricts to the *tautological* bundle on the fibres $\mathbb{P}^n$. $\square$

EXAMPLE 4.3 (Quasi-toric bundles). Let M and X be a quasi-toric manifolds over P and Q with characteristic maps $\lambda\colon \mathcal{F}_P \to \mathbb{Z}^n$, $\mu\colon \mathcal{F}_Q \to \mathbb{Z}^k$. Let $\gamma\colon \mathcal{F}_P \to \mathbb{Z}^k$. To this data, we associate a quasi-toric manifold $\widetilde{M}$ over $\widetilde{P} := P \times Q$ which is a quasi-toric bundle over M with fibre X as follows. Note that $\widetilde{\mathcal{F}} := \mathcal{F}_{\widetilde{P}} = \{P\} \times \mathcal{F}_Q \cup \mathcal{F}_P \times \{Q\}$. Define $\widetilde{\lambda} : \widetilde{\mathcal{F}} \to \mathbb{Z}^{k+n} = \mathbb{Z}^n \times \mathbb{Z}^k$ as follows:

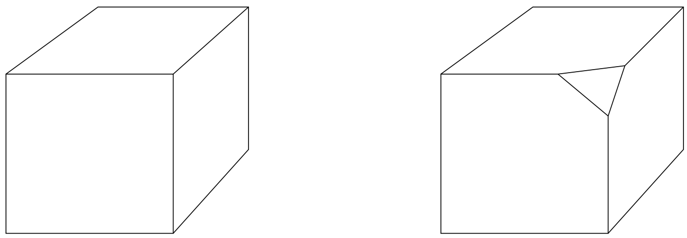
$$\widetilde{\lambda}(P \times F') = (0, \mu(F')) \quad \text{for} \quad F' \in \mathcal{F}_Q$$
$$\widetilde{\lambda}(F \times Q) = (\lambda(F), \gamma(F)) \quad \text{for} \quad F \in \mathcal{F}_P.$$

Denote by G_P, G_Q the tori which act on M and X respectively and set $G := G_P \times G_Q$. The notations $\widetilde{G}_P, \widetilde{G}_Q$ and $\widetilde{G}$ will have similar meaning. Let $\widetilde{\mathcal{Z}} := \mathcal{Z}_{\widetilde{P}} = \widetilde{G} \times \widetilde{P}/\sim$. The projection $\widetilde{G}_P \times \widetilde{G}_Q \times P \times Q \to \widetilde{G}_P \times P$ and the inclusion $\widetilde{G}_Q \times Q \hookrightarrow \widetilde{G}_P \times \widetilde{G}_Q \times P \times Q$ induce a projection map $\widetilde{\eta}\colon \widetilde{\mathcal{Z}} \to \mathcal{Z}_P$ and an inclusion map $\widetilde{\iota}\colon \mathcal{Z}_Q \hookrightarrow \widetilde{\mathcal{Z}}$. The map $\widetilde{\eta}$ is equivariant with respect to the $\widetilde{G}_P$ action and the map $\widetilde{\iota}$ is equivariant with respect to the $\widetilde{G}_Q$. These maps pass to the quotient to yield well-defined maps $\iota\colon X \to \widetilde{M}$ and $\eta : \widetilde{M} \to M$. In fact, it is not difficult to show that η is the projection of the X-bundle associated to the G_Q-bundle $\xi_1 \oplus \cdots \oplus \xi_k$ where $\xi_r = L_1^{a_{1r}} \otimes \cdots \otimes L_d^{a_{dr}}$ where $\gamma(F_j) = \sum_{1 \le r \le k} a_{jr} e_r$, $1 \le j \le d$.

The following proposition, which generalizes Prop. 4.2, is an immediate consequence of Theorem 1.2.

Proposition 4.4. *With the above notations, $K(\widetilde{M})$ is isomorphic to $\mathcal{R}(K(M); \mu)$ with $r_j = [\xi_j]$, $1 \le j \le k$.*

EXAMPLE 4.5 (Connected sum with $\mathbb{P}^n_{\mathbb{C}}$). Let M be a G-quasi-toric manifold over $P \subset \mathbb{R}^n$. Consider the $G = (\mathbb{S}^1)^n$-fixed point $[1, \mathsf{v}] \in M$ corresponding to a vertex $\mathsf{v} \in P$. Let $\lambda\colon \mathcal{F} \to \mathbb{Z}^n$ be the characteristic map of M. Without loss of generality we may (and do) assume that $\lambda(F_i) = e_i$, $1 \le i \le n$, where F_i, $1 \le i \le n$, are the facets which meet at v. Consider the G action on $\mathbb{P}^n_{\mathbb{C}}$ given by $t_i.[z_0 : z_1 : \cdots : z_n] = [z_0 : z_1 : \cdots : t_i z_i : \cdots : z_n]$, $1 \le i \le n$, for t_i in the i-th factor of G. The G-neighbourhood $\{[1 : z_1 : \cdots : z_n] \mid z_i \in \mathbb{C}\}$ is equivariantly diffeomorphic to a G neighbourhood of $[1, \mathsf{v}] \in M$ and hence one has the equivariant connected sum $\widetilde{M} := M \# \mathbb{P}^n_{\mathbb{C}}$ which is a quasi-toric manifold over the polytope the connected sum $\widetilde{P} = P \# \Delta^n$. (See 1.11, [9].) The polytope $\widetilde{P}$ is isomorphic to that obtained from P by removing $P \cap V_>$ where $V_>$ is the open half space containing the vertex v defined by a hyperplane V in $\mathbb{R}^n$ sufficiently close to v and $\mathsf{v} \notin V$. In particular, $\widetilde{\mathcal{F}} := \mathcal{F}_{\widetilde{P}}$



P $\qquad\qquad$ $\widetilde{\mathrm{P}}$

consists of those facets of P which do not meet V, the facets $\widetilde{F}_i = F_i \cap V_{\leq}$, $1 \leq i \leq n$, and one extra facet, $\widetilde{F}_0 := P \cap V$. Here $V_{\leq} = \mathbb{R}^n - V_{>}$.

The characteristic map $\widetilde{\lambda}\colon \widetilde{\mathcal{F}} \to \mathbb{Z}^n$ is defined as

$$\widetilde{\lambda}(\widetilde{F}) = \begin{cases} \lambda(F) & \text{if} \quad \widetilde{F} = F \text{ is a facet of } P, \\ e_i = \lambda(F_i) & \text{if} \quad \widetilde{F} = \widetilde{F}_i,\ 1 \leq i \leq n, \\ -\left(\sum\limits_{1\leq i\leq n} e_i\right) & \text{if} \quad \widetilde{F} = \widetilde{F}_0. \end{cases} \tag{4.1}$$

Set $\widetilde{G}_0 = \mathbb{S}^1 \times \widetilde{G} = (\mathbb{S}^1)^{d+1}$, and let $\widetilde{H} \subset \widetilde{G}_0$ be the subgroup of $\widetilde{G}_0$ generated by $H \subset \widetilde{G} \subset \widetilde{G}_0$ and $\big\{(z_0, z_1, \ldots, z_d) \in \widetilde{G}_0 \bigm| z_j = z_0,\ 1 \leq j \leq n;\ z_j = 1,\ j > n\big\}$. One has $\widetilde{M} = \widetilde{\mathcal{Z}}/\widetilde{H}$ where $\widetilde{\mathcal{Z}} = \mathcal{Z}_{\widetilde{P}} := \widetilde{G}_0 \times \widetilde{P}/\!\sim$. The map $q_0\colon \widetilde{P} \to P$ of polytopes which collapses $\widetilde{F}_0$ to v yields a natural surjection $\widetilde{\mathcal{Z}} \to \mathcal{Z}$ defined as $[z_0 g, \widetilde{\mathsf{p}}] \mapsto [g, q_0(\widetilde{\mathsf{p}})]$ which intertwines the $\widetilde{G}_0$ action on $\widetilde{\mathcal{Z}}$ and $\widetilde{G}$ action on $\mathcal{Z}$. This induces an equivariant collapsing map $q\colon \widetilde{M} \to M$ which maps $\widetilde{M} - \widetilde{M}_0$ diffeomorphically onto $M - \{[1, \mathsf{v}]\}$ and maps $\widetilde{M}_0$ onto $[1, \mathsf{v}]$. Here $\widetilde{M}_0 = \widetilde{\pi}^{-1}\big(\widetilde{F}_0\big)$. The induced map $q^*\colon K(M) \to K\big(\widetilde{M}\big)$ allows us to view $K\big(\widetilde{M}\big)$ as a $K(M)$-algebra. Note that equation (4.1) together with Proposition 3.2 gives explicit description of the ring structure of $K\big(\widetilde{M}\big)$.

Proposition 4.6. *$K\big(\widetilde{M}\big) \cong K(M)[x_0]/J$ where the ideal J is generated by the elements*

$$\prod_{1\leq j\leq n} ((1 - x_j)x_0 + x_j); \quad x_r x_0, \quad r > n,$$

where $x_j = 1 - [L_j] \in K(M)$, $1 \leq j \leq d$.

Proof. Denote by $\widetilde{L}_i$, $0 \leq i \leq d$, the line bundles $\widetilde{\mathcal{Z}} \times_{\widetilde{H}} \mathbb{C}_{\rho_{i,0}} \to \widetilde{M}$ where $\rho_{i,0}$ is the restriction to $\widetilde{H}$ of the i-th projection $\widetilde{\rho}_{i,0}\colon \widetilde{G}_0 \to \mathbb{S}^1$. The L_j, $1 \leq j \leq d$, are the line bundles over M likewise associated to the j-th projection $\widetilde{\rho}_j\colon \widetilde{G} \to \mathbb{S}^1$.

It is easily seen that $q^*(L_j)$ isomorphic to the line bundle associated to $\widetilde{\rho}_j\colon \widetilde{H} \to \mathbb{S}^1$ for $j > n$ and so $q^*(L_j) \cong \widetilde{L}_j$ for $j > n$. However, for $1 \leq j \leq n$, $q^*(L_j)$ is isomorphic to the line bundle associated to the character $\chi_j := \rho_j - \rho_0$ and so $q^*(L_j) \cong \widetilde{L}_j \otimes \widetilde{L}_0^*$. Indeed it is straightforward to verify that $[[z_0 g, \widetilde{\mathfrak{p}}], w] \mapsto [[g, q_0(\widetilde{\mathfrak{p}})], z_0 w]$ is a well-defined map $\widetilde{\widetilde{\mathcal{Z}}} \times_{\widetilde{H}} \mathbb{C}_{\chi_j} \to \mathcal{Z} \times_H \mathbb{C}_{\rho_j}$ which is a 'bundle map' that covers the $q\colon \widetilde{M} \to M$. Hence by Lemma 3.1 [16] it follows that $q^*(L_j) \cong \widetilde{L}_j \otimes \widetilde{L}_0^*$.

It follows that $q^*\colon K(M) \to K\big(\widetilde{M}\big)$ is given by the map

$$1 - x_j \mapsto \begin{cases} (1-\widetilde{x}_0)^{-1}(1-\widetilde{x}_j), & 1 \le j \le n, \\ (1-\widetilde{x}_j), & n < j \le d. \end{cases}$$

It is clear that $K\big(\widetilde{M}\big)$ is generated as an algebra over $q^*(K(M))$ by the element $\widetilde{x}_0$. The relation $\prod_{1\le j\le n} \widetilde{x}_j = 0$ can be rewritten as $\prod_{1\le j\le n}(\widetilde{x}_0(1-q^*(x_j)) + q^*(x_j)) = 0$. Also, since $\widetilde{F}_r \cap \widetilde{F}_0 = \emptyset$ for $r > n$, we have $q^*(x_r).\widetilde{x}_0 = \widetilde{x}_r\widetilde{x}_0 = 0$. Thus q^* extends to a $K(M)$-algebra map $\eta\colon K(M)[x_0]/J \to K\big(\widetilde{M}\big)$ which maps x_0 to $\widetilde{x}_0$. It is straightforward to verify that one has a well-defined ring homomorphism $\mu\colon K\big(\widetilde{M}\big) \to K(M)[x_0]/J$ where

$$\mu(1-\widetilde{x_k}) = \begin{cases} (1-x_0) & \text{if} \quad k = 0 \\ (1-x_k) & \text{if} \quad n < k \le d \\ (1-x_0)(1-x_k) & \text{if} \quad 1 \le k \le n. \end{cases}$$

This is evidently the inverse of η, completing the proof. □

5. Bott-Samelson varieties

In this section we illustrate our theorem in the case of Bott-Samelson manifolds which were first constructed in [4] to study cohomology of generalized flag varieties. M. Demazure and D. Hansen used it to obtain desingularizations of Schubert varieties in generalized flag varieties. M. Grossberg and Y. Karshon [12] constructed Bott towers, which are iterated fibre bundles with fibre at each stage being $\mathbb{P}^1_{\mathbb{C}}$. They also showed that Bott-Samelson variety can be deformed to a toric variety. The 'special fibre,' of this deformation is a Bott tower. The underlying differentiable structure is preserved under the deformation. It follows that Bott-Samelson varieties have the structure of a quasi-toric manifold with quotient polytope being the n-dimensional cube I^n where n is the complex dimension of the Bott-Samelson variety. This quasi-toric structure has been explicitly worked out by Grossberg-Karshon [12] and by M. Willems [19]. In this section we describe the K-ring of the Bott towers in terms of generators and relations using our main theorem. Using their structure as an iterated 2-sphere bundle, one could alternatively use Theorem 2.16, Ch. IV, [14] to obtain the same result. (Indeed this was our originial approach [18].)

We shall recall briefly the construction and the quasi-toric structure of Bott towers.

Let $C = (c_{i,j})$ denote an n-by-n unipotent upper triangular matrix with integer entries. The matrix C determines a Bott tower $M(C)$ of (real) dimension $2n$. Using the notation of §3, it turns out that $\mathcal{Z} = (\mathbb{S}^1)^{2n} \times I^n/\sim$ is the space $(\mathbb{S}^3)^n \subset \mathbb{C}^{2n}$.

The quasi-toric manifold $M(C)$ is the quotient of $(\mathbb{S}^3)^n$ by the action of $H \cong (\mathbb{S}^1)^n$ on the right of $\mathcal{Z}$ where

$$(z_1, w_1, \ldots, z_n, w_n).t_i = \left(z_1, w_1, \ldots, t_i^{-1} z_i, t_i^{-1} w_i, \ldots, z_j, t_i^{-c_{i,j}} w_j, \ldots, z_n, t_i^{-c_{i,n}} w_n\right)$$

for t_i in the i-th factor of H. The action of $\widetilde{G} = (\mathbb{S}^1)^{2n}$ on (the left of) $\mathcal{Z}$ is such that the first n factors act on the z-coordinates and the last n on the w-coordinates. Thus $H \subset \widetilde{G} = (\mathbb{S}^1)^{2n}$ is identified as the subgroup of $\widetilde{G}$ generated by the 1-parameter subgroups corresponding to the vectors $\varepsilon_i := e_i + e_{i+n} + \sum_{i<j\leq n} c_{i,j} e_{j+n} \in \mathbb{Z}^{2n} = \mathrm{Hom}(S^1, \widetilde{G})$. Note that the elements $e_1, \ldots, e_n, \varepsilon_1, \ldots \varepsilon_n$ forms a $\mathbb{Z}$-basis for $\mathbb{Z}^{2n}$. Let $G \subset \widetilde{G}$ be the group generated by the 1-parameter subgroups $e_1, \ldots, e_n$. Then $G \cap H = \{1\}$ and so we may identify G with the quotient $\widetilde{G}/H$ via the projection $\Lambda\colon \widetilde{G} \to \widetilde{G}/H$.

The group G acts on $M(C)$ with quotient $M/G \cong I^n$. The projection $\pi\colon M(C) \to I^n$ is the map $[z_1, w_1, \ldots, z_n, w_n] \mapsto (|z_1|, \ldots, |z_n|)$. Denote by F_i^0, F_i^1 the facet of I^n with i-th coordinate $0, 1$ respectively. One has the inclusion $I^n \subset M(C)$ is given by $(z_1, \ldots, z_n) \mapsto [z_1, w_1, \ldots, z_n, w_n]$ where $w_i := \sqrt{1 - z_i^2}$, $z_i \in I$, $1 \leq i \leq n$.

Lemma 5.1. *With the above notations $M(C)$ is a quasi-toric manifold over I^n where the characteristic map is given by*

$$\lambda(F_i^0) = v_i^0 := e_i,$$
$$\lambda(F_i^1) = v_i^1 := -e_i - \sum_{i<j\leq n} c_{i,j} v_j^1 \tag{5.1}$$

for $1 \leq i \leq n$ where $v_n^1 := -e_n$.

Proof. First we show that the isotropy group for the G-action on $M(C)$ at any interior point of F_i^ϵ is the (image of the) 1-parameter subgroup $v_i^\epsilon \in \mathbb{Z}^n = \mathrm{Hom}(\mathbb{S}^1, G)$ for $1 \leq i \leq n$, $\epsilon = 0, 1$. Indeed set $\mathsf{a}_i^\epsilon \in F_i^\epsilon$ to be such that all its coordinates are $1/\sqrt{2}$ except $z_i = \epsilon$, $w_i = 1 - \epsilon$, for $1 \leq i \leq n$, $\epsilon = 0, 1$. It is readily seen that the isotropy at a_i^0 is the 1-parameter subgroup $e_i =: v_i^0$, $1 \leq i \leq n$.

The isotropy at a_i^1 is the image under the projection $\Lambda\colon \widetilde{G} \to G$ of the stabilizer—call it S—of $p^{-1}(\mathsf{a}_i^1) \subset \mathcal{Z}$. Let $\widetilde{\lambda}\colon \mathbb{Z}^{2n} \to \mathbb{Z}^n$ be the map induced by Λ between the 1-parameter subgroups of $\widetilde{G}$ and G. Then $\ker(\widetilde{\lambda})$ is the group of 1-parameter subgroups of $H = \ker(\Lambda)$ and hence is generated by $\left\{e_i + e_{i+n} + \sum_{i<j\leq n} c_{i,j} e_{j+n} \;\middle|\; 1 \leq i \leq n\right\}$. Note that S contains the 1-parameter subgroup e_{i+n} since w_i-coordinate of any point of $p^{-1}(\mathsf{a}_i^1)$ is zero. Evidently $e_{i+n} \notin \ker(\widetilde{\lambda})$. By dimension considerations, the isotropy subgroup at $\mathsf{a}_i^1 \in M(C)$ *equals* the 1-parameter subgroup $v_i^1 := \widetilde{\lambda}(e_{i+n}) \in \mathbb{Z}^n$, $1 \leq i \leq n$. Since $\widetilde{\lambda}\left(e_i + e_{i+n} + \sum_{i<j\leq n} c_{i,j} e_{j+n}\right) = 0$, it follows that $v_i^1 = -e_i - \sum_{i<j\leq n} c_{i,j} \widetilde{\lambda}(e_{j+n}) = -e_i - \sum_{i<j\leq n} c_{i,j} v_j^1$. This establishes (5.1).

To complete the proof that $M(C)$ is a quasi-toric manifold we need to verify that condition (1.1) is satisfied. Let $\underline{\epsilon} = (\epsilon_1, \cdots, \epsilon_n) \in I^n$, $\epsilon_i \in \{0,1\}$, be any vertex of I^n. We need to show that the set $\{v_i^{\epsilon_i} \mid 1 \leq i \leq n\}$ is a basis for $\mathbb{Z}^n$. It can be seen from (5.1) that the matrix relating this set and the standard basis $e_1, \ldots, e_n$ is lower triangular with i-th diagonal entry being ± 1, completing the proof. $\square$

Denote by L_i, $1 \leq i \leq 2n$, the canonical bundles of the quasi-toric manifold $M(C)$. Recall that L_i is associated to i-th projection $\widetilde{G} = (\mathbb{S}^1)^{2n} \to \mathbb{S}^1$.

Lemma 5.2. *For any $1 \leq i \leq n$, one has the bundle isomorphism*

$$L_{i+n} \cong L_i \prod_{1 \leq j < i} L_j^{c_{j,i}}. \tag{5.2}$$

Proof. Let $u_1, \ldots, u_n$ denote the basis of $\mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z}) = \mathrm{Hom}(G, \mathbb{S}^1)$ dual to the basis $\lambda(F_i^1)$, $1 \leq i \leq n$. Fix an i. Set $u = u_i$, and let $\widetilde{u} = u \circ \widetilde{\lambda} \in \mathrm{Hom}(\mathbb{Z}^{2n}, \mathbb{Z}) = \mathrm{Hom}(\widetilde{G}, \mathbb{S}^1)$. From (3.2), one has the bundle isomorphism

$$L_{\widetilde{u}} \cong \prod_{1 \leq j \leq n} L_j^{u(v_j^0)} \prod_{1 \leq j \leq n} L_{j+n}^{u(v_j^1)}.$$

Note that $u(v_j^1) = \delta_{i,j}$ by the very definition of u and that, in view of (5.1), we have $u(v_j^0) = u(e_j) = u\big(-v_j^1 - \sum_{j<k\leq n} c_{j,k} v_k^1\big)$ which equals -1 if $i = j$ and equals $-c_{j,i}$ when $j < i$. On the other hand $L_{\widetilde{u}}$ is isomorphic to the trivial line bundle since $\widetilde{u}$ (viewed as a character) is trivial on H. Therefore the above isomorphism yields

$$\mathbf{1} \cong L_{i+n} L_i^{-1} \prod_{1 \leq j < i} L_i^{-c_{j,i}}$$

from which the lemma follows. $\square$

Theorem 5.3. *We keep the above notations. Let C be any $n \times n$ unipotent upper triangular matrix over $\mathbb{Z}$. The map $y_i \mapsto [L_i]$, $1 \leq i \leq n$, defines an isomorphism of rings $\mathbb{Z}\big[y_1^{\pm 1}, \ldots, y_n^{\pm 1}\big]\big/ J$ to $K(M(C))$ where J is the ideal generated by the elements: $(y_i - 1)\big(y_i - y_0 y_1^{-c_{1,i}} \cdots y_{i-1}^{-c_{i-1,i}}\big)$, $1 \leq i \leq n$, where $y_0 := 1$.*

Proof. Since $F_i^0 \cap F_i^1 = \emptyset$ we obtain from equation (3.3) that, $([L_{i+n}] - 1)([L_i] - 1) = 0$, $1 \leq i \leq n$. Substituting for L_{i+n} from the isomorphism of Lemma 5.2, leads to

$$(y_i - 1)\big(y_i - y_{i-1}^{-c_{i-1,i}} \cdots y_1^{-c_{1,i}}\big) = 0$$

where $y_i = [L_i] \in K(M(C))$. In the case of the n-cube I^n, all relations of the form (1.2) are consequences of $F_i^0 \cap F_i^1 = \emptyset$, $1 \leq i \leq n$. Also all relations of the form (1.3) are consequences of (5.2) above. In view of the fact that the classes of line bundles are invertible elements of the K-ring, it follows from Prop. 3.2 that $K(M(C)) \cong \mathbb{Z}\big[y_1^{\pm 1}, \ldots, y_n^{\pm 1}\big]\big/J$. $\square$

Let $\mathcal{G}$ be a complex semisimple linear algebraic group, B a Borel subgroup. Fix an *algebraic* maximal torus $T \cong (\mathbb{C}^*)^l$, l being the rank of $\mathcal{G}$, contained in B and let Φ^+, Δ denote the corresponding system of positive roots and simple roots respectively. Denote by W the Weyl group of $\mathcal{G}$ with respect to T and $S \subset W$ the set of simple reflections $s_\alpha, \alpha \in \Delta$. For $\gamma \in \Delta$, denote by $P_\gamma \supset B$ the minimal parabolic subgroup corresponding to γ so that $P_\gamma/B \cong \mathbb{P}^1_{\mathbb{C}}$. Let $\alpha_1, \ldots, \alpha_n$ be any sequence of simple roots. Consider the Bott-Samelson variety $M = P_{\alpha_1} \times_B \cdots \times_B P_{\alpha_n} \times_B \{\text{pt.}\}$. Explicitly M is the quotient of $\mathcal{P} := P_{\alpha_1} \times \cdots \times P_{\alpha_n}$ by the action of B^n given by $(p_1, \ldots, p_n).b = \big(p_1 b_1, b_1^{-1} p_2 b_2, \ldots, b_{n-1}^{n-1} p_n b_n\big)$ for $(p_1, \ldots, p_n) \in \mathcal{P}$, $(b_1, \ldots, b_n) \in B^n$. When $w = s_{\alpha_1} \cdots s_{\alpha_n} \in W$ is a reduced expression for w, one has a surjective birational morphism $M \to X(w)$ which maps $[p_1, \ldots, p_n]$ to the coset $p_1 \cdots p_n.B$ in the Schubert variety $X(w) \subset \mathcal{G}/B$. In this case, M is the Bott-Samelson-Demazure-Hansen [4], [10], [13] desingularization of the Schubert variety associated to the reduced expression $w = s_{\alpha_1} \cdots s_{\alpha_n}$. The Bott tower, which arises as the special fibre of a certain deformation of the complex structure of M, is associated to the unipotent upper triangular matrix (c_{ij}) where $c_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle$, $i < j$. The polytope which arises as the quotient of the Bott tower by the n-dimensional (*compact*) torus action is the n-cube I^n. See [12] or [19] for details. Feeding this data into Theorem 5.3, we obtain an explicit description of the K-ring of a Bott-Samelson variety which is diffeomorphic to the Bott tower. Alternatively one could use Propsition 4.2 and induction to obtain the same result.

The Bott-Samelson variety M has an algebraic cell decomposition, i.e., a CW structure where the open cells are affine spaces contained in M. The closed cells of real codimension 2 are the divisors M_j of M, defined as the image of $\{(p_1, \ldots, p_n) \in \mathcal{P} \mid p_j = 1\}$ under the canonical map $\mathcal{P} \to M$. These submanifolds correspond to the facets $F_j^0 \subset \mathcal{F}_{I^n}$.

Note that the integral cohomology ring of M is generated by the dual cohomology classes $[M_j] \in H^2(M; \mathbb{Z})$, $1 \leq j \leq n$.

Lemma 4.2 of [17] implies that the forgetful map $\mathcal{K}(M) \to K(M)$ is an isomorphism of rings where $\mathcal{K}(M)$ is the Grothendieck $\mathcal{K}$-ring of M. The following theorem is established using Theorem 5.3.

Theorem 5.4. *Let M be the* (*generalized*) *Bott-Samelson variety $P_{\alpha_1} \times_B \cdots \times_B P_{\alpha_n} \times_B \{\text{pt.}\}$. Let $c_{i,j} = \langle \alpha_i, \alpha_j^\vee \rangle$, $1 \leq i < j \leq n$. The Grothendieck ring $\mathcal{K}(M)$ of algebraic vector bundles on M is isomorphic to $\mathbb{Z}\big[y_1^{\pm 1}, \cdots, y_n^{\pm 1}\big]\big/\big\langle (y_i - 1)\big(y_i - y_0 y_1^{-c_{1,i}} \cdots y_{i-1}^{-c_{i-1,i}}\big); 1 \leq i \leq n\big\rangle$ where $y_0 := 1$. The class y_i is represented by the* *algebraic line bundle* $\mathcal{O}(-M_i)$ *for* $1 \leq i \leq n$. *The forgetful ring homomorphism* $\mathcal{K}(M) \to K(M)$ *is an isomorphism.*

REMARK 5.5. One has a well-defined involution $y_i \mapsto y_i^{-1} =: w_i$ of the algebra $K(M(C))$. Indeed multiplying the two factors in generating relation $(y_i - 1)\bigl(y_i - y_1^{-c_{1,i}} \cdots y_{i-1}^{-c_{i-1,i}}\bigr) = 0$ by y_i^{-1} and $y_i^{-1} y_1^{c_{1,i}} \cdots y_{i-1}^{c_{i,i-1}} = 0$ we get the same relation with the y_j's replaced by $y_j^{-1} = w_j$: that is, $(w_i - 1)\bigl(w_i - w_1^{-c_{1,i}} \cdots w_{i-1}^{-c_{i-1,i}}\bigr) = 0$. Consequently, one could let y_i to be the class of $\mathcal{O}(M_i)$ in the above theorem.

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Parameswaran Sankaran
Institute of Mathematical Sciences
CIT Campus
Chennai 600 113
India
e-mail: sankaran@imsc.res.in

V. Uma
Institut Fourier
Universite Grenoble
38402 St.-Martin d'Heres Cedex
France
Current address:
Department of Mathematics
Indian Institute Technology,
Chennai-600 036
India
e-mail: vuma@iitm.ac.in
e-mail: uma@cmi.ac.in