Masuda, M. and Panov, T.
Osaka J. Math.
43 (2006), 711–746

ON THE COHOMOLOGY OF TORUS MANIFOLDS

MIKIYA MASUDA and TARAS PANOV

(Received January 21, 2005, revised January 20, 2006)

Abstract

A *torus manifold* is an even-dimensional manifold acted on by a half-dimensional torus with non-empty fixed point set and some additional orientation data. It may be considered as a far-reaching generalisation of *toric manifolds* from algebraic geometry. The orbit space of a torus manifold has a rich combinatorial structure, e.g., it is a *manifold with corners* provided that the action is *locally standard*. Here we investigate relationships between the cohomological properties of torus manifolds and the combinatorics of their orbit quotients. We show that the cohomology ring of a torus manifold is generated by two-dimensional classes if and only if the quotient is a *homology polytope*. In this case we retrieve the familiar picture from toric geometry: the equivariant cohomology is the *face ring* of the nerve simplicial complex and the ordinary cohomology is obtained by factoring out certain linear forms. In a more general situation, we show that the odd-degree cohomology of a torus manifold vanishes if and only if the orbit space is *face-acyclic*. Although the cohomology is no longer generated in degree two under these circumstances, the equivariant cohomology is still isomorphic to the face ring of an appropriate *simplicial poset*.

1. Introduction

Since the 1970s algebraic geometers have studied equivariant algebraic compactifications of the *algebraic torus* $(\mathbb{C}^*)^n$, nowadays known as *complete toric varieties*. The study quickly grew into a separate branch of algebraic geometry, "toric geometry", incorporating many topological and convex-geometrical ideas and constructions, and producing a spectacular array of applications. A toric variety is a (normal) algebraic variety on which an algebraic torus acts with a dense orbit. The variety and the action are fully determined by a combinatorial object called a *fan* [7].

With the appearance of the pioneering work [6] of Davis and Januszkiewicz in the beginning of the 1990s, the ideas of toric geometry have started penetrating into topology. The orbit space of a non-singular projective toric variety with respect to the action of the compact torus $T^n \subset (\mathbb{C}^n)^*$ can be identified with the simple polytope "dual" to the corresponding fan. Moreover, the action of the compact torus on a non-singular toric variety is "locally standard," that is, locally modelled by the standard

2000 Mathematics Subject Classification. 57R91.

The second author was partially supported by the Russian Foundation for Basic Research, grant no. 01-01-00546.

action on $\mathbb{C}^n$. Davis and Januszkiewicz took these two characteristic properties as a starting point for their topological generalisation of toric varieties, namely *quasitoric manifolds*. A quasitoric manifold is a compact manifold M^{2n} with a locally standard action of T^n whose orbit space is (combinatorially) a simple polytope. (Davis and Januszkiewicz used the term "toric manifold," but by the time their work appeared the latter had already been used in algebraic geometry as a synonym of "non-singular toric variety.") According to one of the main results of [6], the cohomology ring of a quasitoric manifold M has the same structure as that of a non-singular complete toric variety, and is isomorphic to the quotient of the Stanley-Reisner face ring of the orbit space by certain linear forms. In particular, the cohomology of M is generated by degree-two elements.

In contrast, the convex-geometrical notion of polytope, while playing a very important role in geometrical considerations related to toric geometry, appears to be less relevant in the topological study of torus actions. The orbit quotient $Q = M/T$ of a non-singular compact toric variety M locally looks like the positive cone $\mathbb{R}^n_+$ and thereby acquires a specific face decomposition. This combinatorial structure on Q is known to differential topologists as that of a *manifold with corners*. Moreover, all faces of Q, including Q itself, and all their intersections are acyclic. We call such a manifold with corners a *homology polytope*. It is a genuine polytope provided that the toric variety is projective, but in general may fail to be so. This implies, in particular, that the class of quasitoric manifolds does not include all non-singular compact toric varieties (see [3, §5.2] for more discussion on the relationships between toric varieties and quasitoric manifolds). On the other hand we might expect that all the topological properties of quasitoric manifolds would still hold under a weaker assumption that the orbit space of the torus action is a homology polytope. This is justified by some results of the present paper (see Theorem 8.3).

An alternative far-reaching topological generalisation of complete nonsingular toric varieties was introduced in [13] and [11] under the name of *torus manifolds* (or *unitary toric manifolds* in the earlier terminology). A torus manifold is an even-dimensional manifold M acted on by a half-dimensional torus T with non-empty fixed point set; we also specify certain orientation data on M from the beginning, in order to make certain isomorphisms canonical. Particular examples of torus manifolds include non-singular complete toric varieties (otherwise known as toric manifolds) and the quasitoric manifolds of Davis and Januszkiewicz. On the other hand, the conditions on the action are significantly weakened in comparison to quasitoric manifolds. Surprisingly, torus manifolds admit a combinatorial treatment similar to toric varieties. It relies on the notions of *multi-fans* and *multi-polytopes*, developed in [11] as an alternative to fans associated with toric varieties.

The notion of torus manifold appears to be an appropriate concept for investigating relationships between the topology of torus action and the combinatorics of orbit quotient, which is the main theme of the current paper. Our first main result (Theorem 8.3) measures the extent of the analogy between the cohomological structure of non-singular complete toric varieties and torus manifolds:

Theorem 1. *The cohomology of a torus manifold M is generated by its degree-two part if and only if M is locally standard and the orbit space Q is a homology polytope.*

The cohomology ring itself may also be calculated and has a structure familiar from toric geometry: it is isomorphic to the Stanley-Reisner face ring of Q modulo certain linear forms.

Next we study a more general class of torus manifolds: those with vanishing odd-degree cohomology. Under these circumstances the equivariant cohomology of M is a free finitely generated module over the equivariant cohomology of point, $H_T^*(pt) = \mathbb{Z}[t_1, \ldots, t_n]$. This condition is known to algebraists as *Cohen-Macaulayness* and is equivalent to M being *equivariantly formal* in the terminology of [9]. The orbit space of a torus manifold with $H^{odd}(M) = 0$ may fail to be a homology polytope, as a simple example of torus acting on an even-dimensional sphere shows (see Example 3.2). We introduce a weaker notion of *face-acyclic* manifold with corners Q, in which all the faces are still acyclic, but their intersections may fail to be connected, and prove

Theorem 2. *The odd-degree cohomology of M vanishes if and only if M is locally standard and the orbit space Q is face-acyclic.*

This result is stated as Theorem 9.3 in our paper. We also show that the equivariant cohomology is isomorphic to the face ring of the simplicial poset of faces of Q and identify the ordinary cohomology accordingly (Theorem 7.7 and Corollary 7.8). The face ring of a simplicial poset is not generated by its degree-two elements in general.

At the end we prove Stanley's conjecture on the characterisation of h-vectors of Gorenstein* simplicial posets in the particular case of face posets of orbit quotients for torus manifolds (Theorem 10.1). Unlike the case of Gorenstein* simplicial complexes (which can be considered as an "algebraic approximation" to triangulations of spheres), the conditions for an integer vector to be an h-vector of a Gorenstein* simplicial poset are relatively weak. Such an h-vector must have non-negative entries h_i and satisfy the *Dehn-Sommerville equations* $h_i = h_{n-i}$, $i = 0, \ldots, n$. There are no other conditions for odd n. In even dimensions there is one other troublesome condition; the middle-dimensional entry of the h-vector must be even if at least one other entry is zero. It is not hard to check that these conditions are sufficient, by providing the corresponding examples of simplicial posets. We show that these simplicial posets can be realised as the face posets of orbit quotients for torus manifolds with $H^{odd}(M) = 0$ (so that the h-vectors of posets are the even Betti vectors of torus manifolds). Stanley's conjecture [17] was that those three conditions are also necessary. In this paper we establish the necessity for h-vectors of posets associated to torus manifolds with $H^{odd}(M) = 0$. This is done through the calculation of the Stiefel-Whitney classes of torus manifolds. Similar topological ideas were used by the first author to prove the Stanley conjecture in full generality in [14].

We note that the characterisation of h-vectors for Gorenstein* simplicial complexes, as well as for sphere triangulations, remains wide open.

The paper is organised as follows. In Section 2 we establish the notation concerning torus actions on manifolds and prove three pivotal statements (Lemmas 2.1–2.3) describing different properties of fixed point sets. In Section 3 we introduce the concept of torus manifold, give a few examples, and establish some basic facts about them. In Section 4 we discuss locally standard torus actions. The main result here is Theorem 4.1 showing that a torus manifold M is locally standard provided that $H^{odd}(M) = 0$. We also introduce a canonical model for a torus manifold with given orbit space Q and the distribution of circle subgroups fixing characteristic submanifolds. Then we show that a torus manifold is equivariantly diffeomorphic to its canonical model provided that $H^2(Q) = 0$. This extends the corresponding result for quasitoric manifolds due to Davis and Januszkiewicz. In Section 5 we develop the necessary apparatus of "combinatorial commutative algebra." Here we introduce face rings of manifolds with corners and simplicial posets, and list their main algebraic properties. We try not to overload the notation with poset terminology, but a reader familiar with posets will recognise the notions of (semi)lattice, meet, join, etc. In Section 6 we turn to the equivariant cohomology of torus manifolds. We introduce certain key concepts and construct a map from the face ring of the orbit quotient to the equivariant cohomology of the torus manifold, which is later shown to be an isomorphism under certain conditions. Sections 7–9 contain the proofs of the main results quoted above. In Section 10 we prove the above mentioned particular case of Stanley's conjecture on Gorenstein* simplicial posets.

2. Preliminaries

We start with recalling some basic theory of G-spaces, referring to [1, Ch. II] for the proofs of the corresponding statements. Let X be a topological space with a left action of a compact topological group G. The action is *effective* if unit is the only element of G that acts trivially, and is *free* if the *isotropy subgroup* $G_x = \{g \in G\colon gx = x\}$ is trivial for all $x \in X$. The fixed point set is denoted X^G. There exists a contractible free right G-space EG called the *universal G-space*; the quotient $BG := EG/G$ is called the *classifying space* for free G-actions. The product $EG \times X$ is a free left G-space by $g \cdot (e, x) = (eg^{-1}, gx)$; the quotient $EG \times_G X := (EG \times X)/G$ is called the *Borel construction* on X or the *homotopy quotient* of X. The *equivariant cohomology* with coefficients in a ring $\mathbf{k}$ is defined as

$$H^*_G(X; \mathbf{k}) := H^*(EG \times_G X; \mathbf{k}).$$

The map ρ collapsing X to a point induces a homomorphism

$$\rho^*\colon H^*_G(pt;\mathbf{k}) = H^*(BG;\mathbf{k}) \to H^*_G(X;\mathbf{k}) \tag{2.1}$$

thereby defining a canonical $H^*(BG;\mathbf{k})$-module structure on $H^*_G(X;\mathbf{k})$. The Borel construction can also be applied to a G-vector bundle. For instance, if E is an oriented G-vector bundle over a G-space X, then the Borel construction on E produces an oriented vector bundle over $EG \times_G X$ and its Euler class is called the *equivariant Euler class* of E and denoted by $e^G(E)$. Note that $e^G(E)$ lies in $H^*_G(X;\mathbb{Z})$. Below we use integer coefficients, unless another coefficient ring is specified.

If G is a commutative group (e.g., a compact torus $T = T^k$), then the notions of left and right G-spaces coincide. As is well known, $H^*(BT)$ is a polynomial ring in k variables of degree two, in particular $H^{odd}(BT) = 0$. All manifolds M in this paper are closed connected smooth and orientable.

Lemma 2.1. *Let M be a manifold with a smooth action of T such that the fixed point set M^T is finite and non-empty. Then $H^*_T(M)$ is free as an $H^*(BT)$-module if and only if $H^{odd}(M) = 0$. In this case $H^*_T(M) \cong H^*(BT) \otimes H^*(M)$ as $H^*(BT)$-modules.*

Proof. Assume $H^{odd}(M) = 0$. Then the Serre spectral sequence of the fibration $ET \times_T M \to BT$ collapses and $H^*(M)$ has no torsion, so $H^*_T(M)$ is isomorphic to $H^*(BT) \otimes H^*(M)$ and thus is a free $H^*(BT)$-module. This proves the "if" part.

To prove the "only if" part, we use the Eilenberg-Moore spectral sequence of the bundle $ET \times_T M \to BT$ with fibre M. It converges to $H^*(M)$ and has

$$E_2^{*,*} = \operatorname{Tor}^{*,*}_{H^*(BT)}\big(H^*_T(M), \mathbb{Z}\big).$$

Since $H^*_T(M)$ is free as an $H^*(BT)$-module, we have

$$\begin{aligned}\operatorname{Tor}^{*,*}_{H^*(BT)}\big(H^*_T(M), \mathbb{Z}\big) &= \operatorname{Tor}^{0,*}_{H^*(BT)}\big(H^*_T(M), \mathbb{Z}\big)\\ &= H^*_T(M) \otimes_{H^*(BT)} \mathbb{Z}\\ &= H^*_T(M)\big/\big(\rho^*\big(H^{>0}(BT)\big)\big).\end{aligned}$$

Therefore, $E_2^{0,*} = H^*_T(M)/(\rho^*(H^{>0}(BT)))$ and $E_2^{-p,*} = 0$ for $p > 0$. It follows that the Eilenberg-Moore spectral sequence collapses at the E_2 term and

$$H^*(M) = H^*_T(M)\big/\big(\rho^*\big(H^{>0}(BT)\big)\big). \tag{2.2}$$

On the other hand, it follows from the localisation theorem (see [12]) that the kernel of the restriction map

$$H^*_T(M) \to H^*_T(M^T) = H^*(BT) \otimes H^*(M^T)$$

is the $H^*(BT)$-torsion subgroup and hence the restriction map is injective in our case. Therefore $H_T^{odd}(M) = 0$ because M^T is a finite set of isolated points. This fact together with (2.2) proves that $H^{odd}(M) = 0$. □

Two classes of T-manifolds, namely those having zero odd degree cohomology or even cohomology generated in degree two, are of particular importance in this paper. Next we prove two technical lemmas showing that these cohomological properties are inherited by the fixed point set M^H for any subtorus $H \subseteq T$. These lemmas will be used in inductive arguments later in the paper.

Lemma 2.2. *Let M be a T-manifold, H a subtorus of T and N a connected component of M^H. If $H^{odd}(M) = 0$, then $H^{odd}(N) = 0$ and $N^T \neq \varnothing$.*

Proof. We first prove that $H^{odd}(M^H) = 0$. Note that for a generic circle subgroup $S \subseteq H$ we have $M^S = M^H$. Let p be a prime and G be an order p subgroup in S. The induced action of G on $H^*(M)$ is trivial because G is contained in the connected group S. Then $\dim H^{odd}(M^G; \mathbb{Z}/p) \leqslant \dim H^{odd}(M; \mathbb{Z}/p)$ by [1, Theorem VII.2.2]. Therefore, $H^{odd}(M^G; \mathbb{Z}/p) = 0$ by the assumption. Repeating the same argument for M^G with the induced action of S/G, which is again a circle group, we conclude that $H^{odd}(M^G; \mathbb{Z}/p) = 0$ for any p-subgroup G of S. However, $M^G = M^S = M^H$ if the order of G is sufficiently large, so we have $H^{odd}(M^H; \mathbb{Z}/p) = 0$. Since p is an arbitrary prime, this implies that $H^{odd}(M^H) = 0$.

Now since $H^{odd}(N) = 0$, the Euler characteristic $\chi(N)$ of N is non-zero. As is well-known $\chi(N) = \chi(N^T)$, which implies that N^T is non-empty. □

Lemma 2.3. *Let M, H, N be as in* Lemma 2.2. *If $H^*(M)$ is generated by its degree-two part* (*as a ring*), *then the restriction map $H^*(M) \to H^*(N)$ is surjective*; *in particular*, *$H^*(N)$ is also generated by its degree-two part*.

Proof. Since $H^{odd}(M) = 0$, we have $H^{odd}(N) = 0$ by Lemma 2.2; so it suffices to prove that the restriction map $H^*(M; \mathbb{Z}/p) \to H^*(N; \mathbb{Z}/p)$ is surjective for any prime p.

The argument below is similar to that used in the proof of Theorem VII.3.1 in [1]. As in the proof of Lemma 2.2, let S be a generic circle subgroup of H (so that $M^S = M^H$) and let G be the subgroup of S of prime order p. Then the restriction map $H_G^k(M; \mathbb{Z}/p) \to H_G^k(M^G; \mathbb{Z}/p)$ is an isomorphism for sufficiently large k by [1, Theorem VII.1.5]. Hence, for any connected component N' of M^G the restriction $r\colon H_G^k(M; \mathbb{Z}/p) \to H_G^k(N'; \mathbb{Z}/p)$ is surjective if k is sufficiently large. Now consider the commutative diagram

$$\begin{tikzcd} H_G^*(M; \mathbb{Z}/p) \arrow[r, "r"] \arrow[d] & H_G^*(N'; \mathbb{Z}/p) \cong H^*(BG; \mathbb{Z}/p) \otimes H^*(N'; \mathbb{Z}/p) \arrow[d] \\ H^*(M; \mathbb{Z}/p) \arrow[r, "s"] & H^*(N'; \mathbb{Z}/p) \end{tikzcd} .$$

Choose a basis $v_1, \dots, v_d \in H^2(M;\mathbb{Z}/p)$; then these elements are multiplicative generators for $H^*(M;\mathbb{Z}/p)$. Since $H^{odd}(M;\mathbb{Z}/p) = H^{odd}(M^G;\mathbb{Z}/p) = 0$ and $\chi(M) = \chi(M^T) = \chi(M^G)$, we have $\sum \dim H^i(M;\mathbb{Z}/p) = \sum \dim H^i(M^G;\mathbb{Z}/p)$. By [1, Theorem VII.1.6] the Serre spectral sequence of the fibration $EG \times_G M \to BG$ collapses. Therefore, the vertical map $H_G^*(M;\mathbb{Z}/p) \to H^*(M;\mathbb{Z}/p)$ in the above diagram is surjective. Let $\xi_j \in H_G^*(M;\mathbb{Z}/p)$ be a lift of v_j, and $w_j := s(v_j)$. Let t be a generator of $H^2(BG;\mathbb{Z}/p) \cong \mathbb{Z}/p$. Since the above diagram is commutative and $H^1(N';\mathbb{Z}/p) = 0$ by Lemma 2.2, we have $r(\xi_j) = \alpha_j t + w_j$ for some $\alpha_j \in \mathbb{Z}/p$. Now let $a \in H^*(N';\mathbb{Z}/p)$ be an arbitrary element. Then there exist l and a polynomial $P(\xi_1, \dots, \xi_d)$ such that

$$r(P(\xi_1, \dots, \xi_d)) = t^l a.$$

On the other hand,

$$r(P(\xi_1, \dots, \xi_d)) = P(\alpha_1 t + w_1, \dots, \alpha_d t + w_d) = \sum_{k \geqslant 0} t^k Q_k(w_1, \dots, w_d)$$

for some polynomials Q_k. Therefore, $a = Q_l(w_1, \dots, w_d)$, the restriction map $H^*(M;\mathbb{Z}/p) \to H^*(N';\mathbb{Z}/p)$ is surjective, and $H^*(N';\mathbb{Z}/p)$ is generated by the degree-two elements $w_1, \dots, w_d$.

Now we can repeat the same argument for N' with the induced action of S/G, which is again a circle group. It follows that the restriction map $H^*(M;\mathbb{Z}/p) \to H^*(N';\mathbb{Z}/p)$ is surjective for any connected component N' of M^G with G any p-subgroup of S. However, if the order of G is sufficiently large, then $M^G = M^S = M^H$ and hence $N' = N$, so it follows that the restriction map $H^*(M;\mathbb{Z}/p) \to H^*(N;\mathbb{Z}/p)$ is surjective for any connected component N of M^H. Since the prime p is arbitrary, the proof is finished. □

3. Torus manifolds

The notion of torus manifold was introduced in [11] and [13], and here we follow the notation of these papers with some additional specifications.

A *torus manifold* is a $2n$-dimensional closed connected orientable smooth manifold M with an effective smooth action of an n-dimensional torus $T = (S^1)^n$ such that $M^T \neq \emptyset$. Since $\dim M = 2 \dim T$ and M is compact, the fixed point set M^T is a finite set of isolated points.

A codimension-two connected component of the set fixed pointwise by a circle subgroup of T is called a *characteristic submanifold* of M. The existence of a T-fixed point is required for the definition of characteristic submanifold in [11] and [13] but not in this paper. However, when $H^{odd}(M) = 0$, these two definitions agree by Lemma 2.2.

Since M is compact, there are only finitely many characteristic submanifolds, and we denote them by M_i, $i = 1, \dots, m$. Each characteristic submanifold M_i is orientable as a connected component of the fixed point set for a circle action on an orientable manifold. Following [4], we say that M is *omnioriented* if an orientation is specified for M and for every characteristic submanifold M_i. There are 2^{m+1} choices of omniorientations. It is extremely convenient, although not absolutely necessary to assume that all torus manifolds are omnioriented (in [11] a choice of omniorientation for characteristic submanifolds was a part of the definition of torus manifold).

Here are two typical examples of torus manifolds.

EXAMPLE 3.1. A complex projective space $\mathbb{C}P^n$ has a natural T-action defined in the homogeneous coordinates by

$$(t_1, \dots, t_n) \cdot (z_0 : z_1 : \cdots : z_n) = (z_0 : t_1 z_1 : \cdots : t_n z_n).$$

It has $(n+1)$ characteristic submanifolds $\{z_0 = 0\}, \dots, \{z_n = 0\}$ and $(n+1)$ fixed points $(1 : 0 : \cdots : 0), \dots, (0 : \cdots : 0 : 1)$. In this example the intersection of any set of characteristic submanifolds is connected.

EXAMPLE 3.2. Let S^{2n} be the $2n$-sphere identified with the following subset in $\mathbb{C}^n \times \mathbb{R}$:

$$\left\{(z_1, \dots, z_n, y) \in \mathbb{C}^n \times \mathbb{R} \colon |z_1|^2 + \cdots + |z_n|^2 + y^2 = 1\right\}.$$

Define a T-action by

$$(t_1, \dots, t_n) \cdot (z_1, \dots, z_n, y) = (t_1 z_1, \dots, t_n z_n, y).$$

It has n characteristic submanifolds $\{z_1 = 0\}, \dots, \{z_n = 0\}$, and two fixed points $(0, \dots, 0, \pm 1)$. The intersection of any k characteristic submanifolds is connected if $k \leqslant n-1$, but consists of two disjoint fixed points if $k = n$.

If M is an (omnioriented) torus manifold, then both M and M_i are oriented, and the Gysin homomorphism $H_T^*(M_i) \to H_T^{*+2}(M)$ in equivariant cohomology is defined. Denote by $\tau_i \in H_T^2(M)$ the image of the identity element in $H_T^0(M_i)$. We may think of τ_i as the Poincaré dual of M_i in equivariant cohomology.

Proposition 3.3 (See section 1 of [13]). *Let M be a torus manifold.*
1. *For each characteristic submanifold M_i with $(M_i)^T \neq \varnothing$, there is a unique element $a_i \in H_2(BT)$ such that*

$$\rho^*(t) = \sum_i \langle t, a_i \rangle \tau_i \quad \textit{modulo} \quad H^*(BT)\textit{-torsions}$$

for any element $t \in H^2(BT)$. Here the sum is taken over all characteristic submanifolds M_i with $(M_i)^T \neq \varnothing$ and ρ^* denotes the homomorphism (2.1).

2. *The circle subgroup fixing M_i with $(M_i)^T \neq \varnothing$ coincides with the one determined by $a_i \in H_2(BT)$ through the identification $H_2(BT) = \mathrm{Hom}(S^1, T)$.*
3. *If n different characteristic submanifolds $M_{i_1}, \dots, M_{i_n}$ have a T-fixed point in their intersection, then the elements $a_{i_1}, \dots, a_{i_n}$ form a basis of $H_2(BT)$ over $\mathbb{Z}$.*

The next lemma provides a sufficient cohomological condition for the intersections of characteristic submanifolds to be connected (compare Examples 3.1 and 3.2).

Lemma 3.4. *Suppose that $H^*(M)$ is generated in degree two. Then all non-empty multiple intersections of the characteristic submanifolds are connected and have cohomology generated in degree two.*

Proof. Since every characteristic submanifold M_i is a connected component of the fixed point set of a circle subgroup of T, the cohomology $H^*(M_i)$ is generated by the degree-two part and the restriction map $H^*(M) \to H^*(M_i)$ is onto by Lemma 2.3. It follows that the restriction map $H_T^*(M) \to H_T^*(M_i)$ in equivariant cohomology is also onto.

Now we prove the connectedness of multiple intersections. Suppose that $M_{i_1} \cap \cdots \cap M_{i_k} \neq \varnothing$, $(1 < k \leqslant n)$, and pick a connected component N of the intersection. Since N is fixed by a subtorus, it contains a T-fixed point by Lemma 2.2. For each $i \in \{i_1, \dots, i_k\}$ there are embeddings $\varphi_i \colon N \to M_i$, $\psi_i \colon M_i \to M$, and the corresponding Gysin homomorphisms in equivariant cohomology:

$$H_T^0(N) \xrightarrow{\varphi_{i!}} H_T^{2k-2}(M_i) \xrightarrow{\psi_{i!}} H_T^{2k}(M).$$

Since the restriction $\psi_i^* \colon H_T^*(M) \to H_T^*(M_i)$ is surjective, we have $\varphi_{i!}(1) = \psi_i^*(u)$ for some $u \in H_T^{2k-2}(M)$. Now we calculate

$$(\psi_i \circ \varphi_i)_!(1) = \psi_{i!}(\varphi_{i!}(1)) = \psi_{i!}\big(\psi_i^*(u)\big) = \psi_{i!}(1)u = \tau_i u.$$

Hence, $(\psi_i \circ \varphi_i)_!(1)$ is divisible by τ_i for every $i \in \{i_1, \dots, i_k\}$. By Proposition 3.4 of [13], the degree-$2k$ part of $H_T^*(M)$ is additively generated by the monomials $\tau_{j_1}^{k_1} \dots \tau_{j_p}^{k_p}$ such that $M_{j_1} \cap \cdots \cap M_{j_p} \neq \varnothing$ and $k_1 + \cdots + k_p = k$. It follows that $(\psi_i \circ \varphi_i)_!(1)$ is a non-zero integral multiple of $\tau_{i_1} \cdots \tau_{i_k} \in H_T^{2k}(M)$. By the definition of Gysin map, $(\psi_i \circ \varphi_i)_!(1)$ goes to zero under the restriction map $H_T^*(M) \to H_T^*(x)$ for every point $x \in (M \backslash N)^T$. On the other hand, the image of $\tau_{i_1} \cdots \tau_{i_k}$ under the restriction map $H_T^*(M) \to H_T^*(x)$ is non-zero for every T-fixed point $x \in M_{i_1} \cap \cdots \cap M_{i_k}$. Thus, N is the only connected component of the latter intersection. The fact that $H^*(N)$ is generated by its degree-two part follows from Lemma 2.3. $\square$

4. Locally standard torus manifolds and orbit spaces

4.1. Locally standardness. We say that a torus manifold M is *locally standard* if every point in M has an invariant neighbourhood U weakly equivariantly diffeomorphic to an open subset $W \subset \mathbb{C}^n$ invariant under the standard T^n-action on $\mathbb{C}^n$. The latter means that there is an automorphism $\psi\colon T \to T$ and a diffeomorphism $f\colon U \to W$ such that $f(ty) = \psi(t)f(y)$ for all $t \in T$, $y \in U$.

The following statement gives a sufficient cohomological condition for local standardness.

Theorem 4.1. *A torus manifold M with $H^{odd}(M) = 0$ is locally standard.*

Proof. We first show that there are no non-trivial finite isotropy subgroups for the T-action on M. Assume the opposite, i.e., the isotropy group T_x is finite and non-trivial for some $x \in M$. Then T_x contains a non-trivial cyclic subgroup G of some prime order p. Let N be the connected component of M^G containing x. Since N contains x and T_x is finite, the principal isotropy group of N is finite. Like in the proof of Lemma 2.2, it follows from [1, Theorem VII.2.2] that $H^{odd}(N;\mathbb{Z}/p) = 0$. In particular, the Euler characteristic of N is non-zero, and therefore, N has a T-fixed point, say y. The tangential T-representation $\mathcal{T}_yM$ at y is faithful, $\dim M = 2\dim T$ and $\mathcal{T}_yN$ is a proper T-subrepresentation of $\mathcal{T}_yM$. It follows that there is a subtorus T' (of positive dimension) which fixes $\mathcal{T}_yN$ and does not fix the complement of $\mathcal{T}_yN$ in $\mathcal{T}_yM$. Clearly, T' is the principal isotropy group of N, which contradicts the above observation that the principal isotropy group of N is finite.

If the isotropy group T_x is trivial, M is obviously locally standard near x. Suppose that T_x is non-trivial. Then it cannot be finite and therefore, $\dim T_x > 0$. Let H be the identity component of T_x, and N the connected component of M^H containing x. By Lemma 2.2, N has a T-fixed point, say y. Looking at the tangential representation at y, we observe that the induced action of T/H on N is effective. By the previous argument, no point of N has a non-trivial finite isotropy group for the induced action of T/H, which implies that $T_x = H$. Since x and y are both in the same connected component N fixed pointwise by T_x, the T_x-representation in $\mathcal{T}_xM$ agrees with the restriction of the tangential T-representation $\mathcal{T}_yM$ to T_x. This implies that M is locally standard near x. $\square$

In the rest of this section we assume that M is locally standard.

Let $Q := M/T$ denote the orbit space of M and $\pi\colon M \to Q$ the quotient projection. Since M is locally standard, any point in the orbit space Q has a neighbourhood diffeomorphic to an open subset in the positive cone

$$\mathbb{R}^n_{\geqslant} = \{(y_1, \dots, y_n) \in \mathbb{R}^n\colon y_i \geqslant 0,\ \ i = 1, \dots, n\}.$$

This identifies Q as a *manifold with corners*, see e.g. [5, §6], and faces of Q can be defined in a natural way. The vertices of Q, that is, the 0-dimensional faces, correspond to the T-fixed points of M through the quotient projection π. Codimension one faces of Q are called the *facets* of Q. They are the π images of characteristic submanifolds M_i, $i = 1, \dots, m$. We set $Q_i := \pi(M_i)$. We refer to a non-empty intersection of k facets as a codimension-k *preface*, $k = 1, \dots, n$. In general, prefaces of codimension > 1 may fail to be connected (see Example 3.2). Faces are connected components of prefaces. We also regard Q itself as a codimension-zero face; other faces are called *proper faces*. If $H^{odd}(M) = 0$, then every face has a vertex by Lemma 2.2. Moreover, if $H^*(M)$ is generated in degree two, then all prefaces are connected by Lemma 3.4; so prefaces are faces in this case.

A space X is *acyclic* if $\widetilde{H}_i(X) = 0$ for all i. We say that Q is *face-acyclic* if all of its faces (including Q itself) are acyclic. It is not difficult to see that if Q is face-acyclic, then every face of Q has a vertex. We call Q a *homology polytope* if all its prefaces are acyclic (in particular, connected), in other words, Q is a homology polytope if and only if it is face-acyclic and all non-empty multiple intersections of characteristic submanifolds are connected.

REMARK. A *simple convex polytope* is an example of a manifold with corners and is a homology polytope. A *quasitoric manifold* [6], [3] can be defined as a locally standard torus manifold whose orbit space is a simple convex polytope with the standard face structure.

EXAMPLE 4.2. Torus manifold $\mathbb{C}P^n$ with the T-action from Example 3.1 is locally standard and the map

$$(z_0 : z_1 : \cdots : z_n) \to \frac{1}{\sum_{i=0}^n |z_i|^2}\left(|z_1|^2, \dots, |z_n|^2\right)$$

induces a face preserving homeomorphism from the orbit space $\mathbb{C}P^n/T$ to a standard n-simplex. The latter is a simple polytope, in particular, a homology polytope.

EXAMPLE 4.3. Torus manifold S^{2n} with the T-action from Example 3.2 is also locally standard and the map

$$(z_1, \dots, z_n, y) \to (|z_1|, \dots, |z_n|, y)$$

induces a face preserving homeomorphism from the orbit space S^{2n}/T to the space

$$\left\{(x_1, \dots, x_n, y) \in \mathbb{R}^{n+1} \colon x_1^2 + \cdots + x_n^2 + y^2 = 1,\ \ x_1 \geqslant 0, \dots, x_n \geqslant 0\right\}.$$

This space is not a homology polytope, but is a face-acyclic manifold with corners.

4.2. Canonical model. In this paragraph we reconstruct the torus manifold M from the orbit space Q and a map Λ defined below using a "canonical model" $M_Q(\Lambda)$, which generalises a result of Davis-Januszkiewicz [6, Prop. 1.8].

Remember that $M_i = \pi^{-1}(Q_i)$ is fixed by a circle subgroup of T. We choose a map

$$\Lambda\colon \{Q_1, \dots, Q_m\} \to H_2(BT) = \mathrm{Hom}(S^1, T) \cong \mathbb{Z}^n \tag{4.1}$$

such that $\Lambda(Q_i)$ is primitive and determines the circle subgroup of T fixing M_i. When M_i has a T-fixed point, $\Lambda(Q_i)$ coincides with the element a_i introduced in Proposition 3.3 up to sign. The following lemma follows immediately from the local standardness of M.

Lemma 4.4. *If $Q_{i_1} \cap \dots \cap Q_{i_k}$ is non-empty, then $\Lambda(Q_{i_1}), \dots, \Lambda(Q_{i_k})$ is a part of basis for the integral lattice* $\mathrm{Hom}(S^1, T) \cong \mathbb{Z}^n$.

Given a point $x \in Q$, the smallest face which contains x is an intersection $Q_{i_1} \cap \dots \cap Q_{i_k}$ of some facets, and we define $T(x)$ to be the subtorus of T generated by the circle subgroups corresponding to $\Lambda(Q_{i_1}), \dots, \Lambda(Q_{i_k})$. Now introduce the identification space

$$M_Q(\Lambda) := T \times Q/\sim, \tag{4.2}$$

where $(t, x) \sim (t', x')$ if and only if $x = x'$ and $t^{-1}t' \in T(x)$. The space $M_Q(\Lambda)$ admits a natural action of T and is a closed manifold (this follows from Lemma 4.4 and the fact that Q is a manifold with corners). The following is a straightforward generalisation of a [6, Prop. 1.8].

Lemma 4.5. *Let M be a locally standard torus manifold with orbit space Q, and the map Λ defined by* (4.1). *If $H^2(Q) = 0$, then there is an equivariant homeomorphism*

$$M_Q(\Lambda) \to M$$

covering the identity on Q.

Proof. The idea is to construct a continuous map $f\colon T \times Q \to M$ taking $T \times q$ onto $\pi^{-1}(q)$ for each point $q \in Q$. This is done by subsequent "blowing up the singular strata." The condition on the second cohomology group guarantees that the resulting principal T-bundle over Q is trivial. Then the map f descends to the required equivariant homeomorphism. See [6] for details. □

REMARK. Like in the case of quasitoric manifolds, it follows that a torus manifold whose orbit quotient Q satisfies $H^2(Q) = 0$ is determined by Q and Λ.

5. Face rings of manifolds with corners and simplicial posets

Before we proceed with describing the ordinary and equivariant cohomology rings of torus manifolds we need an algebraic digression. Here we review a notion of face ring generalising the classical Stanley-Reisner face ring [18] to combinatorial structures more general than simplicial complexes. We consider two cases, which are in a sense dual to each other: "nice" manifolds with corners and simplicial posets. The latter one is more general, however the former one is more convenient for applications to torus manifolds. The face ring of a manifold with corners is also a little easier to visualise, so we start with considering this case.

The relationship between nice manifolds with corners and simplicial posets is similar to that between simple polytopes and simplicial complexes. Face rings of simplicial posets were introduced and studied in [17]. Most of the statements in this section follow from the general theory of ASL's (*algebras with straightening law*) and *Hodge algebras* as explained in [17] and [2, Ch. 7], however our treatment is independent and geometrical.

5.1. Nice manifolds with corners. To begin, we assume that Q is a homology polytope (or even a simple convex polytope) with m facets $Q_1, \dots, Q_m$. Let $\mathbf{k}$ be a ground commutative ring with unit, and assign a degree-two polynomial generator v_{Q_i} to each facet Q_i. We refer to the quotient ring

$$\mathbf{k}[Q] = \mathbf{k}\big[v_{Q_1}, \dots, v_{Q_m}\big]\big/\big(v_{Q_{i_1}} \cdots v_{Q_{i_k}} = 0 \quad \text{if} \quad Q_{i_1} \cap \dots \cap Q_{i_k} = \varnothing\big).$$

as the *face ring* of Q. In coincides with the Stanley-Reisner face ring [18] of the nerve simplicial complex K.

For arbitrary pair of faces G, H of Q the intersection $G \cap H$ is a unique maximal face contained in both G and H. There is also a unique minimal face that contains both G and H, which we denote $G \vee H$. Let $\mathbf{k}[v_F : F \text{ a face}]$ be the graded polynomial ring with one $2k$-dimensional generator v_F for every proper codimension-k face F. We also identify v_Q with the unit and $v_\varnothing$ with zero. The following proposition gives another presentation of $\mathbf{k}[Q]$, by extending both the set of generators and relations. It will be used for a subsequent generalisation of $\mathbf{k}[Q]$ to arbitrary manifolds with corners.

Proposition 5.1. *There is a canonical isomorphism of rings*

$$\mathbf{k}[v_F : F \textit{ a face}]/\mathcal{I}_Q \cong \mathbf{k}[Q],$$

where $\mathcal{I}_Q$ is the ideal generated by all elements

$$v_G v_H - v_{G\vee H} v_{G\cap H}.$$

Proof. The identification is established by the map sending v_F to $\prod_{Q_i \supseteq F} v_{Q_i}$. □

Now let Q be an arbitrary connected manifold with corners. We also assume that Q is *nice*, that is, every codimension-k face is contained in exactly k facets. Note that the orbit space of a locally standard torus manifold is always nice. In a nice manifold with corners, all faces containing a given face form a Boolean lattice (like in the case of $\mathbb{R}^n_{\geqslant}$).

REMARK. By the definition of manifold with corners, every codimension-k face is contained in at most k facets. A 2-disc with one 0-face and one 1-face on the boundary gives an example of manifold with corners which is not nice.

The intersection of two faces G and H in a manifold with corners may be disconnected, but every its connected component is a face of codimension $\operatorname{codim} G + \operatorname{codim} H$. We regard $G \cap H$ as the set of its connected components; so the notation $E \in G \cap H$ is used below for connected components E of the intersection.

Proposition 5.2. *For every two faces G and H with non-empty intersection, there is a unique minimal face $G \vee H$ that contains both G and H.*

Proof. Take any $E \in G \cap H$. The statement follows from the fact that the poset of faces containing E is a Boolean lattice. □

Now we use the interpretation from Proposition 5.1 to introduce a more general version of $\mathbf{k}[Q]$.

DEFINITION 5.3. The *face ring* $\mathbf{k}[Q]$ of a nice manifold with corners Q is a graded ring defined by

$$\mathbf{k}[Q] := \mathbf{k}[v_F : F \text{ a face}]/\mathcal{I}_Q,$$

where $\deg v_F = 2 \operatorname{codim} F$ and $\mathcal{I}_Q$ is the ideal generated by all elements

$$v_G v_H - v_{G \vee H} \cdot \sum_{E \in G \cap H} v_E.$$

If G and H are transversal, that is, $\operatorname{codim} G \cap H = \operatorname{codim} G + \operatorname{codim} H$, then $G \vee H = Q$, so in $\mathbf{k}[Q]$ we get the identity

$$v_G v_H = \sum_{E \in G \cap H} v_E.$$

Below we give a sequence of statements describing algebraic properties of $\mathbf{k}[Q]$ and emphasising its analogy with the classical Stanley-Reisner face ring.

Lemma 5.4. *Every element $a \in \mathbf{k}[Q]$ can be written as a linear combination*

$$a = \sum_{\substack{G_1 \supset \cdots \supset G_q \\ \alpha_1, \ldots, \alpha_q}} A(G_1 \supset \cdots \supset G_q; \alpha_1, \ldots, \alpha_q) v_{G_1}^{\alpha_1} \cdots v_{G_q}^{\alpha_q}$$

with coefficients $A(G_1 \supset \cdots \supset G_q; \alpha_1, \ldots, \alpha_q) \in \mathbf{k}$. Here $\operatorname{codim} G_i = i$ and G_q is an inclusion minimal face, and the sum is taken over all chains of faces $G_1 \supset \cdots \supset G_q$ with all non-negative integers α_i.

Proof. We may assume that $a = v_{H_1} v_{H_2} \cdots v_{H_k}$ (some H_i may coincide), and it is enough to show that it can be written as $\sum v_{G_1} \cdots v_{G_l}$ with $G_1 \supseteq \cdots \supseteq G_l$ for every summand (without making any assumptions on codimensions, but allowing some G_i to coincide). By induction we may assume that $H_2 \supseteq \cdots \supseteq H_k$. Now we apply the relation from Definition 5.3 and replace a by

$$v_{H_1 \vee H_2} \left(\sum_{E \in H_1 \cap H_2} v_E \right) v_{H_3} \cdots v_{H_k}.$$

The first two faces in every summand above are ordered. Then we replace each $v_E v_{H_3}$ by $v_{E \vee H_3} \left(\sum_{G \in E \cap H_3} v_G \right)$. Since $H_1 \vee H_2 \supseteq E \vee H_3$, we get the first three faces in a linear order. Proceeding in this fashion we finally end up in a sum of monomials corresponding to ordered sets of faces. □

We refer to the presentation from Lemma 5.4 as the *chain decomposition* of an element $a \in \mathbf{k}[Q]$.

For any vertex (0-face) $p \in Q$ we define the *restriction map* s_p by

$$s_p \colon \mathbf{k}[Q] \to \mathbf{k}[Q]/(v_F : F \not\ni p).$$

The next observation is straightforward.

Proposition 5.5. *The image $s_p(\mathbf{k}[Q])$ of the restriction map can be identified with the polynomial ring $\mathbf{k}\left[v_{Q_{i_1}}, \ldots, v_{Q_{i_n}}\right]$ on n degree-two generators, where $Q_{i_1}, \ldots, Q_{i_n}$ are the n different facets containing p.*

Lemma 5.6. *If every face of Q has a vertex, then the sum $s = \bigoplus_p s_p$ of restriction maps over all vertices $p \in Q$ is a monomorphism from $\mathbf{k}[Q]$ to the sum of polynomial rings.*

Proof. Take a non-zero $a \in \mathbf{k}[Q]$ and write it as in Lemma 5.4. Fix a monomial $v_{G_1}^{\alpha_1} \cdots v_{G_n}^{\alpha_n}$ entering the chain decomposition with a non-zero coefficient, and consider the restriction s_p to the vertex $p = G_n$. We claim that $s_p(a) \neq 0$. Identify $s_p(\mathbf{k}[Q])$ with the polynomial ring $\mathbf{k}[t_1, \dots, t_n]$ (so that $t_j := v_{Q_{i_j}}$ in the notation of Proposition 5.5). Then $s_p(v_{G_n}) = t_1 \cdots t_n$ and we may also assume that $s_p(v_{G_j}) = t_1 \cdots t_j$, $j = 1, \dots, n$. Hence,

$$s_p\big(v_{G_1}^{\alpha_1} \cdots v_{G_n}^{\alpha_n}\big) = t_1^{\alpha_1}(t_1t_2)^{\alpha_2} \cdots (t_1 \cdots t_n)^{\alpha_n}.$$

It follows that $s_p(a) \neq 0$ unless some other monomial $v_{H_1}^{\beta_1} \cdots v_{H_n}^{\beta_n}$ hits the same monomial in $\mathbf{k}[t_1, \dots, t_n]$. Note that

$$s_p\big(v_{H_1}^{\beta_1} \cdots v_{H_n}^{\beta_n}\big) = 0 \quad \text{unless} \quad H_k \supseteq G_n \quad \text{for} \quad \beta_k \neq 0.$$

Suppose

$$s_p\big(v_{G_1}^{\alpha_1} \cdots v_{G_n}^{\alpha_n}\big) = s_p\big(v_{H_1}^{\beta_1} \cdots v_{H_n}^{\beta_n}\big). \tag{5.1}$$

We want to prove that $v_{G_1}^{\alpha_1} \cdots v_{G_n}^{\alpha_n} = v_{H_1}^{\beta_1} \cdots v_{H_n}^{\beta_n}$, that is, $\alpha_i = \beta_i$ and $G_i = H_i$ if $\alpha_i \neq 0$, $i = 1, \dots, n$. By induction, we may prove this for $i = j$ assuming that it is true for $i > j$. Then (5.1) turns to the identity

$$\begin{aligned} &s_p\big(v_{G_1}^{\alpha_1} \cdots v_{G_j}^{\alpha_j}\big)(t_1 \cdots t_{j+1})^{\alpha_{j+1}} \cdots (t_1 \cdots t_n)^{\alpha_n} \\ &= s_p\big(v_{H_1}^{\beta_1} \cdots v_{H_j}^{\beta_j}\big)(t_1 \cdots t_{j+1})^{\alpha_{j+1}} \cdots (t_1 \cdots t_n)^{\alpha_n}, \end{aligned}$$

whence $s_p\big(v_{G_1}^{\alpha_1} \cdots v_{G_j}^{\alpha_j}\big) = s_p\big(v_{H_1}^{\beta_1} \cdots v_{H_j}^{\beta_j}\big)$. Suppose that β_l is the last non-zero exponent (so that $\beta_{l+1} = \cdots = \beta_j = 0$). Then we also have $\alpha_{l+1} = \cdots = \alpha_j = 0$, since otherwise $s_p\big(v_{G_1}^{\alpha_1} \cdots v_{G_j}^{\alpha_j}\big)$ would be divisible by $t_1 \cdots t_{l+1}$, while $s_p\big(v_{H_1}^{\beta_1} \cdots v_{H_j}^{\beta_j}\big)$ is not. We also have $\alpha_l = \beta_l$ and $G_l = H_l$ since α_l is the maximal power of $t_1 \dots t_l$ that divides $s_p\big(v_{G_1}^{\alpha_1} \cdots v_{G_j}^{\alpha_j}\big)$. By induction, we conclude that $v_{G_1}^{\alpha_1} \cdots v_{G_n}^{\alpha_n} = v_{H_1}^{\beta_1} \cdots v_{H_n}^{\beta_n}$, whence $s_p(a) \neq 0$. $\square$

REMARK. The same argument as in the proof of Lemma 5.6 shows that for arbitrary Q the sum $s = \bigoplus_G s_G$ of (obviously defined) restriction maps s_G over all minimal faces $G \subset Q$ is a monomorphism.

Corollary 5.7. *The chain decomposition of $a \in \mathbf{k}[Q]$ is unique, and the monomials $v_{G_1}^{\alpha_1} \cdots v_{G_q}^{\alpha_q}$ corresponding to all chains $G_1 \supset \cdots \supset G_q$ and all exponents α_i form an additive basis of $\mathbf{k}[Q]$.*

The *f-vector* of Q is defined as $\boldsymbol{f}(Q) = (f_0, \dots, f_{n-1})$ where f_i is the number of faces of codimension $i + 1$ (so that $f_0 = m$ is the number of facets). The equivalent information is contained in the *h-vector* $\boldsymbol{h}(Q) = (h_0, \dots, h_n)$ determined by the equation

$$h_0t^n + \cdots + h_{n-1}t + h_n = (t - 1)^n + f_0(t - 1)^{n-1} + \cdots + f_{n-1}. \tag{5.2}$$

In particular, $h_0 = 1$ and $h_n = (-1)^n + (-1)^{n-1} f_0 + \cdots + f_{n-1}$, which is equal to 1 when Q is face-acyclic.

EXAMPLE 5.8. We turn again to the T^n-action on S^{2n} from Examples 3.2 and 4.3 and set $n = 2$ there. Then Q is a 2-ball with two 0-faces (say, p and q) and two 1-faces (say, G and H). Then $\boldsymbol{f}(Q) = (2, 2)$, $\boldsymbol{h}(Q) = (1, 0, 1)$ and

$$\mathbf{k}[Q] = \mathbf{k}[v_G, v_H, v_p, v_q]/(v_G v_H = v_p + v_q,\ \ v_p v_q = 0),$$

where $\deg v_G = \deg v_H = 2$, $\deg v_p = \deg v_q = 4$.

5.2. Simplicial posets. The faces (simplices) in a (finite) simplicial complex K form a poset (partially ordered set) with respect to the inclusion, and the empty simplex $\varnothing$ is the initial element. This poset is called the *face poset* of K, and it carries the same combinatorial information as the simplicial complex itself. A poset $\mathcal{P}$ is called *simplicial* if it has an initial element $\hat{0}$ and for each $x \in \mathcal{P}$ the lower segment $\left[\hat{0}, x\right]$ is a boolean lattice (the face poset of a simplex). The face poset of a simplicial complex is a simplicial poset, but there are simplicial posets that cannot be obtained in this way. In the sequel we identify a simplicial complex with its face poset, thereby regarding simplicial complexes as particular cases of simplicial posets.

To each $x \in \overline{\mathcal{P}} := \mathcal{P} - \{\hat{0}\}$ we assign a geometrical simplex whose face poset is $\left[\hat{0}, x\right]$, and glue these geometrical simplices together according to the order relation in $\mathcal{P}$. We get a cell complex such that the closure of each cell can be identified with a simplex preserving the face structure and all the attaching maps are inclusions. We call it a *simplicial cell complex* and denote its underlying space by $|\mathcal{P}|$. If $\mathcal{P}$ is (the face poset of) a simplicial complex K, then $|\mathcal{P}|$ agrees with the geometric realisation $|K|$ of K. The barycentric subdivision of a simplicial cell complex is obviously defined, and is again a simplicial cell complex.

Proposition 5.9. *The barycentric subdivision of a simplicial cell complex is a* (*geometric realisation of*) *simplicial complex.*

Proof. Indeed, we may identify the barycentric subdivision under question with the geometric realisation of the order complex $\Delta\left(\overline{\mathcal{P}}\right)$ of the poset $\overline{\mathcal{P}}$. $\square$

In the sequel we will not distinguish between simplicial posets and simplicial cell complexes, and call (the face poset of) the order complex $\Delta\left(\overline{\mathcal{P}}\right)$ the *barycentric subdivision* of $\mathcal{P}$. The set of faces of a nice manifold with corners Q forms a simplicial poset with respect to reversed inclusion (so Q is the initial element). We call it the *face poset* of Q. It is a face poset of a simplicial complex if and only if all non-empty multiple intersections of facets of Q are connected.

EXAMPLE 5.10. Let Q be the orbit space from Example 4.3. There are n facets in Q and the intersection of any k facets is connected when $k \leqslant n-1$, but the intersection of n facets consists of two points. The corresponding simplicial cell complex is obtained by gluing two $(n-1)$-simplices along their boundaries.

Let $\mathcal{P}$ be a simplicial poset. When $\bigl[\hat{0}, x\bigr]$ is the face poset of a $(k-1)$-simplex, the rank of $x \in \overline{\mathcal{P}}$, denoted by $\operatorname{rk} x = k$, is defined to be k. The rank of $\mathcal{P}$ is the maximum of ranks of elements in $\overline{\mathcal{P}}$. Introduce the graded polynomial ring $\mathbf{k}\bigl[v_x : x \in \overline{\mathcal{P}}\bigr]$ with $\deg v_x = 2\operatorname{rk} x$. We also write formally $v_{\hat{0}} = 1$. For any two elements $x, y \in \mathcal{P}$ denote by $x \vee y$ the set of their least common upper bounds, and by $x \wedge y$ the set of their greatest common lower bounds. Since $\mathcal{P}$ is simplicial, $x \wedge y$ consists of a single element provided that $x \vee y$ is non-empty. The following is the obvious dualisation of Definition 5.3.

DEFINITION 5.11. The *face ring* of a simplicial poset $\mathcal{P}$ is the quotient

$$\mathbf{k}[\mathcal{P}] := \mathbf{k}[v_x : x \in \mathcal{P}]/\mathcal{I}_{\mathcal{P}},$$

where $\mathcal{I}_{\mathcal{P}}$ is the ideal generated by the elements

$$v_x v_y - v_{x\wedge y} \cdot \sum_{z \in x \vee y} v_z .$$

REMARK. Let Q be a nice manifold with corners and let $\mathcal{P}$ be the face poset of Q. Then $\mathbf{k}[Q] \cong \mathbf{k}[\mathcal{P}]$. Let K be the nerve simplicial complex of the covering of $\partial Q = \bigcup_{i=1}^{m} Q_i$ by the facets, that is, the simplicial complex on m vertices whose $(k-1)$-dimensional simplices correspond to the codimension-k prefaces of Q. If all non-empty multiple intersections of facets in Q are connected, then the Stanley-Reisner face ring $\mathbf{k}[K]$ agrees with $\mathbf{k}[\mathcal{P}]$, but otherwise $\mathbf{k}[K]$ may differ from $\mathbf{k}[\mathcal{P}]$.

The *f-vector* of a simplicial poset $\mathcal{P}$ of rank n is $\boldsymbol{f}(\mathcal{P}) = (f_0, \dots, f_{n-1})$ where f_i is the number of elements of rank i. The *h-vector* $\boldsymbol{h}(\mathcal{P}) = (h_0, \dots, h_n)$ is determined by (5.2). If $\mathcal{P}$ is the face poset of a nice manifold with corners Q then $\boldsymbol{h}(\mathcal{P}) = \boldsymbol{h}(Q)$.

Since we have defined $\deg v_x = 2\operatorname{rk} x$, the face ring $\mathbf{k}[\mathcal{P}]$ has no odd degree part. Its Hilbert series $F(\mathbf{k}[\mathcal{P}]; t) := \sum_i \dim_{\mathbf{k}} \mathbf{k}[\mathcal{P}]_{2i} t^{2i}$, where $\mathbf{k}[\mathcal{P}]_{2i}$ denotes the homogeneous degree $2i$ part of $\mathbf{k}[\mathcal{P}]$, looks exactly as in the case of simplicial complexes.

Theorem 5.12 (Proposition 3.8 of [17]). *Let $\mathcal{P}$ be a simplicial poset of rank n with h-vector $(h_0, h_1, \dots, h_n)$. Then*

$$F(\mathbf{k}[\mathcal{P}]; t) = \frac{h_0 + h_1 t^2 + \dots + h_n t^{2n}}{(1-t^2)^n}.$$

In [6], Davis and Januszkiewicz realised the classical Stanley-Reisner face ring $\mathbf{k}[K]$ of a simplicial complex K as the equivariant cohomology ring of a T-space. The same approach works for a simplicial poset $\mathcal{P}$ as well. The order complex $\Delta\big(\overline{\mathcal{P}}\big)$ is a simplicial complex. Let P be the cone on the geometric realisation $\big|\Delta\big(\overline{\mathcal{P}}\big)\big|$. Since $\big|\Delta\big(\overline{\mathcal{P}}\big)\big| = |\mathcal{P}|$, the "boundary" of P is $|\mathcal{P}|$. For each simplex $\sigma \in \Delta\big(\overline{\mathcal{P}}\big)$, let $F_\sigma \subset P$ denote the geometric realisation of the poset $\big\{\tau \in \Delta\big(\overline{\mathcal{P}}\big) \colon \sigma \subseteq \tau\big\}$. If σ is a $(k-1)$-simplex, then we declare F_σ to be a *face of codimension* k. Therefore, each facet (codimension-one face) can be identified with the star of some vertex in $\Delta\big(\overline{\mathcal{P}}\big)$. Each codimension-$k$ face is a connected component of an intersection of k facets and is acyclic since it is a cone. In the case when $\mathcal{P}$ is a simplicial complex the space P with the face decomposition was called in [6, p.428] a *simple polyhedral complex*.

Suppose that the number of facets of P is m and that we have a map Λ as in (4.1) satisfying the condition form Lemma 4.4. (The existence of such a map Λ is equivalent to the existence of a *linear system of parameters* in the ring $\mathbb{Z}[\mathcal{P}]$, see e.g. [18, Lemma III.2.4].) Then the same construction as $M_Q(\Lambda)$ in (4.2) with Q replaced by P produces a T-space $M_P(\Lambda)$. Since P is not a manifold with corners for arbitrary $\mathcal{P}$, the space $M_P(\Lambda)$ may fail to be a manifold. Nevertheless, a similar argument to that in [6, Theorem 4.8] gives the following result:

Proposition 5.13. *$H_T^*(M_P(\Lambda);\mathbb{Z})$ is isomorphic to $\mathbb{Z}[\mathcal{P}]$ as a ring.*

For an arbitrary nice manifold with corners Q the equivariant cohomology of the canonical model $M_Q(\Lambda)$ may fail to be isomorphic to $\mathbb{Z}[Q]$ as the faces of Q themselves may have complicated cohomology. In the next sections we shall study this question in more details. As the first step in this direction we relate $M_Q(\Lambda)$ to $M_P(\Lambda)$ in our last statement of this paragraph.

Proposition 5.14. *Let Q be a nice manifold with corners, and P the space associated with the face poset $\mathcal{P}$ of Q. Then there is a map $Q \to P$ which preserves the face structure. It is covered by a canonical equivariant map*

$$\Phi\colon M_Q(\Lambda) \to M_P(\Lambda).$$

Proof. The map $Q \to P$ is constructed inductively, starting from an identification of vertices and extending the map on each higher-dimensional face by a degree-one map. Every face of P is a cone, so there are no obstructions to such extensions. Since the map between orbit spaces preserves the face structure, it is covered by an equivariant map of the identification spaces

$$M_Q(\Lambda) = T \times Q/\!\sim\; \to T \times P/\!\sim\; = M_P(\Lambda)$$

by the definition of identification spaces, see (4.2). □

6. Axial functions and Thom classes

Here we relate the equivariant cohomology ring of a torus manifolds M to the face ring of the orbit space Q. We construct a natural ring homomorphism from $\mathbb{Z}[Q]$ to $H_T^*(M)$ modulo $H^*(BT)$-torsions. In the next section we show that this is an isomorphism when $H^{odd}(M) = 0$. In this and next sections we assume that M is locally standard for simplicity, but the arguments will work without this assumption with a little modification.

6.1. Axial functions. Like in the algebraic situation of the previous section, we have the restriction map to a sum of polynomial rings:

$$r = \bigoplus_{p \in M^T} r_p \colon H_T^*(M) \to H_T^*\big(M^T\big) = \bigoplus_{p \in M^T} H^*(BT). \tag{6.1}$$

The kernel of r is the $H^*(BT)$-torsion subgroup of $H_T^*(M)$, so r is injective when $H^{odd}(M) = 0$ by Lemma 2.1.

We identify M^T with the vertices of Q. The 1-skeleton of Q, consisting of vertices (0-faces) and edges (1-faces) of Q, is an n-valent graph. Denote by $E(Q)$ the set of oriented edges. Given an element $e \in E(Q)$, denote the initial point and terminal point of e by $i(e)$ and $t(e)$ respectively. Then $M_e := \pi^{-1}(e)$ is a 2-sphere fixed by a codimension-one subtorus in T (here $\pi : M \to Q$ is the quotient map). It contains two T-fixed points $i(e)$ and $t(e)$. The 2-dimensional subspace $\mathcal{T}_{i(e)}M_e \subseteq \mathcal{T}_{i(e)}M$ is an irreducible component of the tangential T-representation $\mathcal{T}_{i(e)}M$. The same is true for the other T-fixed point $t(e)$, and the T-representations $\mathcal{T}_{i(e)}M$ and $\mathcal{T}_{t(e)}M$ are isomorphic. There is a unique characteristic submanifold, say M_i, intersecting M_e at $i(e)$ transversally. Assuming both M and M_i are oriented, we choose a compatible orientation for the normal bundle ν_i of M_i and therefore, for $\mathcal{T}_{i(e)}M_e$. The orientation on $\mathcal{T}_{i(e)}M_e$ determines a complex structure, so that $\mathcal{T}_{i(e)}M_e$ can be viewed as a complex 1-dimensional T-representation. This defines an element of $\mathrm{Hom}(T, S^1) = H^2(BT)$, which we denote by $\alpha(e)$.

Let $e^T(\nu_i)$ be the equivariant Euler class in $H_T^2(M_i)$ and denote its restriction to $p \in M_i^T$ by $e^T(\nu_i)|_p \in H_T^2(p) = H^2(BT)$. Then

$$e^T(\nu_i)|_p = \alpha(e), \tag{6.2}$$

where e is the unique edge such that $i(e) = p$ and $e \notin Q_i = \pi(M_i)$. Following [10], we call the map

$$\alpha \colon E(Q) \to H^2(BT)$$

an *axial function*.

Lemma 6.1. *The axial function α has the following properties*:
(1) $\alpha(\overline{e}) = \pm\alpha(e)$ *for all* $e \in E(Q)$, *where* $\overline{e}$ *denotes* e *with the opposite orientation*;
(2) *for each vertex* (*or a* T*-fixed point*) p, *the set* $\alpha_p := \{\alpha(e) : i(e) = p\}$ *is a basis of* $H^2(BT)$ *over* $\mathbb{Z}$.
(3) *for* $e \in E(Q)$, *we have* $\alpha_{i(e)} \equiv \alpha_{t(e)} \mod \alpha(e)$.

Proof. Property (1) follows from the fact that $\mathcal{T}_{i(e)}M_e$ and $\mathcal{T}_{t(e)}M_e$ are isomorphic as real T-representations, and (2) from that the T-representation $\mathcal{T}_{i(e)}M$ is faithful of complex dimension n. Let T_e be the codimension one subtorus fixing M_e. Then the T-representations $\mathcal{T}_{i(e)}M$ and $\mathcal{T}_{t(e)}M$ are isomorphic as T_e-representations, since the points $i(e)$ and $t(e)$ are contained in the same connected component M_e of the T_e-fixed point set. This implies (3). $\square$

REMARK. In [10], the property $\alpha(\overline{e}) = -\alpha(e)$ is required in the definition of axial function, but we allow $\alpha(\overline{e}) = \alpha(e)$. For example, $\alpha(\overline{e}) = \alpha(e)$ for the T^2-action on S^4 from Example 3.2.

Lemma 6.2. *Fix* $\eta \in H_T^*(M)$; *then* $r_{i(e)}(\eta) - r_{t(e)}(\eta)$ *is divisible by* $\alpha(e)$ *for all* $e \in E(Q)$.

Proof. Consider the commutative diagram of restrictions

$$\begin{array}{ccc} H_T^*(M) & \longrightarrow & H_T^*(i(e)) \oplus H_T^*(t(e)) = H^*(BT) \oplus H^*(BT) \\ \downarrow & & \downarrow \\ H_{T_e}^*(M_e) & \longrightarrow & H_{T_e}^*(i(e)) \oplus H_{T_e}^*(t(e)) = H^*(BT_e) \oplus H^*(BT_e) \end{array}.$$

Since $H_{T_e}^*(M_e) = H^*(BT_e) \otimes H^*(M_e)$, the two components of the image of η in $H^*(BT_e) \oplus H^*(BT_e)$ above coincide. Therefore it follows from the commutativity of the above diagram that the restrictions of $r_{i(e)}(\eta)$ and $r_{t(e)}(\eta)$ to $H^*(BT_e)$ coincide. Since the kernel of the restriction map $H^*(BT) \to H^*(BT_e)$ is the ideal generated by $\alpha(e)$, the lemma follows. $\square$

6.2. Thom classes. The preimage $M_F := \pi^{-1}(F)$ of a codimension-k face $F \subset Q$ is a connected component of an intersection of k characteristic submanifolds. The orientations of M and characteristic submanifolds M_i determine compatible orientations for the normal bundles ν_i of M_i. These orientations determine an orientation on the normal bundle ν_F of M_F, and thereby on M_F itself, since M is oriented. With this convention on orientations, we consider the Gysin homomorphism $H_T^0(M_F) \to H_T^{2k}(M)$ in the equivariant cohomology and denote the image of the identity element by τ_F. The element τ_F may be thought of as the Poincaré dual of M_F in equivariant cohomology and is called the *Thom class* of M_F. The restriction of $\tau_F \in H_T^{2k}(M)$ to

$H_T^{2k}(M_F)$ is the equivariant Euler class of ν_F, and $r_p(\tau_F) = 0$ unless $p \in (M_F)^T$. It follows from (6.2) that

$$r_p(\tau_F) = \begin{cases} \prod_{i(e)=p,\, e\not\subseteq F} \alpha(e), & \text{if} \quad p \in (M_F)^T \\ 0, & \text{otherwise.} \end{cases} \tag{6.3}$$

We set

$$\widehat{H}_T^*(M) := H_T^*(M)/H^*(BT)\text{-torsions.}$$

The restriction map (6.1) induces a monomorphism $\widehat{H}_T^*(M) \to H_T^*(M^T)$, which we also denote by r. Therefore, $\tau_F = 0$ in $\widehat{H}_T^*(M)$ if M_F has no T-fixed point. The following lemma shows that the relations from Definition 5.3 hold in $\widehat{H}_T^*(M)$ with v_F replaced by τ_F.

Lemma 6.3. *For any two faces G and H of Q, the relation*

$$\tau_G \tau_H = \tau_{G\vee H} \cdot \sum_{E\in G\cap H} \tau_E,$$

holds in $\widehat{H}_T^*(M)$, where we set $\tau_\varnothing = 0$.

Proof. Since the restriction map $r\colon \widehat{H}_T^*(M) \to H_T^*(M^T)$ is injective, it suffices to show that r_p maps both sides of the identity to the same element for all $p \in M^T$.

Let $p \in M^T$. For a face F such that $p \in F$, we set

$$N_p(F) := \{e \in E(Q)\colon i(e) = p,\ \ e \notin F\},$$

which may be thought of as the set of directions normal to F at p. We also set $N_p(F) = \varnothing$ if $p \notin F$. Then the identity (6.3) can be written as

$$r_p(\tau_F) = \prod_{e\in N_p(F)} \alpha(e) \tag{6.4}$$

where the right hand side is understood to be zero if $N_p(F) = \varnothing$. If $p \notin G \cap H$, then $N_p(E) = \varnothing$ for any connected component E of $G \cap H$ and either $N_p(G) = \varnothing$ or $N_p(H) = \varnothing$. Therefore, both sides of the identity from the lemma map to zero by r_p. If $p \in G \cap H$, then

$$N_p(G) \cup N_p(H) = N_p(G \vee H) \cup N_p(E)$$

where E is the connected component of $G \cap H$ containing p, and $N_p(E') = \varnothing$ for any other connected component of $G \cap H$. This together with (6.4) shows that both sides of the identity map to the same element by r_p. $\square$

By virtue of the above lemma, the map $\mathbb{Z}[v_F : F \text{ a face}] \to H_T^*(M)$ sending v_F to τ_F induces a homomorphism

$$\varphi : \mathbb{Z}[Q] \to \widehat{H}_T^*(M). \tag{6.5}$$

Lemma 6.4. *The homomorphism φ is injective if every face of Q has a vertex.*

Proof. We have $s = r \circ \varphi$, where s is the map from Lemma 5.6. Since s is injective if every face of Q has a vertex, so is φ. □

7. Equivariant cohomology ring of torus manifolds with vanishing odd-degree cohomology

In this section we give a sufficient condition for the monomorphism φ in (6.5) to be an isomorphism (Theorem 7.5). In particular, it turns out that φ is an isomorphism when $H^{odd}(M) = 0$ (Corollary 7.6). Using these results, we give a description of the ring structure in $H^*(M)$ in the case when $H^{odd}(M) = 0$ (Corollary 7.8).

7.1. Ring structure in equivariant cohomology. The following theorem shows that the converse of Lemma 6.2 holds for torus manifolds with vanishing odd degree cohomology.

Theorem 7.1 ([8], see also Chapter 11 in [9]). *Suppose $H^{odd}(M) = 0$ and we are given an element $\eta_p \in H^*(BT)$ for each $p \in M^T$. Then $(\eta_p) \in \bigoplus_{p \in M^T} H^*(BT)$ belongs to the image of the restriction map r in* (6.1) *if and only if $\eta_{i(e)} - \eta_{t(e)}$ is divisible by $\alpha(e)$ for any $e \in E(Q)$.*

Corollary 7.2. *The* 1*-skeleton of any face of Q* (*including Q itself*) *is connected if $H^{odd}(M) = 0$.*

Proof. Since M is connected, the image $r\big(H_T^0(M)\big)$ is one-dimensional. Then it follows from the "if" part of Theorem 7.1 that the 1-skeleton of Q is connected. Similarly, the 1-skeleton of any face F of Q is connected because $M_F = \pi^{-1}(F)$ is also a torus manifold with vanishing odd degree cohomology (see Lemma 2.2). □

REMARK. The connectedness of 1-skeletons of faces of Q can be proven without referring to Theorem 7.1, see remark after Theorem 9.3.

For a face F of Q, we denote by $I(F)$ the ideal in $H^*(BT)$ generated by all elements $\alpha(e)$ with $e \in F$.

Lemma 7.3. *Suppose that the* 1*-skeleton of a face* F *is connected. Given* $\eta \in H_T^*(M)$*, if* $r_p(\eta) \notin I(F)$ *for some vertex* $p \in F$*, then* $r_q(\eta) \notin I(F)$ *for any vertex* $q \in F$*.*

Proof. Suppose $r_q(\eta) \in I(F)$ for some vertex $q \in F$. Then $r_s(\eta) \in I(F)$ for any vertex $s \in F$ joined to q by an edge $f \subseteq F$ because $r_q(\eta) - r_s(\eta)$ is divisible by $\alpha(f)$ by Lemma 6.2. Since the 1-skeleton of F is connected, $\eta(q) \in I(F)$ for any vertex $q \in F$, which contradicts the assumption. $\square$

Proposition 7.4. *If the* 1*-skeleton of every face of* Q *is connected, then* $\widehat{H}_T^*(M)$ *is generated by the elements* τ_F *as an* $H^*(BT)$*-module.*

Proof. Let $\eta \in H_T^{>0}(M)$ be a nonzero element. Set

$$Z(\eta) := \left\{ p \in M^T : r_p(\eta) = 0 \right\}.$$

Take $p \in M^T$ such that $p \notin Z(\eta)$. Then $r_p(\eta) \in H^*(BT)$ is non-zero and we can express it as a polynomial in $\{\alpha(e) \colon i(e) = p\}$ (the latter is a basis of $H^2(BT)$). Let

$$\prod_{i(e)=p} \alpha(e)^{n_e}, \tag{7.1}$$

$n_e \geqslant 0$, be a monomial entering $r_p(\eta)$ with a non-zero coefficient. Let F be the face spanned by the edges e with $n_e = 0$. Then $r_p(\eta) \notin I(F)$ since $r_p(\eta)$ contains the monomial (7.1). Hence, $r_q(\eta) \notin I(F)$, in particular $r_q(\eta) \neq 0$, for every vertex $q \in F$ by Lemma 7.3.

On the other hand, it follows from (6.3) that the monomial (7.1) can be written as $r_p(u_F \tau_F)$ with some $u_F \in H^*(BT)$. Set $\eta' := \eta - u_F \tau_F \in H_T^*(M)$. Since $r_q(\tau_F) = 0$ for every vertex $q \notin F$, we have $r_q(\eta') = r_q(\eta)$ for such q. At the same time, $r_q(\eta) \neq 0$ for every vertex $q \in F$ (see above). It follows that $Z(\eta') \supseteq Z(\eta)$. However, the number of monomials in $r_p(\eta')$ is less than that in $r_p(\eta)$. Therefore, subtracting from η a linear combination of τ_F's with coefficients in $H^*(BT)$, we obtain an element λ such that $Z(\lambda)$ contains $Z(\eta)$ as a proper subset. Repeating this procedure, we end up at an element whose restriction to every vertex is zero. Since the restriction map $r \colon \widehat{H}_T^*(M) \to H_T^*(M^T)$ is injective, this finishes the proof. $\square$

Theorem 7.5. *Let* M *be a* (*locally standard*) *torus manifold with orbit space* Q*. If every face of* Q *has a vertex and its* 1*-skeleton is connected, then the monomorphism* $\varphi \colon \mathbb{Z}[Q] \to \widehat{H}_T^*(M)$ *in* (6.5) *is an isomorphism.*

Proof. To prove that φ is surjective it suffices to show that $\widehat{H}_T^*(M)$ is generated by the elements τ_F as a ring. By Proposition 3.3, $\widehat{H}_T^2(M)$ is generated over $\mathbb{Z}$ by the elements τ_{Q_i} corresponding to the facets Q_i. (Note: the notation τ_i is used for τ_{Q_i} in

Proposition 3.3.) In particular, any element in $H^2(BT) \subset \widehat{H}^*_T(M)$ can be written as a linear combination of τ_{Q_i}'s with coefficients in $\mathbb{Z}$. Hence, any element in $H^*(BT)$ is a polynomial in τ_{Q_i}'s. The rest follows from Proposition 7.4. □

Now assume $H^{odd}(M) = 0$. Then the assumption in Theorem 7.5 is satisfied, and $H^*_T(M)$ is a free $H^*(BT)$-module by Lemma 2.1, whence $\widehat{H}^*_T(M) = H^*_T(M)$.

Corollary 7.6. *For a torus manifold M with vanishing odd degree cohomology, the map $\varphi\colon \mathbb{Z}[Q] \to H^*_T(M)$ in* (6.5) *is an isomorphism.*

Proof. This follows from Corollary 7.2 and Theorem 7.5. □

REMARK. When $H^*(M)$ is generated in degree two, all non-empty multiple intersections of facets are connected by Lemma 3.4. Therefore, the face poset of Q is the face poset of the nerve K of the covering of ∂Q, and $\mathbb{Z}[Q]$ reduces to the classical Stanley-Reisner face ring of a simplicial complex. Therefore, Corollary 7.6 is a generalisation of Proposition 3.4 in [13].

If $\mathcal{P}$ is the face poset of Q, then $\mathbb{Z}[\mathcal{P}] = \mathbb{Z}[Q]$ by the definition. The following statement gives a characterisation of torus manifolds M with vanishing odd degree cohomology (and with cohomology generated in degree two) in terms of the face poset $\mathcal{P}$ associated with M.

Theorem 7.7. *Let M be a torus manifold with orbit space Q, and let $\mathcal{P}$ be the face poset of Q. Then $H^{odd}(M) = 0$ if and only if the following two conditions are satisfied*:

(1) *$H^*_T(M)$ is isomorphic to $\mathbb{Z}[\mathcal{P}](= \mathbb{Z}[Q])$ as a ring*;

(2) *$\mathbb{Z}[\mathcal{P}]$ is Cohen-Macaulay*.

Moreover, $H^*(M)$ is generated by its degree-two part if and only if $\mathcal{P}$ is (*the face poset of*) *a simplicial complex in addition to the above two conditions*.

Proof. If $H^{odd}(M) = 0$, then $H^*_T(M) \cong \mathbb{Z}[Q] = \mathbb{Z}[\mathcal{P}]$ by Corollary 7.6, and $\mathbb{Z}[\mathcal{P}]$ is Cohen-Macaulay because $H^*_T(M)$ is a free $H^*(BT)$-module by Lemma 2.1. This proves the "only if" part of the first statement.

Now we prove the "if" part. Let $\rho\colon ET \times_T M \to BT$ be the projection, and consider the composite map

$$H^*(BT) \xrightarrow{\rho^*} H^*_T(M) \xrightarrow{r} \bigoplus_{p \in M^T} H^*(BT).$$

Its restriction to each summand of the target is the identity, i.e., $r \circ \rho^*$ is a diagonal map. This implies that $\rho^*(t_1), \dots, \rho^*(t_n)$ is a linear system of parameters (an l.s.o.p.), see [2, Theorem 5.1.16]. By the assumption, $H_T^*(M)$ is isomorphic to $\mathbb{Z}[\mathcal{P}]$ and $\mathbb{Z}[\mathcal{P}]$ is Cohen-Macaulay, so every l.s.o.p. is a regular sequence (see [18, Theorem I.5.9]). It follows that $H_T^*(M)$ is a free $H^*(BT)$-module and hence $H^{odd}(M) = 0$ by Lemma 2.1, thus proving the "if" part of the first statement.

It remains to prove the second statement. The "only if" part follows from Lemma 3.4 by the last remark. For the "if" part, if $\mathcal{P}$ is a simplicial poset, then $\mathbb{Z}[\mathcal{P}]$ is generated by its degree-two part. By the first statement of the theorem, $H_T^*(M) \cong \mathbb{Z}[\mathcal{P}]$ is a free $H^*(BT)$-module, whence $H^*(M)$ is a quotient ring of $H_T^*(M)$. It follows that $H^*(M)$ is also generated by its degree-two part. □

The following description of cohomology ring of a torus manifold with vanishing odd degree cohomology generalises that of a complete non-singular toric variety, see [7, p.106].

Corollary 7.8. *For a torus manifold M with vanishing odd degree cohomology,*

$$H^*(M) \cong \mathbb{Z}[v_F : F \text{ a face of } Q]/I \quad \text{as a ring},$$

where I is the ideal generated by the following two types of elements:
(1) $v_G v_H - v_{G\vee H} \sum_{E\in G\cap H} v_E$;
(2) $\sum_{i=1}^m \langle t, a_i\rangle v_{Q_i}$ *for* $t \in H^2(BT)$.
Here Q_i are the facets of Q and the elements $a_i \in H_2(BT)$ are defined in Proposition 3.3.

Proof. Since the Serre spectral sequence of the fibration $\rho\colon ET\times_T M \to BT$ collapses, the restriction map $H_T^*(M) \to H^*(M)$ is surjective and its kernel is the ideal generated by all $\rho^*(t)$ with $t \in H^2(BT)$. Therefore, the statement follows from Proposition 3.3 and Corollary 7.6. □

7.2. Dehn-Sommerville equations. Suppose that $H^{odd}(M) = 0$. Then, since $H_T^*(M) = H^*(BT) \otimes H^*(M)$ by Lemma 2.1 and $H^*(BT)$ is a polynomial ring in n variables of degree two, the Hilbert series of $H_T^*(M)$ is given by

$$F(H_T^*(M); t) = \frac{\sum_{i=0}^n \operatorname{rank}_{\mathbb{Z}} H^{2i}(M) t^{2i}}{(1-t^2)^n}.$$

On the other hand, the Hilbert series of the face ring $\mathbb{Z}[Q]$ is given by Theorem 5.12 and these two series must coincide by Corollary 7.6. It follows that

$$\operatorname{rank}_{\mathbb{Z}} H^{2i}(M) = h_i. \tag{7.2}$$

Since M is a manifold, the Poincaré duality implies that

$$h_i = h_{n-i}, \quad i = 0, \dots, n. \tag{7.3}$$

When every non-empty multiple intersection of facets in Q is connected, $\mathbb{Z}[Q]$ reduces to the classical Stanley-Reisner ring of the nerve of the covering of ∂Q and equations (7.3) are known as the *Dehn-Sommerville equations* for the numbers of faces.

8. Orbit spaces of torus manifolds with cohomology generated in degree two

Using the equivariant cohomology calculations from the previous section, we are finally able to relate the cohomology of a torus manifold M and the cohomology of its orbit space Q. The main result of this section is Theorem 8.3 which gives a cohomological characterisation of torus manifolds whose orbit spaces are homology polytopes. Using this result, in the next section we prove that Q is face-acyclic if $H^{odd}(M) = 0$.

Lemma 8.1. *If $H^{odd}(M) = 0$, then $H^1(Q;\mathbf{k}) = 0$ for any coefficient ring $\mathbf{k}$. In particular, Q is orientable.*

Proof. We use the Leray spectral sequence (with $\mathbf{k}$ coefficient) of the projection map $ET \times_T M \to M/T = Q$ on the second factor. Its E_2 term is given by $E_2^{p,q} = H^p(M/T;\mathcal{H}^q)$ where $\mathcal{H}^q$ is a sheaf with stalk $H^q(BT_x;\mathbf{k})$ over a point $x \in M/T$, and the spectral sequence converges to $H_T^*(M;\mathbf{k})$. Since the T-action on M is locally standard by Theorem 4.1, the isotropy group T_x at $x \in M$ is a subtorus; so $H^{odd}(BT_x;\mathbf{k}) = 0$. Hence, $\mathcal{H}^{odd} = 0$, in particular, $\mathcal{H}^1 = 0$. Moreover, $\mathcal{H}^0 = \mathbf{k}$ (a constant sheaf). Therefore, we have $E_2^{0,1} = 0$ and $E_2^{1,0} = H^1(M/T;\mathbf{k})$, whence $H^1(M/T;\mathbf{k}) \cong H_T^1(M;\mathbf{k})$. On the other hand, since $H^{odd}(M) = 0$ by assumption, $H_T^*(M)$ is a free $H^*(BT)$-module (isomorphic to $H^*(BT) \otimes H^*(M)$ by Lemma 2.1). Therefore, $H_T^{odd}(M;\mathbf{k}) = 0$ by the universal coefficient theorem. In particular, $H_T^1(M;\mathbf{k}) = 0$, thus proving the lemma. □

Lemma 8.2. *If either*

(1) *Q is a homology polytope, or*

(2) *$H^*(M)$ is generated by its degree-two part,*

then the face poset $\mathcal{P}$ of Q is (the face poset of) a simplicial Gorenstein complex. In particular, $\mathbb{Z}[\mathcal{P}]$ is Cohen-Macaulay and the geometric realisation $|\mathcal{P}|$ of $\mathcal{P}$ has the homology of an $(n-1)$-sphere.*

Proof. Under either assumption (1) or (2), all non-empty multiple intersections of facets of Q are connected, so $\mathcal{P}$ agrees with the face poset of the nerve simplicial complex K of the covering of ∂Q. In what follows we identify $\mathcal{P}$ with K.

First we prove that $\mathcal{P}$ is Gorenstein* under assumption (1). According to Theorem II.5.1 of [18] it is enough to show that the link of a simplex σ of $\mathcal{P}$, denoted by $\operatorname{link}\sigma$, has the homology of a sphere of $\dim \operatorname{link}\sigma = n-2-\dim\sigma$. If $\sigma = \varnothing$ then $\operatorname{link}\sigma$ is $\mathcal{P}$ itself and its homology is isomorphic to the homology of the boundary ∂Q of Q, since $\mathcal{P}$ is the nerve of Q and Q is a homology polytope. If $\sigma \neq \varnothing$ then $\operatorname{link}\sigma$

is the nerve of a face of Q. Since any face of Q is again a homology polytope, $\operatorname{link}\sigma$ has the homology of a sphere of $\dim \operatorname{link}\sigma$ by the same argument.

Now we prove that $\mathcal{P}$ is Gorenstein* under assumption (2). Using Theorem II.5.1 of [18] once again, it is enough to show that

(a) $\mathcal{P}$ is Cohen-Macaulay;

(b) every $(n-2)$-dimensional simplex is contained in exactly two $(n-1)$-dimensional simplices;

(c) $\chi(\mathcal{P}) = \chi(S^{n-1})$.

The condition (a) follows from Lemma 2.1 and Corollary 7.6. By definition, every k-dimensional simplex of $\mathcal{P}$ corresponds to a set of $k+1$ characteristic submanifolds having non-empty intersection. By Lemma 3.4, the intersection of any n characteristic submanifolds is either empty or consists of a single T-fixed point. This means that the $(n-1)$-simplices of $\mathcal{P}$ are in one-to-one correspondence with the T-fixed points of M. Now, each $(n-2)$-simplex of $\mathcal{P}$ corresponds to a non-empty intersection of $n-1$ characteristic submanifolds of M. The latter intersection is connected by Lemma 3.4 and has a non-trivial T-action, so it is a 2-sphere. Every 2-sphere contains exactly two T-fixed points, which implies (b). Finally, (c) is just the Dehn-Sommerville equation $h_0 = h_n$, see (5.2) and (7.3). □

Theorem 8.3. *The cohomology of a torus manifold M is generated by its degree-two part if and only if M is locally standard and the orbit space Q is a homology polytope.*

Proof. Let $\mathcal{P}$ be the face poset of Q, and P the cone on $|\mathcal{P}|$ with the face structure associated with $\mathcal{P}$, see end of Subsection 5.2.

We first prove the "if" part. Suppose Q is a homology polytope. Since $H^2(Q) = 0$ and M is locally standard, M is equivariantly homeomorphic to $M_Q(\Lambda)$ by Lemma 4.5; so we may regard the map Φ in (5.14) as a map from M to $M_P := M_P(\Lambda)$. Let $M_{P,i}$ be characteristic subcomplexes of M_P defined similarly to characteristic submanifolds M_i of M. Since the T-actions on $M_P \setminus \bigcup_i M_{P,i}$ and $M \setminus \bigcup_i M_i$ are free, we have

$$H_T^*\left(M_P, \bigcup_i M_{P,i}\right) \cong H^*(P, |\mathcal{P}|), \quad H_T^*\left(M, \bigcup_i M_i\right) \cong H^*(Q, \partial Q).$$

Therefore, the map Φ induces a map between exact sequences

$$\begin{tikzcd}
{} \arrow[r] & H^*(P, |\mathcal{P}|) \arrow[r] \arrow[d] & H_T^*(M_P) \arrow[r] \arrow[d, "\Phi^*"] & H_T^*\left(\bigcup_i M_{P,i}\right) \arrow[r] & {} \\
{} \arrow[r] & H^*(Q, \partial Q) \arrow[r] & H_T^*(M) \arrow[r] & H_T^*\left(\bigcup_i M_i\right) \arrow[r] & {}
\end{tikzcd} \tag{8.1}$$

Each M_i itself is a torus manifold over a homology polytope Q_i. Using induction and a Mayer-Vietoris argument, we may assume that the map $H_T^*\big(\bigcup_i M_{P,i}\big) \to H_T^*\big(\bigcup_i M_i\big)$ above is an isomorphism. By Lemma 8.2, $|\mathcal{P}|$ has the homology of an $(n-1)$-sphere, and since P is the cone over $|\mathcal{P}|$, we have $H^*(P, |\mathcal{P}|) \cong H^*(D^n, S^{n-1})$. We also have $H^*(Q, \partial Q) \cong H^*(D^n, S^{n-1})$ because Q is a homology polytope. Using these isomorphisms, we see from the construction of the map Φ that the induced map $H^*(P, |\mathcal{P}|) \to H^*(Q, \partial Q)$ is the identity map on $H^*(D^n, S^{n-1})$. Therefore, the 5-lemma applied to (8.1) shows that $\Phi^*\colon H_T^*(M_P) \to H_T^*(M)$ is an isomorphism; whence $H_T^*(M) \cong \mathbb{Z}[\mathcal{P}]$ by Proposition 5.13. We also know that $\mathbb{Z}[\mathcal{P}]$ is Cohen-Macaulay by Lemma 8.2. Therefore, the two conditions in Theorem 7.7 are satisfied. It follows that $H^*(M)$ is generated by its degree-two part by Theorem 7.7, which finishes the proof of the "if" part.

Now we prove the "only if" part. Suppose that $H^*(M)$ is generated by the degree-two elements. Then M is locally standard by Theorem 4.1. Since all non-empty multiple intersections of characteristic submanifolds are connected and their cohomology rings are generated in degree two by Lemma 3.4, we may assume by induction that all the proper faces of Q are homology polytopes. In particular, the proper faces are acyclic, whence $H^*(\partial Q) \cong H^*(|\mathcal{P}|)$. This together with Lemma 8.2 shows that

$$H^*(\partial Q) \cong H^*\big(S^{n-1}\big). \tag{8.2}$$

Claim. $H^2(Q) = 0$.

The claim is trivial for $n = 1$. If $n = 2$ then Q is a surface with boundary, hence, $H^2(Q) = 0$ in this case too. Now assume $n \geqslant 3$. Let us consider the exact equivariant cohomology sequence of pair $\big(M, \bigcup_i M_i\big)$, see the bottom row of (8.1). All the maps in the exact sequence are $H^*(BT)$-module maps. By Lemma 2.1, $H_T^*(M)$ is a free $H^*(BT)$-module. On the other hand, $H^*(Q, \partial Q)$ is finitely generated over $\mathbb{Z}$, so it is a torsion $H^*(BT)$-module. It follows that the whole sequence splits in short exact sequences:

$$0 \to H_T^k(M) \to H_T^k\left(\bigcup_i M_i\right) \to H^{k+1}(Q, \partial Q) \to 0 \tag{8.3}$$

Taking $k = 1$ above, we get

$$H_T^1\left(\bigcup_i M_i\right) \cong H^2(Q, \partial Q).$$

The same argument as in Lemma 8.1 shows that the former is isomorphic to $H^1\big(\big(\bigcup_i M_i\big)/T\big) = H^1(\partial Q)$, and the above isomorphism implies (through the projection $(ET \times M)/T \to M/T = Q$) that the coboundary map $H^1(\partial Q) \to H^2(Q, \partial Q)$

in the exact sequence of the pair $(Q, \partial Q)$ is an isomorphism. Therefore, we get the following exact sequence fragment:

$$0 \to H^2(Q) \to H^2(\partial Q) \to H^3(Q, \partial Q).$$

Since $H^2(\partial Q) \cong H^2(S^{n-1})$ by (8.2), we have $H^2(Q) = 0$ if $n \geqslant 4$. When $n = 3$, the coboundary map above is an isomorphism because Q is orientable by Lemma 8.1, whence $H^2(Q) = 0$ again. This completes the proof of the claim.

Since $H^2(Q) = 0$, we have a map $\Phi\colon M \to M_P(\Lambda)$ as in the proof of the "if" part. Let us consider the diagram (8.1) with $\mathbf{k}$ coefficient where $\mathbf{k} = \mathbb{Q}$ or $\mathbb{Z}/p$ with prime p. Using induction and a Mayer-Vietoris argument, we deduce that $H_T^*\bigl(\bigcup_i M_{P,i};\mathbf{k}\bigr) \to H_T^*\bigl(\bigcup_i M_i;\mathbf{k}\bigr)$ is an isomorphism. We know that $H^*(P, |\mathcal{P}|;\mathbf{k}) \cong H^*(D^n, S^{n-1};\mathbf{k})$ by Lemma 8.2, and it follows from the construction of Φ that the induced map

$$H^*(D^n, S^{n-1};\mathbf{k}) \cong H^*(P, |\mathcal{P}|;\mathbf{k}) \to H^*(Q, \partial Q;\mathbf{k}) \tag{8.4}$$

is an isomorphism in degree n, and thus is injective in all degrees. Therefore (an extended version of) the 5-lemma (see [16, p.185]) applied to (8.1) with $\mathbf{k}$ coefficient shows that $\Phi^*\colon H_T^*(M_P;\mathbf{k}) \to H_T^*(M;\mathbf{k})$ is injective. Here, $H_T^*(M) \cong \mathbb{Z}[Q] \cong H_T^*(M_P)$ by Corollary 7.6 (or Proposition 3.4 in [13]) and Proposition 5.13 (or Theorem 4.8 of [6]), so $H_T^*(M_P;\mathbf{k})$ and $H_T^*(M;\mathbf{k})$ have the same dimension over $\mathbf{k}$ in each degree. Therefore, the monomorphism $\Phi^*\colon H_T^*(M_P;\mathbf{k}) \to H_T^*(M;\mathbf{k})$ is actually an isomorphism. Again, the 5-lemma applied to (8.1) with $\mathbf{k}$ coefficients implies that the map (8.4) is an isomorphism, so $H^*(Q, \partial Q;\mathbf{k}) \cong H^*(D^n, S^{n-1};\mathbf{k})$ for any $\mathbf{k}$ and hence $H^*(Q, \partial Q) \cong H^*(D^n, S^{n-1})$. This together with (8.2) (or the Poincaré-Lefschetz duality) gives the acyclicity of Q, thus finishing the proof of the theorem. □

The following statement gives a characterisation of simplicial complexes associated with torus manifolds with cohomology generated in degree two.

Theorem 8.4. *A simplicial complex $\mathcal{P}$ is associated with a torus manifold M whose cohomology is generated by its degree-two part if and only if $\mathcal{P}$ is Gorenstein* and $\mathbb{Z}[\mathcal{P}]$ admits an l.s.o.p.*

Proof. If $H^*(M)$ is generated by its degree-two part, then $\mathcal{P}$ is Gorenstein*, in particular $\mathbb{Z}[\mathcal{P}]$ is Cohen-Macaulay by Lemma 8.2. Moreover $H_T^*(M) \cong \mathbb{Z}[\mathcal{P}]$ by Corollary 7.6 (or Proposition 3.4 in [13]). Since $H_T^*(M) \cong H^*(BT)\otimes H^*(M)$ as an $H^*(BT)$-module by Lemma 2.1, $\mathbb{Z}[\mathcal{P}]$ admits an l.s.o.p.

Now we prove the "if" part. According to Theorem 12.2 of [5], there exists a homology polytope Q whose nerve is $\mathcal{P}$. Since the face ring $\mathbb{Z}[\mathcal{P}]$ admits an l.s.o.p., it is a free module over a polynomial ring $\mathbb{Z}[t_1, \dots, t_n]$ in n variables. We can express any element $t \in H^2(BT) \cong \mathbb{Z}[t_1, \dots, t_n]$ as

$$t = \sum_{i=1}^{m} a_i(t) v_i,$$

where $a_i(t) \in \mathbb{Z}$. Clearly, $a_i(t)$ is linear on t, so a_i can be viewed as an element of the dual space $H_2(BT)$ (see Proposition 3.3). Now define a map Λ (4.1) by sending Q_i to a_i. Then $M := M_Q(\Lambda)$ (see (4.2)) is a torus manifold, and its cohomology is generated in degree two by Theorem 8.3, which finishes the proof. □

9. Orbit spaces of torus manifolds with vanishing odd degree cohomology

Let F be a face of Q. The facial submanifold $M_F = \pi^{-1}(F)$ is a connected component of an intersection of finitely many characteristic submanifolds. The Whitney sum of their normal bundles restricted to M_F gives the normal bundle ν_F of M_F. The orientations for M and characteristic submanifolds determine a T-invariant complex structure on ν_F, so that the complex projective bundle $P(\nu_F)$ of ν_F can be considered. Replacing M_F in M by $P(\nu_F)$, we obtain a new torus manifold $\widetilde{M}$. The passage from M to $\widetilde{M}$ is called the *blowing-up* of M at M_F. (Remark: the normal bundle ν_F admits many invariant complex structures and the following argument works once we choose one.) The orbit space $\widetilde{Q}$ of $\widetilde{M}$ is then the result of "cutting off" the face F from Q, and the simplicial cell complex dual to $\widetilde{Q}$ is obtained from that dual to Q by applying a stellar subdivision of the face dual to F.

Lemma 9.1. *The orbit space $\widetilde{Q}$ is face-acyclic if and only if so is Q.*

Proof. By cutting the face F off Q we obtain a new facet $\widetilde{F} \subset \widetilde{Q}$, and all other new faces of $\widetilde{Q}$ are contained in this facet. The projection map $\widetilde{Q} \to Q$ collapses $\widetilde{F}$ back to F. The face F is a deformation retract of $\widetilde{F}$. Hence, F is acyclic if and only if $\widetilde{F}$ is acyclic. The same is true for any other new face of $\widetilde{Q}$. It is also clear from the construction that Q is a deformation retract of $\widetilde{Q}$. Therefore, $\widetilde{Q}$ is acyclic if and only if so is Q. □

Lemma 9.2. $H^{odd}\big(\widetilde{M}\big) = 0$ *if* $H^{odd}(M) = 0$.

Proof. The facial submanifold $M_F \subset M$ is blown up to a codimension-two facial submanifold $\widetilde{M}_{\widetilde{F}} \subset \widetilde{M}$, namely, $\widetilde{M}_{\widetilde{F}} = P(\nu_F)$. Since $\widetilde{M}_{\widetilde{F}}$ is the total space of a bundle with base M_F and fibre a complex projective space, its cohomology is a free $H^*(M_F)$-module on even-dimensional generators by Dold's theorem (see, e.g., [19, Ch. V]). If $H^{odd}(M) = 0$, then $H^{odd}(M_F) = 0$ by Lemma 2.2 and hence $H^{odd}\big(\widetilde{M}_{\widetilde{F}}\big) = 0$. Let $\widetilde{M} \to$

M be the collapse map and consider the diagram

$$\begin{tikzcd} H^{k-1}(M_F) \arrow[r] \arrow[d] & H^k(M, M_F) \arrow[r] \arrow[d, "\cong"] & H^k(M) \arrow[r] \arrow[d] & H^k(M_F) \arrow[d] \\ H^{k-1}\big(\widetilde{M}_{\widetilde{F}}\big) \arrow[r] & H^k\big(\widetilde{M}, \widetilde{M}_{\widetilde{F}}\big) \arrow[r] & H^k\big(\widetilde{M}\big) \arrow[r] & H^k\big(\widetilde{M}_{\widetilde{F}}\big) \end{tikzcd}$$

where the second vertical arrow is an isomorphism by excision. Assume that k is odd. If $H^{odd}(M) = 0$ then $H^{k-1}(M_F) \to H^k(M, M_F)$ is onto. Therefore, it follows from the above commutative diagram that $H^{k-1}\big(\widetilde{M}_{\widetilde{F}}\big) \to H^k\big(\widetilde{M}, \widetilde{M}_{\widetilde{F}}\big)$ is also onto. Since $H^k\big(\widetilde{M}_{\widetilde{F}}\big) = 0$, this implies $H^k\big(\widetilde{M}\big) = 0$. $\square$

The following main result of this section is an analogue of Theorem 8.3.

Theorem 9.3. *The odd-degree cohomology of M vanishes if and only if M is locally standard and the orbit space Q is face-acyclic.*

Proof. The idea is to reduce to Theorem 8.3 by blowing up sufficiently many facial submanifolds $M_F = \pi^{-1}(F)$. Since the barycentric subdivision is a sequence of stellar subdivisions, by applying sufficiently many blow-ups we get a torus manifold $\widehat{M}$ with orbit space $\widehat{Q}$ such that the face poset of $\widehat{Q}$ is the barycentric subdivision of the face poset of Q. The collapse map $\widehat{M} \to M$ is decomposed into a sequence of collapse maps for single blow-ups:

$$M = M_0 \longleftarrow M_1 \longleftarrow \cdots \longleftarrow M_k = \widehat{M}. \tag{9.1}$$

Assume that $H^{odd}(M) = 0$. Then M is locally standard by Theorem 4.1. By applying Lemma 9.2 several times we get $H^{odd}\big(\widehat{M}\big) = 0$. By construction, all the intersections of faces of $\widehat{Q}$ are connected, so $H^*\big(\widehat{M}\big)$ is generated by its degree-two part by Theorem 7.7 and $\widehat{Q}$ is a homology polytope by Theorem 8.3. In particular, $\widehat{Q}$ is face-acyclic. Finally, by applying Lemma 9.1 inductively we conclude that Q is also face-acyclic.

The scheme of the proof of the "if" part is same as that of Theorem 8.3. But there are two things to be checked. These are

(1) $|\mathcal{P}|$ has the homology of an $(n-1)$-sphere,

(2) $\mathbb{Z}[\mathcal{P}]$ is Cohen-Macaulay.

Let $\widehat{\mathcal{P}}$ be the face poset of $\widehat{Q}$. Since Q is face-acyclic, $\widehat{Q}$ is a homology polytope. Therefore, $\big|\widehat{\mathcal{P}}\big|$ has the homology of an $(n-1)$-sphere by Lemma 8.2. However, $\big|\widehat{\mathcal{P}}\big| = |\mathcal{P}|$, so the first statement above follows. Since $\widehat{Q}$ is a homology polytope, $\mathbb{Z}\big[\widehat{\mathcal{P}}\big]$ is Cohen-Macaulay by Lemma 8.2. This implies that $\mathbb{Z}[\mathcal{P}]$ itself is Cohen-Macaulay by Corollary 3.7 of [17], proving the second statement above. $\square$

REMARK. As one can easily observe, the argument in the "only if" part of the above theorem is independent of Theorem 7.1 and Corollary 7.6. Now, given that Q is face-acyclic, one readily deduces that the 1-skeleton of Q is connected. Indeed, otherwise the smallest face containing vertices from two different connected components of the 1-skeleton would be a manifold with at least two boundary components and thereby non-acyclic. Thus, our reference to Theorem 7.1 was actually irrelevant, although it made the arguments more straightforward.

Finally, we note that the proof of the "if" part of Theorem 9.3 could have been identical to that of the "only if" part if the converse of Lemma 9.2 was true. It is indeed the case, however the only proof we have so far uses quite complicated analysis of Cohen-Macaulay simplicial posets. We are going to write it down elsewhere.

10. Gorenstein simplical posets and Betti numbers of torus manifolds

The barycentric subdivision $\widehat{\mathcal{P}}$ of a simplicial poset $\mathcal{P}$ is (the face poset of) a simplicial complex and $\mathcal{P}$ is called *Gorenstein** if $\widehat{\mathcal{P}}$ is Gorenstein* ([17], [18]). If $\mathcal{P}$ is the simplicial poset associated with a torus manifold M with $H^{odd}(M) = 0$, then the torus manifold $\widehat{M}$ corresponding to $\widehat{\mathcal{P}}$ has cohomology generated by its degree-two part as remarked in the proof of Theorem 9.3. Hence, $\widehat{\mathcal{P}}$ is Gorenstein* by Lemma 8.2 and $\mathcal{P}$ is Gorenstein* by definition. In [17] Stanley proved that any vector satisfying the conditions in Theorem 10.1 below is an h-vector of a Gorenstein* simplicial poset. He also conjectured that those conditions are necessary. In this section we prove this conjecture for Gorenstein* simplicial posets $\mathcal{P}$ associated with torus manifolds M with vanishing odd degree cohomology, and characterize h-vectors of those Gorenstein* simplicial posets. The Stanley conjecture was proved in full generality by the first author in [14].

Since

$$h_i(\mathcal{P}) = \operatorname{rank}_{\mathbb{Z}} H^{2i}(M), \tag{10.1}$$

by (7.2), we need to characterise the Betti numbers of torus manifolds with vanishing odd degree cohomology. We note that

$$h_i(\mathcal{P}) \geqslant 0, \quad h_i(\mathcal{P}) = h_{n-i}(\mathcal{P}) \quad \text{for all} \quad i, \quad \text{and} \quad h_0(\mathcal{P}) = 1.$$

Theorem 10.1. *Let $\boldsymbol{h} = (h_0, h_1, \dots, h_n)$ be a vector of non-negative integers with $h_i = h_{n-i}$ for all i and $h_0 = 1$. Any of the following (mutually exclusive) conditions is sufficient for the existence of a rank n Gorenstein* simplicial poset $\mathcal{P}$ that is associated with a $2n$-dimensional torus manifold with vanishing odd degree cohomology and has h-vector $\boldsymbol{h}$:*

(1) *n is odd,*

(2) *n is even and $h_{n/2}$ is even,*

(3) *n is even, $h_{n/2}$ is odd, and $h_i > 0$ for all i.*
Moreover, if $\boldsymbol{h}$ is the h-vector of a simplicial poset of the above described type, then it satisfies one of the above three conditions.

Proof. For a torus manifold M, we set $h_i(M) = \operatorname{rank}_{\mathbb{Z}} H^{2i}(M)$. Thanks to (10.1), we may use $h_i(M)$ instead of $h_i(\mathcal{P})$ to prove the theorem.

We shall prove the sufficiency first. Examples 3.1 and 3.2 produce torus manifolds $\mathbb{C}P^n$, S^{2n} and $S^{2n-2k} \times S^{2k}$ with $1 \leqslant k \leqslant n-1$. In all three cases the odd-degree cohomology is zero. If M_1 and M_2 are torus manifolds (of same dimension) with vanishing odd degree cohomology, then their equivariant connected sum $M_1 \# M_2$ at two fixed points with isomorphic tangential representations produces a torus manifold with vanishing odd degree cohomology. We have

$$h_i(M_1 \# M_2) = h_i(M_1) + h_i(M_2) \quad \text{for} \quad 1 \leqslant i \leqslant n-1.$$

Using this identity, one easily gets any vector satisfying the conditions in the theorem by taking equivariant connected sum of $\mathbb{C}P^n$, S^{2n} and $S^{2n-2k} \times S^{2k}$.

Now we prove the necessity. Let M be a torus manifold of dimension $2n$. It suffices to prove that $h_{n/2}(M)$ is even if n is even and $h_i(M) = 0$ for some $i > 0$.

Let G be the 2-torus subgroup of T of rank n (that is, $G \cong (\mathbb{Z}/2)^n$). Then the equivariant total Stiefel-Whitney class of M with the restricted G-action is defined to be the ordinary total Stiefel-Whitney class of the vector bundle $EG \times_G \mathcal{T}M \to EG \times_G M$, and is denoted by $w^G(M)$. By definition, $w^G(M)$ lies in $H^*_G(M;\mathbb{Z}/2)$. We denote by τ_i the image of the identity under the equivariant Gysin map $H^0_G(M_i;\mathbb{Z}/2) \to H^2_G(M;\mathbb{Z}/2)$, where M_i $(i = 1, \dots, m)$ are characteristic submanifolds of M.

Claim. $w^G(M) = \prod_{i=1}^m (1 + \tau_i)$.

The proof of the claim is similar to that of Theorem 3.1 in [13], where the same formula was proved for the total equivariant Chern class. Since $H^{odd}(M;\mathbb{Z}/2) = 0$ and $M^G = M^T$ is isolated, we have

$$\dim H^*(M;\mathbb{Z}/2) = \chi(M) = \chi\left(M^T\right) = \chi\left(M^G\right) = \dim H^*\left(M^G;\mathbb{Z}/2\right).$$

Therefore, $H^*_G(M;\mathbb{Z}/2)$ is a free $H^*(BG;\mathbb{Z}/2)$-module (see [1, Theorem VII.1.6]). It follows from the localisation theorem that the restriction map

$$H^*_G(M;\mathbb{Z}/2) \to H^*_G\left(M^G;\mathbb{Z}/2\right) \tag{10.2}$$

is injective. Given $p \in M^G = M^T$, set $I(p) := \{i : p \in M_i\}$. The cardinality of $I(p)$ is n and the tangential G-representation $\mathcal{T}_pM$ decomposes as

$$\mathcal{T}_pM = \bigoplus_{i \in I(p)} \nu_i|_p$$

where ν_i is the normal bundle of M_i to M and $\nu_i|_p$ is its restriction to p. It follows that

$$w^G(M)|_p = \prod_{i \in I(p)} w^G(\nu_i|_p). \tag{10.3}$$

Since ν_i is orientable of real dimension two, $w_1^G(\nu_i) = 0$ and $w_2^G(\nu_i)$ is the mod 2 reduction of the equivariant Euler class of ν_i. Therefore, we have $w_2^G(\nu_i|_p) = \tau_i|_p$ for $i \in I(p)$. Moreover, $\tau_i|_p = 0$ for $i \notin I(p)$ by a property of equivariant Gysin homomorphism. Thus, the identity (10.3) gives

$$w^G(M)|_p = \prod_{i \in I(p)} (1 + \tau_i)|_p = \prod_{i=1}^{m} (1 + \tau_i)|_p.$$

This together with the injectivity of the restriction map in (10.2) proves the claim.

The forgetful map $H_G^*(M; \mathbb{Z}/2) \to H^*(M; \mathbb{Z}/2)$ takes the equivariant Stiefel-Whitney class $w^G(M)$ to the (ordinary) Stiefel-Whitney class $w(M)$ of M. Since τ_i is of degree two, the above claim shows that $w_{2n}(M)$ is a polynomial in degree two elements. Assume $h_i(M) = 0$ for some $i > 0$. Then $w_{2n}(M) = 0$. The mod 2 reduction of the Euler characteristic $\chi(M)$ of M agrees with $w_{2n}(M)$ evaluated on the mod 2 fundamental class of M. Hence, $w_{2n}(M) = 0$ implies that $\chi(M)$ is even. Here $\chi(M) = \sum_{i=0}^{n} h_i(M)$ and $h_i(M) = h_{n-i}(M)$ by the Poincaré duality, thus $h_{n/2}(M)$ must be even for even n. $\square$

References

[1] G.E. Bredon: Introduction to Compact Transformation Groups, Academic Press, New York–London, 1972.

[2] W. Bruns and J. Herzog: Cohen-Macaulay Rings, revised edition, Cambridge Studies in Adv. Math. **39**, Cambridge Univ. Press, Cambridge, 1998.

[3] V.M. Buchstaber and T.E. Panov: Torus Actions and Their Applications in Topology and Combinatorics, University Lecture Series **24**, Amer. Math. Soc., Providence, R.I., 2002.

[4] V.M. Buchstaber and N. Ray: *Tangential structures on toric manifolds, and connected sums of polytopes*, Internat. Math. Res. Notices **4** (2001), 193–219.

[5] M.W. Davis: *Groups generated by reflections and aspherical manifolds not covered by Euclidean space*, Ann. of Math. **117** (1983), 293–324.

[6] M.W. Davis and T. Januszkiewicz: *Convex polytopes, Coxeter orbifolds and torus actions*, Duke Math. J. **62** (1991), 417–451.

[7] W. Fulton: An Introduction to Toric Varieties, Ann. of Math. Studies **113**, Princeton Univ. Press, Princeton, N.J., 1993.

[8] M. Goresky, R. Kottwitz and R. MacPherson: *Equivariant cohomology, Koszul duality and the localisation theorem*, Invent. Math. **131** (1998), 25–83.

[9] V.W. Guillemin and S. Sternberg: Supersymmetry and Equivariant de Rham Theory, Springer, Berlin–Heidelberg–New York, 1999.

[10] V.W. Guillemin and C. Zara: *One-skeleta, Betti numbers and equivariant cohomology*, Duke Math. J. **107** (2001), 283–349.
[11] A. Hattori and M. Masuda: *Theory of multi-fans*, Osaka J. Math. **40** (2003), 1–68.
[12] W.-Y. Hsiang: Cohomology Theory of Topological Transformation Groups, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band **85**, Springer, Berlin–Heidelberg–New York, 1976.
[13] M. Masuda: *Unitary toric manifolds, multi-fans and equivariant index*, Tohoku Math. J. **51** (1999), 237–265.
[14] M. Masuda: *h-vectors of Gorenstein$*$ simplicial posets*, Adv. Math. **194** (2005), 332–344.
[15] J.R. Munkres: *Topological results in combinatorics*, Michigan Math. J. **31** (1984), 113–128.
[16] E.H. Spanier: Algebraic Topology, McGraw-Hill Book Company, New York, 1966.
[17] R.P. Stanley: *f-vectors and h-vectors of simplicial posets*, J. Pure Appl. Algebra **71** (1991), 319–331.
[18] R.P. Stanley: Combinatorics and Commutative Algebra, second edition, Progress in Math. **41**, Birkhäuser, Boston, 1996.
[19] R.E. Stong: Notes on Cobordism Theory, Princeton Univ. Press, Princeton, N.J., 1968.

Mikiya Masuda
Department of Mathematics
Osaka City University
Sumiyoshi-ku, Osaka 558-8585
Japan
e-mail: masuda@sci.osaka-cu.ac.jp

Taras Panov
Department of Mathematics and Mechanics
Moscow State University
Leninskiye Gory, Moscow 119992
Russia

Institute for Theoretical and Experimental Physics
Moscow 117259
Russia
e-mail: tpanov@mech.math.msu.su