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SEMICOLOCAL PAIRS AND FINITELY COGENERATED INJECTIVE MODULES

Dedicated to Professor Yukio Tsushima on his 60th birthday

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Let P and Q be rings, and ${}_PM$, N_Q and ${}_PV_Q$ a left P-module, a right Q-module and a P-Q-bimodule, respectively. Let $\varphi : M \times N \to V$ be a P-Q-bilinear map. Then we say that $({}_PM, N_Q)$ is a pair with respect to φ or simply a pair (see [12], [14], [10] or [1, Section 24]). For elements $x \in M$, $y \in N$ and for submodules ${}_PX \leq {}_PM$, $Y_Q \leq N_Q$, by xy we denote the element $\varphi(x, y)$, and by XY we denote the P-Q-subbimodule of ${}_PV_Q$ generated by $\{xy | x \in X, y \in Y\}$. A pair $({}_PM, N_Q)$ is said to be colocal if ${}_PMN_Q$ is colocal both as a left P-module and as a right Q-module. In [10] and [7], we studied colocal pairs related to some results in [5] and [4].

We shall define a semicolocal pair $({}_PM, N_Q)$ as a generalization of a colocal pair. A P-Q-bimodule ${}_PU_Q$ is said to be semicolocal if (i) the rings P and Q have complete sets $\{e_1, e_2, \dots, e_m\}$ and $\{f_1, f_2, \dots, f_n\}$ of orthogonal idempotents, respectively such that each e_iU_Q and each ${}_PUf_j$ are colocal modules and (ii) the socle of ${}_PU$ coincides with the socle of U_Q. Moreover a pair $({}_PM, N_Q)$ is said to be semicolocal if ${}_PMN_Q$ is semicolocal. Anh and Menini investigated semicolocal modules with some conditions related to duality (see [2]). In this note, we shall give some generalizations of results of [10] and [7] using the term "semicolocal pairs", and in particular give characterizations of finitely cogenerated injective modules (Theorems 2.4 and 2.5).

Throughout this note, P, Q and R are rings with identity and all modules are unitary. Let M be a module. Then $L \leq M$ ($L < M$) signifies that L is a (proper) submodule of M. By $\mathrm{S}(M)$, $\mathrm{T}(M)$ and $|M|$, we denote the socle, the top and the composition length of M, respectively. Moreover by $\mathrm{Pi}(R)$, we denote the set of primitive idempotents of R. Every homomorphism is written on the side opposite to the scalars.

1. Semicolocal pairs

A module M_R is said to be colocal if M_R has an essential simple socle.

Lemma 1.1. *Let f be an idempotent of R and M_R a colocal module with $\mathrm{S}(M_R) \cong \mathrm{T}(hR_R)$ for some $h \in \mathrm{Pi}(Q)$, where $Q = fRf$. Then Mf_Q is a colocal module with $\mathrm{S}(Mf_Q) = \mathrm{S}(M_R)f = \mathrm{S}(M_R)hQ$.*

Proof. Let $0 \neq x = xf \in \mathrm{S}(M_R)f$ and $0 \neq y = yf \in Mf$. Then $xR = \mathrm{S}(M_R) \leq yR$, so $xQ \leq yQ$. This shows that Mf_Q is a colocal module and $\mathrm{S}(Mf_Q) = \mathrm{S}(M_R)f$. Moreover $\mathrm{S}(M_R)hQ = \mathrm{S}(M_R)f$ holds since $0 \neq \mathrm{S}(M_R)hQ \leq \mathrm{S}(M_R)f$. □

A P-Q-bimodule ${}_PU_Q$ is said to be colocal (resp. faithful) if both ${}_PU$ and U_Q are colocal (resp. faithful).

REMARK 1. For a P-Q-bimodule ${}_PU_Q$, the following hold.
(1) Both $\mathrm{S}({}_PU)$ and $\mathrm{S}(U_Q)$ are subbimodules of ${}_PU_Q$.
(2) If ${}_PU_Q$ is a colocal bimodule, then $\mathrm{S}({}_PU) = \mathrm{S}(U_Q)$.
(3) For any idempotents $e \in P$ and $f \in Q$, $\mathrm{S}(eU_Q) = e\,\mathrm{S}(U_Q)$ and $\mathrm{S}({}_PUf) = \mathrm{S}({}_PU)f$.

A finite set $\{e_1, e_2, \dots, e_n\}$ of orthogonal idempotents of R is said to be complete if $e_1 + e_2 + \cdots + e_n = 1 \in R$.

Let P and Q be rings. Then a P-Q-bimodule ${}_PU_Q$ is said to be semicolocal if the following conditions (i) and (ii) are satisfied.
(i) The rings P and Q have complete sets $\{e_1, e_2, \dots, e_m\}$ and $\{f_1, f_2, \dots, f_n\}$ of orthogonal idempotents, respectively such that each e_iU_Q and each ${}_PUf_j$ are colocal modules.
(ii) $\mathrm{S}({}_PU) = \mathrm{S}(U_Q)$.

Let ${}_PM$ and N_Q be modules and $({}_PM, N_Q)$ a pair and put $U = {}_PMN_Q$. Then the pair $({}_PM, N_Q)$ is said to be semicolocal if ${}_PU_Q$ is a semicolocal bimodule.

REMARK 2. If ${}_PU_Q$ is a bimodule and e and e' are idempotents of P with $eP \cong e'P$, then $eU_Q \cong e'U_Q$. This is easily seen since there exist elements $a = eae'$ and $b = e'be$ in P such that $ab = e$ and $ba = e'$.

REMARK 3. Let P and Q be semiperfect rings. Then by Remark 2, a bimodule ${}_PU_Q$ is semicolocal if and only if for each $g \in \mathrm{Pi}(P)$ and each $h \in \mathrm{Pi}(Q)$ with $gU \neq 0$ and $Uh \neq 0$, gU_Q and ${}_PUh$ are colocal modules and $\mathrm{S}({}_PU) = \mathrm{S}(U_Q)$.

Let R be a semiperfect ring and e and f idempotents of R. Then in [16], Xue defined a Nakayama pair (eR, Rf) as a generalization of an i-pair in [4] (also see [5, Theorem 3.1]). We define a Nakayama pair (eU, Uf) for a bimodule ${}_PU_Q$ and idempotents $e \in P$ and $f \in Q$ (see the condition 4 in [2, Theorem 3.3]). An idempotent e of R is said to be local if eRe is a local ring.

Let P and Q be rings and ${}_PU_Q$ a P-Q-bimodule. First, for local idempotents $g \in P$ and $h \in Q$, (gU, Uh) is called a Nakayama pair if gU_Q and ${}_PUh$ are colocal modules and $\mathrm{S}(gU_Q) \cong \mathrm{T}(hQ_Q)$ and $\mathrm{S}({}_PUh) \cong \mathrm{T}({}_PPg)$. Generally for idempotents $e \in P$ and $f \in Q$ with semiperfect rings ePe and fQf, (eU, Uf) is called a

Nakayama pair if for each $g \in \mathrm{Pi}(ePe)$ (resp. $h \in \mathrm{Pi}(fQf)$) there exists $h \in \mathrm{Pi}(fQf)$ (resp. $g \in \mathrm{Pi}(ePe)$) such that (gU, Uh) is a Nakayama pair (see Remark 2).

Let ${}_PM_R$ be a P-R-bimodule and f an idempotent of R and put $Q = fRf$. Then we always assume that a pair $({}_PM, Rf_Q)$ signifies the pair with respect to the P-Q-bilinear map $\varphi : M \times Rf \to Mf$ defined by $\varphi(x, af) = xaf$; $x \in M$, $af \in Rf$.

Let $({}_PM, N_Q)$ be a pair. Then for any subsets $A \subseteq M$ and $B \subseteq N$, we define submodules $r(A)$ $(= r_N(A)) \leq N_Q$ and $l(B)$ $(= l_M(B)) \leq {}_PM$, as follows: $r(A) = \{y \in N \mid Ay = 0\}$ and $l(B) = \{x \in M \mid xB = 0\}$. We say that the pair $({}_PM, N_Q)$ is left faithful (resp. right faithful) if $l(N) = 0$ (resp. $r(M) = 0$) holds, and $({}_PM, N_Q)$ is faithful if it is left and right faithful.

Let M_R and N_R be semisimple modules. Then by $M_R \sim N_R$, we mean that any simple submodule of M_R is isomorphic to a submodule of N_R and the converse is also satisfied.

Lemma 1.2. *Let M_R be a module and f an idempotent of R such that $({}_PM, Rf_Q)$ is a left faithful pair, where $P = \mathrm{End}\, M_R$, $Q = fRf$. If Mf_Q is colocal, then M_R is colocal with $\mathrm{S}(M_R) = \mathrm{S}(Mf_Q)R$.*

Therefore, if Q is a semiperfect ring and $({}_PM, Rf_Q)$ is a faithful semicolocal pair, then M_R is a direct sum of a finite number of colocal right R-modules and $\mathrm{S}(M_R) \sim \mathrm{T}(fR_R)$ holds, and in particular M_R is finitely cogenerated.

Proof. Let $0 \neq x = xf \in \mathrm{S}(Mf_Q)$ and $0 \neq y \in M_R$. Since $({}_PM, Rf_Q)$ is left faithful, we have $ya \neq 0$ for some $a = af \in Rf$. Hence $xQ \leq yaQ$, so $xR \leq yaR \leq yR$. This shows that M_R is a colocal module with $\mathrm{S}(M_R) = \mathrm{S}(Mf_Q)R$. Assume that Q is a semiperfect ring and $({}_PM, Rf_Q)$ is a faithful semicolocal pair. Since ${}_PMf_Q$ is a faithful bimodule, P and Q have complete sets $G = \{g_1, g_2, \dots, g_m\}$ and $H = \{h_1, h_2, \dots, h_n\}$ of orthogonal primitive idempotents, respectively such that each g_iMf_Q and each ${}_PMfh_j$ are colocal modules and $\mathrm{S}({}_PMf) = \mathrm{S}(Mf_Q)$. Hence for any $g \in G$ (resp. $h \in H$) there exists $h \in H$ (resp. $g \in G$) such that $\mathrm{S}(gMf_Q)h = g\,\mathrm{S}(Mf_Q)h = g\,\mathrm{S}({}_PMf)h \neq 0$, so $\mathrm{S}(gM_R) \cong \mathrm{T}(hR_R)$ by using the first assertion. Thus $\mathrm{S}(M_R) \sim \mathrm{T}(fR_R)$ holds. $\square$

For local idempotents g and h of R, (gR, Rh) is a Nakayama pair if and only if $({}_{gRg}gR, Rh_{hRh})$ is a faithful colocal pair (e.g. see [7, Lemma 3.2]). In the following proposition, the equivalence (1) $\Longleftrightarrow$ (3) is a generalization of this fact.

Proposition 1.3. *Let ${}_PU_Q$ be a bimodule and g and h local idempotents of P and Q, respectively. Then the following are equivalent.*

(1) *(gU, Uh) is a Nakayama pair.*

(2) *Both gU_Q and ${}_PUh$ are colocal and $g\,\mathrm{S}(U_Q)h = g\,\mathrm{S}({}_PU)h \neq 0$ holds.*

(3) *$({}_{gPg}gU, Qh_{hQh})$ is a left faithful pair and $({}_{gPg}gP, Uh_{hQh})$ is a right faithful*

pair and ${}_{gPg}gUh_{hQh}$ is a colocal bimodule.

Proof. (1) $\Longrightarrow$ (2). By assumption, $g\,\mathrm{S}(U_Q)h = \mathrm{S}(gU_Q)h \neq 0$. Since $\mathrm{S}(U_Q)h$ is a non-zero submodule of a colocal module ${}_PUh$, $\mathrm{S}({}_PU)h = \mathrm{S}({}_PUh) \leq \mathrm{S}(U_Q)h$, so $g\,\mathrm{S}({}_PU)h \leq g\,\mathrm{S}(U_Q)h$. Hence $g\,\mathrm{S}({}_PU)h = g\,\mathrm{S}(U_Q)h \neq 0$ by symmetry.

(2) $\Longrightarrow$ (3). By Lemma 1.1.

(3) $\Longrightarrow$ (1). By Lemma 1.2. □

In the following proposition we give characterizations of a semicolocal bimodule ${}_PU_Q$ for semiperfect rings P and Q. The proposition is essentially due to [2, Theorems 3.3 and 3.4] (also see [16, Theorem 3.4], [8, Proposition 1.11] and [9, Theorem 2.2]).

Proposition 1.4. *Let P and Q be semiperfect rings and ${}_PU_Q$ a bimodule such that $gU \neq 0$ and $Uh \neq 0$ for any $g \in \mathrm{Pi}(P)$ and any $h \in \mathrm{Pi}(Q)$. Then the following are equivalent.*

(1) *${}_PU_Q$ is semicolocal.*

(2) *(U, U) $(= (1_PU, U1_Q))$ is a Nakayama pair.*

(3) *Both ${}_PU$ and U_Q have essential socles, and ${}_PU_Q$-duals of simple modules are simple.*

(4) *For each $g \in \mathrm{Pi}(P)$ and each $h \in \mathrm{Pi}(Q)$, gU_Q and ${}_PUh$ are colocal, and ${}_P\mathrm{S}({}_PU) \sim {}_P\mathrm{T}({}_PP)$ and $\mathrm{S}(U_Q)_Q \sim \mathrm{T}(Q_Q)_Q$.*

Proof. (1) $\Longrightarrow$ (2) $\Longrightarrow$ (4). These are clear (see Remark 3).

(4) $\Longrightarrow$ (1). By assumption, for any $h \in \mathrm{Pi}(Q)$ we have $\mathrm{S}({}_PU)h = \mathrm{S}({}_PUh) \leq {}_P\mathrm{S}(U_Q)h$ since ${}_P\mathrm{S}(U_Q)h$ is a non-zero submodule of a colocal module ${}_PUh$. This shows $\mathrm{S}({}_PU) \leq \mathrm{S}(U_Q)$ and by symmetry $\mathrm{S}({}_PU) = \mathrm{S}(U_Q)$.

(1) $\Longrightarrow$ (3). Let $g \in \mathrm{Pi}(P)$. Then we have $\mathrm{Hom}_P(\mathrm{T}({}_PPg), U)_Q \cong gr_U(\mathrm{rad}(P))_Q = g\,\mathrm{S}({}_PU)_Q$. Hence $\mathrm{Hom}_P(\mathrm{T}({}_PPg), U)_Q \cong \mathrm{S}(gU_Q)_Q$ is simple, and by symmetry ${}_P\mathrm{Hom}_Q(\mathrm{T}(hQ_Q), U)$ is simple for any $h \in \mathrm{Pi}(Q)$.

(3) $\Longrightarrow$ (1). Let $g \in \mathrm{Pi}(P)$. Since $\mathrm{Hom}_P(\mathrm{T}({}_PPg), U)_Q \cong g\,\mathrm{S}({}_PU)_Q$, $g\,\mathrm{S}({}_PU)_Q$ is a simple submodule of gU_Q. Hence we have $g\,\mathrm{S}({}_PU)_Q \leq \mathrm{S}(gU_Q)_Q = g\,\mathrm{S}(U_Q)_Q$. This shows $\mathrm{S}({}_PU)_Q \leq \mathrm{S}(U_Q)_Q$ and by symmetry $\mathrm{S}({}_PU) = \mathrm{S}(U_Q)$. Therefore $\mathrm{S}(gU_Q)_Q = g\,\mathrm{S}({}_PU)_Q$ is simple and similarly ${}_P\mathrm{S}({}_PUh)$ is simple for any $h \in \mathrm{Pi}(Q)$. Thus gU_Q and ${}_PUh$ are colocal. □

In Proposition 1.4, the condition (3) is equivalent to the following condition (3)' since in the proof of (3) $\Longrightarrow$ (1), for any $g \in \mathrm{Pi}(P)$, $g\,\mathrm{S}({}_PU)_Q$ is a simple submodule of a colocal module gU_Q and $g\,\mathrm{S}({}_PU)_Q = g\,\mathrm{S}(U_Q)_Q$ holds.

(3)' For each $g \in \mathrm{Pi}(P)$ and each $h \in \mathrm{Pi}(Q)$, gU_Q and ${}_PUh$ are colocal, and ${}_PU_Q$-duals of simple left P-modules are simple.

Lemma 1.5. *Let $({}_PM, N_Q)$ be a semicolocal pair and $Y' < Y \leq N_Q$ with $Y' = rl(Y')$. If Y/Y'_Q is simple, then ${}_Pl(Y')/l(Y)$ is also simple and $Y = rl(Y)$.*

Proof. Put $U = {}_PMN_Q$, $X = l(Y)$ and $X' = l(Y')$. Since ${}_PU_Q$ is semicolocal and Y/Y' is simple, there exist an idempotent $f \in Q$ and an element $y = yf \in Y$ such that ${}_PUf$ is colocal and $Y = yQ+Y' \leq N_Q$. From $rl(Y') = Y' < Y \leq rl(Y)$, we obtain $X = l(Y) < l(Y') = X'$. For any $x \in X'$, the left multiplication map $\hat{x} : Y/Y'_Q \to xY_Q$ by x is an epimorphism. This shows that $xY_Q \leq \mathrm{S}(U_Q)$, so $X'Y_Q \leq \mathrm{S}(U_Q)$. Therefore we have $0 \neq X'y \leq \mathrm{S}(U_Q)f = \mathrm{S}({}_PU)f = \mathrm{S}({}_PUf)$. Thus ${}_PX'y = \mathrm{S}({}_PUf)$ is a simple left P-module. On the other hand, the map $\eta : {}_PX'/X \to {}_PX'y$ defined by $(x + X)\eta = xy$ is a monomorphism. Thus ${}_Pl(Y')/l(Y)$ $(= {}_PX'/X)$ is simple. By the same argument, it follows that $rl(Y)/rl(Y')_Q$ is simple. Hence we have $Y = rl(Y)$ from $rl(Y') = Y' < Y \leq rl(Y)$. □

We say that a pair $({}_PM, N_Q)$ satisfies l-ann (resp. r-ann) if $lr(X) = X$ (resp. $rl(Y) = Y$) hold for any $X \leq {}_PM$ (resp. $Y \leq N_Q$), and $({}_PM, N_Q)$ is dual if $({}_PM, N_Q)$ satisfies l-ann and r-ann.

In the following theorem, the implications (1) $\Longleftrightarrow$ (2) $\Longrightarrow$ (3) are essentially due to [12, Theorem 1.1] (and [14, Theorem 1.1]).

Theorem 1.6. *Let P and Q be rings and $({}_PM, N_Q)$ a faithful semicolocal pair, and consider the following conditions.*

(1) $|N_Q| < \infty$.

(2) $|{}_PM| < \infty$.

(3) *$({}_PM, N_Q)$ is a dual pair.*

Then the implications (1) $\Longleftrightarrow$ (2) $\Longrightarrow$ (3) *hold, and in case either P or Q is a perfect ring, the conditions are equivalent.*

Proof. The implications (1) $\Longleftrightarrow$ (2) $\Longrightarrow$ (3) are easily seen from Lemma 1.5 (see the proof of [10, Theorem 1.4]).

Assume that $({}_PM, N_Q)$ is a dual pair and P is a perfect ring. Then any factor module of ${}_PM$ has finite Goldie dimension (see [3, Corollary 1.6] or [11, Theorem 1.7]). Hence by the proof of [13, Propositions 2.9 and 2.12] (or [11, Lemma 1.9]) ${}_PM$ has finite length. □

2. Finitely cogenerated injective modules

Throughout this section, we always assume that R is a semiperfect ring.

Let M_R and L_R be right R-module modules. Following Harada [6], M is said to be L-simple-injective if for any submodule K of L_R, any homomorphism $\theta : K_R \to M_R$ can be extended to a homomorphism $\eta : L_R \to M_R$. Moreover M is said to be simple-injective if M is N-simple-injective for any right R-module N.

Lemma 2.1 (see [7, Lemma 4.1]). *Let M_R be a finitely cogenerated module with $\mathrm{S}(M_R) \sim \mathrm{T}(fR_R)$ and assume that Mf_Q has finite Loewy length, where f is an idempotent of R and $Q = fRf$. If M_R is R-simple-injective, then M_R is injective.*

Proof. Since $\mathrm{S}(M_R)$ is essential in M_R and $\mathrm{S}(M_R) \sim \mathrm{T}(M_R)$, $l_M(Rf) = 0$ holds. Hence $Lf \neq 0$ for any non-zero submodule $L \leq M_R$ because $Lf = LRf$. Let I be a non-zero right ideal of R and $\theta : I \to M$ a non-zero homomorphism and put $J = \mathrm{rad}(R)$. Then $0 \neq \theta(I(fJf)^k) = \theta(I)(fJf)^k \leq \mathrm{S}(Mf_Q)$ for some integer $k \geq 0$. Put $K = I(fJf)^k R$. Since $\mathrm{S}(Mf_Q)R = l_{Mf}(fJf)R \leq l_M(J) = \mathrm{S}(M_R)$, $\theta(K) \leq \mathrm{S}(M_R)$ holds. By assumption we have $\mathrm{S}(M_R) = S_1 \oplus \cdots \oplus S_n$ for a finite number of simple modules S_i $(1 \leq i \leq n)$. Hence the restriction map $\theta|_K : K \to M$ of θ can be represented as $\theta|_K = \theta_1 + \cdots + \theta_n$ for some homomorphisms $\theta_i : K \to M$ with $\mathrm{Im}\,\theta_i \leq S_i$ $(1 \leq i \leq n)$. Therefore we have $(\theta - \hat{x})(K) = 0$ with left multiplicaltion $\hat{x} : R \to M$ by some element $x \in M$. If $\theta - \hat{x} : I \to M$ is a non-zero homomorphism, then $(\theta - \hat{x})(I)f \neq 0$ (i.e. $k \geq 1$) and $0 \neq (\theta - \hat{x})(I(fJf)^m) \leq \mathrm{S}(Mf_Q)$ for some integer m with $k > m \geq 0$. Iterating the above argument, we have $(\theta - \hat{y})(I) = 0$ for some element $y \in M$. Thus M_R is injective. □

The following lemma is related to [9, Theorem 1.6].

Lemma 2.2 (see [10, Corollary 2.6]). *Let U_Q be a module with $P = \mathrm{End}\, U_Q$ and g and h local idempotents of P and Q, respectively. If gU_Q is a U-simple-injective module and $0 \neq x = gxh \in \mathrm{S}(gU_Q)$, then gU_Q and ${}_PUh$ are colocal modules with $\mathrm{S}(gU_Q) = xQ \cong \mathrm{T}(hQ_Q)$ and $\mathrm{S}({}_PUh) = Px \cong \mathrm{T}({}_PPg)$. Therefore, for any idempotents $e \in P$ and $f \in Q$ with semiperfect rings ePe and fQf, if eU_Q is a U-simple-injective module and $\mathrm{S}(eU_Q)$ is essential in eU_Q with $\mathrm{S}(eU_Q) \sim \mathrm{T}(fQ_Q)$, then (eU, Uf) is a Nakayama pair.*

Proof. By [10, Lemma 2.2] (or [7, Lemma 3.6]), gU_Q is a colocal module with $\mathrm{S}(gU_Q) = xQ$. Let $0 \neq y \in Uh$. Then we have $r_{hQ}(y) \leq hJ = r_{hQ}(x)$, where $J = \mathrm{rad}(Q)$. Hence the map $\theta : yQ \to gU$ via $\theta(yc) = xc$ $(c \in Q)$ is well-defined. Therefore by U-simple-injectivity of gU_Q we have $x = ay$ for some $a \in \mathrm{Hom}_Q(U, gU) = gP$. Thus $x \in Py$, which implies that ${}_PUh$ is a colocal module with $\mathrm{S}({}_PUh) = Px$. □

Lemma 2.3. *Let M be a finitely cogenerated simple-injective right R-module with $\mathrm{S}(M_R) \sim \mathrm{T}(fR_R)$, where f is an idempotent of R, and assume that* $\mathrm{End}\, M$ *is a semiperfect ring. Then $({}_PM, Rf_Q)$ is a faithful semicolocal pair, where $P = \mathrm{End}\, M$ and $Q = fRf$.*

Proof. By Lemma 2.2 for each $g \in \mathrm{Pi}(P)$ (resp. $h \in \mathrm{Pi}(Q)$), there exists $h \in \mathrm{Pi}(Q)$ (resp. $g \in \mathrm{Pi}(P)$) such that (gM, Mh) is a Nakayama pair. Therefore by Lemma 1.1 for a bimodule ${}_PMf_Q$, (gMf, Mfh) is a Nakayama pair. On the other hand by the proof of [10, Lemma 2.1], $({}_{gPg}gM, Rfh_{hQh})$ is faithful. This shows that $({}_PM, Rf_Q)$ is a faithful semicolocal pair by Proposition 1.4. □

Generalizing [10, Theorem 2.7] and [7, Theorem 4.2], we have the following theorem.

Theorem 2.4. *Let M be a finitely cogenerated right R-module with $\mathrm{S}(M_R) \sim \mathrm{T}(fR_R)$, where f is an idempotent of R, and put $P = \mathrm{End}\, M_R$ and $Q = fRf$. Consider the following conditions.*
(1) *M_R is injective.*
(2) *M_R is simple-injective and P is a semiperfect ring.*
(3) *$({}_PM, Rf_Q)$ is a faithful semicolocal pair satisfying r-ann.*
(4) *M_R is R-simple-injective.*

Then the implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) hold. Moreover, in case Mf_Q has finite Loewy length, these conditions are equivalent.

Proof. Note that in case $({}_PM, Rf_Q)$ is left faithful, $l_M(I) = l_M(If)$ holds for any right ideal of I of R.

(1) $\Longrightarrow$ (2). This is clear.

(2) $\Longrightarrow$ (3). By Lemma 2.3 $({}_PM, Rf_Q)$ is a faithful semicolocal pair. Let L_Q be a submodule of Rf_Q. Assume that $L < rl(L)$. Then $(rl(L)R/LR)f \neq 0$, so $(rl(L)R/LR)h \neq 0$ for some $h \in \mathrm{Pi}(Q)$. Hence there exist right ideals I and K of R such that $LR \leq K < I \leq rl(L)R_R$ and $I/K_R \cong \mathrm{T}(hR)$. Therefore $l(L) \geq l(Kf) \geq l(If) \geq lrl(L) = l(L)$. Thus $l_M(K) = l_M(Kf) = l_M(If) = l_M(I)$. On the other hand $I/K_R(\cong \mathrm{T}(hR))$ is isomorphic to a direct summand of $\mathrm{S}(M)$. Hence we have a map $\theta : I \to M$ such that $\mathrm{Im}\,\theta$ simple and $\mathrm{Ker}\,\theta = K$. Then by simple-injectivity of M, there exists an element x of M such that $xc = \theta(c)$ for each $c \in I$. This implies that $x \in l_M(K) - l_M(I)$, a contradiction. Thus $L = rl(L)$ and $({}_PM, Rf_Q)$ satisfies r-ann.

(3) $\Longrightarrow$ (4). Let I be a right ideal of R and $\theta : I \to M$ a homomorphism with $\mathrm{Im}\,\theta$ simple, and put $K = \mathrm{Ker}\,\theta$. Then $I/K \cong \mathrm{T}(hR)$ for some $h \in \mathrm{Pi}(Q)$. Hence we have $Kf < If$ because of $Kh < Ih$. Since $({}_PM, Rf_Q)$ satisfies r-ann, $l_M(K) = l_M(Kf) > l_M(If) = l_M(I)$. Thus we have an element $x \in l_M(K) - l_M(I)$. Since $I/K \cong \mathrm{T}(hR)$, $I = aR + K$ for some $a = ah \in I$. Put $y = \theta(a)$ and $z = xa$. Then y and z are non-zero elements of ${}_P\mathrm{S}(M_R)h$. By assumption ${}_P\mathrm{S}(M_R)f = l_M(J)f \leq l_{Mf}(fJf) = {}_P\mathrm{S}(Mf_Q) = {}_P\mathrm{S}({}_PMf)$ holds; where $J = \mathrm{rad}(R)$, and ${}_P\mathrm{S}({}_PMf)h = {}_P\mathrm{S}({}_PMh)$ is simple. Hence ${}_P\mathrm{S}(M_R)h = {}_P(\mathrm{S}(M_R)f)h = {}_P\mathrm{S}({}_PMf)h$ is simple, so we have $Py = Pz$ $(= {}_P\mathrm{S}(M_R)h)$ and in particular $y = \varphi(z)$ for some $\varphi \in P$. Therefore we have $\theta(a) = \varphi(z) = \varphi(x)a$. Thus M is R-simple-injective.

(4) $\Longrightarrow$ (1). By Lemma 2.1. □

REMARK 4. In Theorem 2.4, the condition "$P = \operatorname{End} M_R$" can be replaced by the condition "${}_P M_R$ is a P-R-bimodule" except for the implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3).

REMARK 5. For the implication (2) $\Longrightarrow$ (3) in Theorem 2.4, we can give another proof by using Propositions 1.4 and 1.3, [11, Lemma 2.4] and [7, Lemma 3.4].

The following theorem is related to [16, Theorem 3.4].

Theorem 2.5. *Let M be a right R-module and f an idempotent of R and put $P = \operatorname{End} M_R$, $Q = fRf$. If $|Rf_Q| < \infty$ is satisfied, then the following are equivalent.*
(1) *M_R is finitely cogenerated injective with $\mathrm{S}(M_R) \sim \mathrm{T}(fR_R)$.*
(2) *$({}_P M, Rf_Q)$ is a faithful semicolocal pair.*

Proof. In case (2) is satisfied, by Theorem 1.6 and Lemma 1.2, $({}_P M, Rf_Q)$ satisfies r-ann and M_R is a finitely cogenerated module with $\mathrm{S}(M_R) \sim \mathrm{T}(fR_R)$. Thus the assertion follows from Theorem 2.4 since $|Q_Q| \leq |Rf_Q| < \infty$. □

The following proposition is related to [5, Theorem 3.1], [4, Theorem 3], [16, Theorem 3.4] and [2, Theorem 3.4]. The "only if" part of this proposition is well-known (see e.g. [1, Theorem 30.4 or Exercise 24.8]). However, for the benefit of the reader we provide a direct proof.

Proposition 2.6. *Let M_R be a finitely generated right R-module and f an idempotent of R and assume that $({}_P M, Rf_Q)$ is a faithful pair with $|Rf_Q| < \infty$, where $P = \operatorname{End} M_R$ and $Q = fRf$. Then the bimodule ${}_P Mf_Q$ defines a Morita duality if and only if $({}_P M, Rf_Q)$ is semicolocal.*

Proof. "If" part. By Theorem 2.5, M_R is injective. Hence $P \cong \operatorname{End} Mf_Q$ by [5, Lemma 2.1] (this lemma is valid for a semiperfect ring R). By assumption, Q is a right artinian ring and Mf_Q is finitely generated. Since $({}_P Mf, Q_Q)$ is a faithful semicolocal pair with $|Q_Q| < \infty$, Mf_Q is injective by Theorem 2.5. Thus the bimodule ${}_P Mf_Q$ defines a Morita duality.

"Only if" part. Since Q is a right artinian ring, Mf_Q is a finitely generated injective cogenerator. Hence by Lemma 2.3, $({}_P Mf, Q_Q)$ or equivalently $({}_P M, Rf_Q)$ is a semicolocal pair. □

REMARK 6. Let $({}_P M, N_Q)$ be a pair which satisfies (i) $({}_P M, N_Q)$ is a semicolocal dual pair with a faithful bimodule ${}_P U_Q$, where ${}_P U_Q = {}_P MN_Q$, (ii) Q is a right artinian ring and (iii) N_Q has finite length. However, this situation does not necessarily imply that ${}_PU_Q$ is a dual bimodule or equivalently $(P, {}_PU_Q, Q)$ is a Baer duality (see [8] and [2], respectively for the definitions a dual bimodule and a Baer duality). Let R be a right artinian ring such that an injective hull E_R of $\mathrm{T}(R_R)$ is not finitely generated, (see e.g. [15, Remark 2.9] for such a ring R). Then by Lemma 2.3 and Theorem 1.6, $({}_PE, R_R)$ is a semicolocal dual pair, where $P = \mathrm{End}\, E_R$. But by Theorem 1.6, $({}_PP, E_R)$ is not a dual pair, so ${}_PE_R$ is not a dual bimodule. Moreover, this example shows that in Proposition 2.6, the assumption "M_R is finitely generated" can not be removed.

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