

COLOCAL PAIRS IN PERFECT RINGS

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Our main aim of the present note is to provide several sufficient conditions for a colocal module L over a left or right perfect ring A to be injective. Also, by developing the previous works [8] and [5], we will extend recent results of Baba [1, Theorems 1 and 2] to left perfect rings and provide simple proofs of them.

Throughout this note, rings are associative rings with identity and modules are unitary modules. For a ring A we denote by $\text{Mod } A$ (resp. $\text{Mod } A^{\text{op}}$) the category of left (resp. right) A -modules, where A^{op} denotes the opposite ring of A . Sometimes, we use the notation ${}_A L$ (resp. L_A) to signify that the module L considered is a left (resp. right) A -module. For a module L , we denote by $\text{soc}(L)$ the socle, by $\text{rad}(L)$ the Jacobson radical, by $E(L)$ an injective envelope and by $\ell(L)$ the composition length of L . For a subset X of a right module L_A and a subset M of A , we set $l_X(M) = \{x \in X \mid xM = 0\}$ and $r_M(X) = \{a \in M \mid Xa = 0\}$. Also, for a subset X of A and a subset M of a left module ${}_A L$ we set $l_X(M) = \{a \in X \mid aM = 0\}$ and $r_M(X) = \{x \in M \mid Xx = 0\}$. We abbreviate the ascending (resp. descending) chain condition as the ACC (resp. DCC).

Recall that a module L is called colocal if it has simple essential socle. We call a bimodule ${}_H U_R$ colocal if both ${}_H U$ and U_R are colocal. Let A be a semiperfect ring with Jacobson radical J . Let L_A be a colocal module with $H = \text{End}_A(L_A)$ and $f \in A$ a local idempotent with $\text{soc}(L_A) \cong fA/fJ$. In case L_A has finite Loewy length, we will show that L_A is injective if and only if ${}_H L f f A f$ is a colocal bimodule and $M = r_{A f}(l_L(M))$ for every submodule M of $A f f A f$. Also, in case A is left or right perfect and $\ell(A f / r_{A f}(L) f A f) < \infty$, we will show that the following are equivalent: (1) L_A is injective; (2) ${}_H L f f A f$ is a colocal bimodule and $r_{A f}(L) = 0$; and (3) ${}_H L f f A f$ is a colocal bimodule and $M = r_{A f}(l_L(M))$ for every submodule M of $A f f A f$.

Recall that a module L_A is called M -injective if for any submodule N of M_A every $\theta : N_A \rightarrow L_A$ can be extended to some $\phi : M_A \rightarrow L_A$. Dually, a module L_A is called M -projective if for any factor module N of M_A every $\theta : L_A \rightarrow N_A$ can be lifted to some $\phi : L_A \rightarrow M_A$. In case L is L -injective (resp. L -projective), L is called quasi-injective (resp. quasi-projective). Let A be a left perfect ring with Jacobson radical J and $e, f \in A$ local idempotents. Assume $\ell(A f / r_{A f}(eA) f A f) < \infty$. Then we will show that eA_A is quasi-injective with $\text{soc}(eA_A) \cong fA/fJ$ if and only if ${}_A E = E({}_A A e / J e)$ is quasi-projective with ${}_A E / J E \cong A f / J f$ (cf. [1, Theorem 1]).

We call a pair (eA, Af) of a right ideal eA and a left ideal Af in A a colocal pair if $e, f \in A$ are local idempotents and ${}_{eAe}eAf_{fAf}$ is a colocal bimodule. We will see that $\ell({}_{eAe}eA/l_{eA}(Af)) = \ell(Af/r_{Af}(eA)_{fAf})$ for every colocal pair (eA, Af) in A . In case $\ell({}_{eAe}eA/l_{eA}(Af)) = \ell(Af/r_{Af}(eA)_{fAf}) < \infty$, a colocal pair (eA, Af) in A is called finite. Let A be a left perfect ring with Jacobson radical J and $e, f_1, f_2, \dots, f_n \in A$ local idempotents. Put $E = E({}_A Ae/Je)$. Assume (eA, Af_i) is a finite colocal pair in A for all $1 \leq i \leq n$. Then we will show that $\text{soc}(eA_A) \cong \bigoplus_{i=1}^n f_i A/f_i J$ if and only if ${}_A E/JE \cong \bigoplus_{i=1}^n Af_i/Jf_i$ (cf. [1, Theorem 2]).

Following Harada [4], we call a module L_A M -simple-injective if for any submodule N of M_A every $\theta : N_A \rightarrow L_A$ with $\text{Im } \theta$ simple can be extended to some $\phi : M_A \rightarrow L_A$. In case L is L -simple-injective, L is called simple-quasi-injective. We will show that a left perfect ring A is left artinian if A satisfies the ascending chain condition on annihilator right ideals and eA_A is simple-quasi-injective for every local idempotent $e \in A$.

1. Preliminaries

In this section, we collect several basic results which we need in later sections. We refer to Bass [2] for perfect rings.

Lemma 1.1. *Let A be a left or right perfect ring and $f \in A$ an idempotent. Assume $\ell(Af_{fAf}) < \infty$. Then ${}_A Af$ has finite Loewy length.*

Proof. Denote by J the Jacobson radical of A . Consider first the case of A being left perfect. Since the descending chain $Af \supset Jf \supset \dots$ terminates, there exists $n \geq 1$ such that $J^n f = J^{n+1} f$. Thus $J^n f = 0$. Assume next that A is right perfect. Then, since the ascending chain $\text{soc}({}_A Af) \subset \text{soc}^2({}_A Af) \subset \dots$ terminates, there exists $n \geq 1$ such that $\text{soc}^n({}_A Af) = Af$. Thus $J^n f = J^n(\text{soc}^n({}_A Af)) = 0$. □

Lemma 1.2. *Let $e \in A$ be an idempotent. Then for a module $L \in \text{Mod } A$ with $r_L(eA) = 0$ the following hold.*

- (1) *If ${}_A L$ is simple, so is ${}_{eAe}eL$.*
- (2) *${}_{eAe}eE({}_A L) \cong E({}_{eAe}eL)$.*
- (3) *The canonical homomorphism ${}_A E({}_A L) \rightarrow {}_A \text{Hom}_{eAe}(eA, eE({}_A L)), x \mapsto (a \mapsto ax)$, is an isomorphism.*

Proof. (1) See e.g. [5, Lemma 1.1].

(2) See e.g. [5, Lemmas 1.2 and 1.3].

(3) See e.g. [5, Lemma 1.3]. □

Recall that a module L_A is called M -injective if for any submodule N of M_A every $\theta : N_A \rightarrow L_A$ can be extended to some $\phi : M_A \rightarrow L_A$. Dually, a module L_A

is called M -projective if for any factor module N of M_A every $\theta : L_A \rightarrow N_A$ can be lifted to some $\phi : L_A \rightarrow M_A$. In case L is L -injective (resp. L -projective), L is called quasi-injective (resp. quasi-projective).

Lemma 1.3 ([6, Theorem 1.1]). *Let $L \in \text{Mod } A^{\text{op}}$ and put $H = \text{End}_A(E(L_A))$. Then L_A is quasi-injective if and only if $HL = L$. In particular, if L_A is quasi-injective, then we have a surjective ring homomorphism $\rho_L : \text{End}_A(E(L_A)) \rightarrow \text{End}_A(L_A)$, $h \mapsto h|_L$.*

The equivalence (1) \Leftrightarrow (2) of the next lemma is due to Wu and Jans [11, Propositions 2.1, 2.2 and 2.4].

Lemma 1.4 ([11]). *Let A be a left perfect ring. Then for a module $L \in \text{Mod } A$ the following are equivalent.*

- (1) ${}_A L$ is indecomposable quasi-projective.
- (2) There exist a local idempotent $f \in A$ and a two-sided ideal I of A such that ${}_A L \cong Af/I f$.
- (3) There exists a local idempotent $f \in A$ such that ${}_A L \cong Af/l_A(L)f$.

Proof. (1) \Rightarrow (2). By [11, Proposition 2.4] there exists an epimorphism $\pi : {}_A Af \rightarrow {}_A L$ with $f \in A$ a local idempotent. Put $K = \text{Ker } \pi$. Then by [11, Proposition 2.2] $KfAf = K$ and ${}_A L \cong Af/I f$ with $I = KfA$ a two-sided ideal of A .

(2) \Rightarrow (1). Since ${}_{A/I} Af/I f \cong {}_{A/I} (A/I)f$ is projective, ${}_A Af/I f$ is quasi-projective.

(2) \Rightarrow (3). Since $I f = l_A(Af/I f)f$, ${}_A L \cong Af/l_A(L)f$.

(3) \Rightarrow (2). Obvious. □

Recall that an object L of an abelian category \mathcal{A} in which arbitrary direct products exist is called linearly compact if for any inverse system of epimorphisms $\{\pi_\lambda : L \rightarrow L_\lambda\}_{\lambda \in \Lambda}$ in \mathcal{A} the induced morphism $\varprojlim \pi_\lambda : L \rightarrow \varprojlim L_\lambda$ is epic. In case $\mathcal{A} = \text{Mod } A$, there is an equivalent definition of linear compactness. Recall that, for a family of submodules $\{L_\lambda\}_{\lambda \in \Lambda}$ in a module ${}_A L$, a system of congruences $\{x \equiv x_\lambda \pmod{L_\lambda}\}_{\lambda \in \Lambda}$ is said to be finitely solvable if for any nonempty finite subset F of Λ there exists $x_F \in L$ such that $x_F \equiv x_\lambda \pmod{L_\lambda}$ for all $\lambda \in F$, and to be solvable if there exists $x_0 \in L$ such that $x_0 \equiv x_\lambda \pmod{L_\lambda}$ for all $\lambda \in \Lambda$.

For the benefit of the reader, we include a proof of the following.

Proposition 1.5. *For a module $L \in \text{Mod } A$ the following are equivalent.*

- (1) ${}_A L$ is linearly compact.
- (2) For any family of submodules $\{L_\lambda\}_{\lambda \in \Lambda}$ in ${}_A L$, every finitely solvable system of congruences $\{x \equiv x_\lambda \pmod{L_\lambda}\}_{\lambda \in \Lambda}$ is solvable.

Proof. (1) \Rightarrow (2). Let $\{L_\lambda\}_{\lambda \in \Lambda}$ be a family of submodules in L and $\{x \equiv x_\lambda \pmod{L_\lambda}\}_{\lambda \in \Lambda}$ a finitely solvable system of congruences. Denote by $\phi_\lambda : L \rightarrow L/L_\lambda$ the canonical epimorphism for each $\lambda \in \Lambda$ and set $\phi : L \rightarrow \prod_{\lambda \in \Lambda} L/L_\lambda$, $x \mapsto (\phi_\lambda(x))$. Put $\hat{x} = (\phi_\lambda(x_\lambda)) \in \prod_{\lambda \in \Lambda} L/L_\lambda$. We claim that $\hat{x} \in \text{Im } \phi$. Let \mathcal{F} be the directed set of nonempty finite subsets of Λ . For each $F \in \mathcal{F}$, denote by $p_F : \prod_{\lambda \in \Lambda} L/L_\lambda \rightarrow \prod_{\lambda \in F} L/L_\lambda$ the projection and put $\hat{x}_F = p_F(\hat{x}) \in \prod_{\lambda \in F} L/L_\lambda$ and $X_F = (p_F \circ \phi)^{-1}(A\hat{x}_F)$. Note that for any $F \in \mathcal{F}$, since $\{x \equiv x_\lambda \pmod{L_\lambda}\}_{\lambda \in \Lambda}$ is finitely solvable, $p_F \circ \phi : L \rightarrow \prod_{\lambda \in F} L/L_\lambda$ induces an epimorphism $\varphi_F : X_F \rightarrow A\hat{x}_F$. For each $F \in \mathcal{F}$, take a push-out of $\varphi_F : X_F \rightarrow A\hat{x}_F$ along with the inclusion $X_F \rightarrow L$:

$$\begin{array}{ccc} X_F & \longrightarrow & L \\ \varphi_F \downarrow & & \downarrow \pi_F \\ A\hat{x}_F & \longrightarrow & Y_F. \end{array}$$

Then we get an inverse system of epimorphisms $\{\pi_F : L \rightarrow Y_F\}_{F \in \mathcal{F}}$. Also, since \varprojlim is left exact, we get a pull-back square

$$\begin{array}{ccc} \varprojlim X_F & \longrightarrow & L \\ \varprojlim \varphi_F \downarrow & & \downarrow \varprojlim \pi_F \\ \varprojlim A\hat{x}_F & \longrightarrow & \varprojlim Y_F. \end{array}$$

Since L is linearly compact, $\varprojlim \pi_F$ is epic, so is $\varprojlim \varphi_F$. Note that $\varprojlim X_F \xrightarrow{\sim} \bigcap_{F \in \mathcal{F}} X_F$. Also, $\varprojlim p_F : \prod_{\lambda \in \Lambda} L/L_\lambda \rightarrow \varprojlim \prod_{\lambda \in F} L/L_\lambda$ is an isomorphism and hence induces an isomorphism $A\hat{x} \xrightarrow{\sim} \varprojlim A\hat{x}_F$. It follows that $\phi(\bigcap_{F \in \mathcal{F}} X_F) = A\hat{x}$. Thus $\hat{x} \in \text{Im } \phi$.

(2) \Rightarrow (1). Let $\{\pi_\lambda : L \rightarrow L_\lambda\}_{\lambda \in \Lambda}$ be an inverse system of epimorphisms in $\text{Mod } A$. We may consider $\varprojlim L_\lambda$ as a submodule of $\prod_{\lambda \in \Lambda} L_\lambda$. Let $(x_\lambda) \in \varprojlim L_\lambda$ and for each $\lambda \in \Lambda$ choose $y_\lambda \in L$ with $\pi_\lambda(y_\lambda) = x_\lambda$. Then, since for any nonempty finite subset F of Λ there exists $\lambda_0 \in \Lambda$ such that $\lambda_0 \geq \lambda$ for all $\lambda \in F$, the system of congruences $\{x \equiv y_\lambda \pmod{\text{Ker } \pi_\lambda}\}_{\lambda \in \Lambda}$ is finitely solvable and thus solvable. Hence $\varprojlim \pi_\lambda : L \rightarrow \varprojlim L_\lambda$ is an epimorphism. □

Let ${}_H U_R$ be a bimodule and $K \in \text{Mod } R^{\text{op}}$. For a pair of a subset X of $(K_R)^*$ and a subset M of K_R , we set $r_M(X) = \{a \in M \mid h(a) = 0 \text{ for all } h \in X\}$ and $l_X(M) = \{h \in X \mid h(a) = 0 \text{ for all } a \in M\}$, where $(\)^* = \text{Hom}_R(-, {}_H U_R)$.

The next lemma is due essentially to [7, Lemma 4].

Lemma 1.6. *Let ${}_H U_R$ be a bimodule and $K \in \text{Mod } R^{\text{op}}$ a module such that U_R is K -injective. Assume $X = l_{K^*}(r_K(X))$ for every submodule X of $(K_R)^*$. Then $(K_R)^*$ is linearly compact.*

Proof. Let $\{\pi_\lambda : K^* \rightarrow X_\lambda\}_{\lambda \in \Lambda}$ be an inverse system of epimorphisms in $\text{Mod } H$. For $\lambda \in \Lambda$, put $Y_\lambda = \text{Ker } \pi_\lambda$ and $M_\lambda = r_K(Y_\lambda)$, and let $j_\lambda : M_\lambda \rightarrow K$ be the inclusion. Then for each $\lambda \in \Lambda$, since $\text{Ker } j_\lambda^* \cong l_{K^*}(M_\lambda) = Y_\lambda$, and since $j_\lambda^* : K^* \rightarrow M_\lambda^*$ is epic, there exists an isomorphism $\phi_\lambda : M_\lambda^* \rightarrow X_\lambda$ with $\pi_\lambda = \phi_\lambda \circ j_\lambda^*$. Since $\varinjlim j_\lambda$ is monic, $\varprojlim j_\lambda^* \cong (\varinjlim j_\lambda)^*$ is epic. Also, $\varprojlim \phi_\lambda$ is an isomorphism. Thus $\varprojlim \pi_\lambda = (\varprojlim \phi_\lambda) \circ (\varprojlim j_\lambda^*)$ is epic. \square

Corollary 1.7. *Let A be a left or right perfect ring. Assume A_A is injective and $I = l_A(r_A(I))$ for every left ideal I of A . Then A is quasi-Frobenius.*

Proof. It follows by Lemma 1.6 that ${}_A A$ is linearly compact. Thus by [10, Propositions 2.9 and 2.12] A is left noetherian. \square

2. Bilinear maps into colocal bimodules

In this section, as further preliminaries, we modify results of [8, Section 1]. For a left H -module ${}_H L$, a right R -module K_R and an H - R -bimodule ${}_H U_R$, we call a map $\varphi : {}_H L \times K_R \rightarrow {}_H U_R$ H - R -bilinear if $K_R \rightarrow U_R, a \mapsto \varphi(x, a)$, is R -linear for every $x \in L$ and ${}_H L \rightarrow {}_H \text{Hom}_R(K_R, {}_H U_R), x \mapsto (a \mapsto \varphi(x, a))$, is H -linear.

Throughout this section, $\varphi : {}_H L \times K_R \rightarrow {}_H U_R$ is a fixed H - R -bilinear map. For a pair of a subset X of L and a subset M of K we set $r_M(X) = \{a \in M \mid \varphi(x, a) = 0 \text{ for all } x \in X\}$ and $l_X(M) = \{x \in X \mid \varphi(x, a) = 0 \text{ for all } a \in M\}$. We denote by $\mathcal{A}_l(L, K)$ the lattice of submodules X of ${}_H L$ with $X = l_L(r_K(X))$ and by $\mathcal{A}_r(L, K)$ the lattice of submodules M of K_R with $M = r_K(l_L(M))$.

REMARKS (see e.g. [3, Part I] for details). (1) Let X be a subset of L . Then $\varphi(X, r_K(X)) = 0$ implies $X \subset l_L(r_K(X))$ and thus $r_K(l_L(r_K(X))) \subset r_K(X)$. Also, $\varphi(l_L(r_K(X)), r_K(X)) = 0$ implies $r_K(X) \subset r_K(l_L(r_K(X)))$. Thus $r_K(X) = r_K(l_L(r_K(X)))$ and $r_K(X) \in \mathcal{A}_r(L, K)$.

(2) Let X be a subset of L . For any $Y \in \mathcal{A}_l(L, K)$ with $X \subset Y, l_L(r_K(X)) \subset l_L(r_K(Y)) = Y$. Thus $l_L(r_K(X))$ is the smallest module in $\mathcal{A}_l(L, K)$ containing X .

(3) Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family of submodules of ${}_H L$. For any $\mu \in \Lambda$, since $\bigcap_{\lambda \in \Lambda} X_\lambda \subset X_\mu \subset \sum_{\lambda \in \Lambda} X_\lambda, r_K(\sum_{\lambda \in \Lambda} X_\lambda) \subset r_K(X_\mu) \subset r_K(\bigcap_{\lambda \in \Lambda} X_\lambda)$. Thus $r_K(\sum_{\lambda \in \Lambda} X_\lambda) \subset \bigcap_{\lambda \in \Lambda} r_K(X_\lambda)$ and $\sum_{\lambda \in \Lambda} r_K(X_\lambda) \subset r_K(\bigcap_{\lambda \in \Lambda} X_\lambda)$. Let $a \in \bigcap_{\lambda \in \Lambda} r_K(X_\lambda)$. Since $\varphi(X_\lambda, a) = 0$ for all $\lambda \in \Lambda$, and since ${}_H L \rightarrow {}_H U, x \mapsto \varphi(x, a)$, is H -linear, $\varphi(\sum_{\lambda \in \Lambda} X_\lambda, a) = 0$ and $a \in r_K(\sum_{\lambda \in \Lambda} X_\lambda)$. Thus $r_K(\sum_{\lambda \in \Lambda} X_\lambda) = \bigcap_{\lambda \in \Lambda} r_K(X_\lambda)$.

(4) Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family of submodules of ${}_H L$ with the $X_\lambda \in \mathcal{A}_l(L, K)$. Then by (3) $\bigcap_{\lambda \in \Lambda} X_\lambda = \bigcap_{\lambda \in \Lambda} l_L(r_K(X_\lambda)) = l_L(\sum_{\lambda \in \Lambda} r_K(X_\lambda))$. Thus $r_K(\bigcap_{\lambda \in \Lambda} X_\lambda) = r_K(l_L(\sum_{\lambda \in \Lambda} r_K(X_\lambda)))$ and by (2) $r_K(\bigcap_{\lambda \in \Lambda} X_\lambda)$ is the smallest module in $\mathcal{A}_r(L, K)$ containing $\sum_{\lambda \in \Lambda} r_K(X_\lambda)$, so that $r_K(\bigcap_{\lambda \in \Lambda} X_\lambda) = \sum_{\lambda \in \Lambda} r_K(X_\lambda)$ whenever $\sum_{\lambda \in \Lambda} r_K(X_\lambda) \in \mathcal{A}_r(L, K)$.

(5) We have an anti-isomorphism of lattices $\mathcal{A}_l(L, K) \rightarrow \mathcal{A}_r(L, K)$, $X \mapsto r_K(X)$. In particular, $\mathcal{A}_l(L, K)$ satisfies the ACC (resp. DCC) if and only if $\mathcal{A}_r(L, K)$ satisfies the DCC (resp. ACC).

Recall that a module is called colocal if it has simple essential socle. We call a bimodule ${}_H U_R$ colocal if both ${}_H U$ and U_R are colocal modules.

Lemma 2.1. *Let ${}_H U_R$ be a colocal bimodule. Then $\text{soc}({}_H U) = \text{soc}(U_R)$.*

Proof. Since $\text{soc}({}_H U)$ is a subbimodule of ${}_H U_R$, $\text{soc}(U_R) \subset \text{soc}({}_H U)$. Similarly, $\text{soc}({}_H U) \subset \text{soc}(U_R)$. Thus $\text{soc}({}_H U) = \text{soc}(U_R)$. □

Throughout the rest of this section, ${}_H U_R$ is assumed to be a colocal bimodule with ${}_H S_R = \text{soc}({}_H U) = \text{soc}(U_R)$, and $(\)^*$ denotes both the U -dual functors.

Lemma 2.2. *The following hold.*

- (1) *The canonical ring homomorphisms $H \rightarrow \text{End}_R(S_R)$ and $R \rightarrow \text{End}_H({}_H S)^{\text{op}}$ are surjective.*
- (2) *$({}_H S)^* \cong S_R$ and $(S_R)^* \cong {}_H S$.*

Proof. (1) Let $0 \neq u \in S$. Then $S = Hu = uR$. For any $h \in \text{End}_R(S_R)$, $h(u) = au$ for some $a \in H$ and $h(ub) = h(u)b = (au)b = a(ub)$ for all $b \in R$. Thus the canonical ring homomorphism $H \rightarrow \text{End}_R(S_R)$ is surjective. Similarly, the canonical ring homomorphism $R \rightarrow \text{End}_H({}_H S)^{\text{op}}$ is surjective.

(2) Let $\pi : R_R \rightarrow S_R$ be an epimorphism. We have a monomorphism $\mu : (S_R)^* \rightarrow {}_H U$ such that $\mu(h) = (\pi^*(h))(1)$ for $h \in (S_R)^*$. Put $u = \pi(1)$. Then $\mu(h) = h(u) \in S$ for all $h \in (S_R)^*$ and $\text{Im } \mu = {}_H S$, so that $(S_R)^* \cong {}_H S$. Similarly, $({}_H S)^* \cong S_R$. □

Lemma 2.3. *Let $N \subset M$ be submodules of K_R with $N = r_K(l_L(N))$ and M/N_R simple. Then the following hold.*

- (1) *$M/N \cong S_R$ and $l_L(N)/l_L(M) \cong (M/N)^* \cong {}_H S$.*
- (2) *$M = r_K(l_L(M))$.*

Proof. (1) Since $M \neq N = r_K(l_L(N))$, $l_L(M) \subset l_L(N)$ with $l_L(N)/l_L(M) \neq 0$. Let $j : N_R \rightarrow M_R$ be the inclusion. Then we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & l_L(M) & \longrightarrow & L & \longrightarrow & M^* \\
 & & \downarrow & & \parallel & & \downarrow j^* \\
 0 & \longrightarrow & l_L(N) & \longrightarrow & L & \longrightarrow & N^*.
 \end{array}$$

Thus $0 \neq l_L(N)/l_L(M)$ embeds in $\text{Ker } j^* \cong (M/N)^*$. Hence $(M/N)^* \neq 0$, so that $M/N \cong S_R$ and by Lemma 2.2(2) $(M/N)^* \cong {}_H S$.

(2) Since $l_L(M) \subset l_L(N)$ with $l_L(N)/l_L(M)$ simple, one can apply the part (1) to conclude that $r_K(l_L(M))/r_K(l_L(N))$ is simple. Thus, since $r_K(l_L(N)) = N \subset M \subset r_K(l_L(M))$ with both M/N and $r_K(l_L(M))/r_K(l_L(N))$ simple, it follows that $M = r_K(l_L(M))$. □

Lemma 2.4. *Let M be a submodule of K_R with $r_K(L) \subset M$ and $\ell(M/r_K(L)_R) < \infty$. Then the following hold.*

- (1) *Every composition factor of $M/r_K(L)_R$ is isomorphic to S_R .*
- (2) *$M = r_K(l_L(M))$.*

Proof. Since $r_K(L) = r_K(l_L(r_K(L)))$, Lemma 2.3 enables us to make use of induction on $\ell(M/r_K(L)_R)$. □

Lemma 2.5 ([8, Lemma 1.3]). $\ell({}_H L/l_L(K)) = \ell(K/r_K(L)_R)$.

Proof. By symmetry we may assume $\ell({}_H L/l_L(K)) < \infty$. Let $l_L(K) = L_0 \subset L_1 \subset \dots \subset L_n = L$ be a chain of submodules of ${}_H L$ with the L_{i+1}/L_i simple. Then by Lemma 2.3 we get a chain of submodules $r_K(L) = r_K(L_n) \subset \dots \subset r_K(L_1) \subset r_K(L_0) = K$ in K_R with the $r_K(L_i)/r_K(L_{i+1})$ simple. □

Lemma 2.6. *Assume R is left perfect. Then the following are equivalent.*

- (1) $\ell(K/r_K(L)_R) < \infty$.
- (2) $\mathcal{A}_r(L, K)$ satisfies both the ACC and the DCC.
- (3) $\mathcal{A}_r(L, K)$ satisfies the ACC.

Proof. (1) \Rightarrow (2) \Rightarrow (3). Obvious.

(3) \Rightarrow (1). It follows by Lemma 2.4 that there exists a maximal element K_0 in the set of submodules M of K_R with $r_K(L) \subset M$ and $\ell(M/r_K(L)_R) < \infty$. We claim $K_0 = K$. Otherwise, there exists a submodule M of K_R with $K_0 \subset M$ and M/K_0 simple, a contradiction. □

3. Simple-injective colocal modules

Throughout the rest of this note, A stands for a ring with Jacobson radical J . For any pair of a right module L_A and a left ideal K of A , we have a canonical bilinear map ${}_H L \times K_R \rightarrow {}_H L K_R$, $(x, a) \mapsto xa$, where $H = \text{End}_A(L_A)$ and $R = \text{End}_A(AK)^{\text{op}}$, so that, in case ${}_H L K_R$ is a colocal bimodule, we can apply results of the preceding section.

Lemma 3.1. *Let $L \in \text{Mod } A^{\text{op}}$ be a colocal module and $f \in A$ a local idempotent with $\text{soc}(L_A) \cong fA/fJ$. Then the following hold.*

- (1) $l_L(Af) = 0$.
- (2) $l_L(I f) = l_L(I)$ for every right ideal I of A .
- (3) Lf_{fAf} is colocal with $\text{soc}(Lf_{fAf}) = \text{soc}(L_A)f$.

Proof. (1) For any $0 \neq x \in L$, since $\text{soc}(L_A) \subset xA$, $0 \neq \text{soc}(L_A)f \subset xAf$ and thus $x \notin l_L(Af)$.

(2) We have $l_L(I) \subset l_L(I f)$. For any $x \in l_L(I f)$, since $xIAf = xIf = 0$, by the part (1) $xI \subset l_L(Af) = 0$ and $x \in l_L(I)$. Thus $l_L(I f) \subset l_L(I)$.

(3) Let $0 \neq x \in \text{soc}(L_A)f$. For any $0 \neq y \in Lf$, since $xA \subset yA$, $xAf = yAf \subset yAf = yfAf$. Thus Lf_{fAf} is colocal and $\text{soc}(Lf_{fAf}) = \text{soc}(L_A)f$. □

Lemma 3.2. *Let $L \in \text{Mod } A^{\text{op}}$ and $f \in A$ a local idempotent. Then the following are equivalent.*

- (1) L_A is colocal with $\text{soc}(L_A) \cong fA/fJ$.
- (2) Lf_{fAf} is colocal and $l_L(Af) = 0$.

Proof. (1) \Rightarrow (2). By (3) and (1) of Lemma 3.1.

(2) \Rightarrow (1). Since by Lemma 1.2(2) $E(L_A)f_{fAf} \cong E(Lf_{fAf}) \cong E(fAf/fJf_{fAf}) \cong E(fA/fJ_A)f_{fAf}$, by Lemma 1.2(3) $E(L_A) \cong \text{Hom}_{fAf}(Af, E(L_A)f)_A \cong \text{Hom}_{fAf}(Af, E(fA/fJ_A)f)_A \cong E(fA/fJ_A)$. Thus L_A is colocal with $\text{soc}(L_A) \cong fA/fJ$. □

Corollary 3.3. *Let $e, f \in A$ be local idempotents. Then the following are equivalent.*

- (1) $eA/l_{eA}(Af)_A$ is colocal with $\text{soc}(eA/l_{eA}(Af)_A) \cong fA/fJ$.
- (2) eAf_{fAf} is colocal.

Proof. Put $L = eA/l_{eA}(Af)_A$. Then $l_L(Af) = 0$ and, since $l_{eA}(Af)f = 0$, $Lf_{fAf} \cong eAf_{fAf}$. Thus Lemma 3.2 applies. □

Following Harada [4], we call a module L_A M -simple-injective if for any submodule N of M_A every $\theta : N_A \rightarrow L_A$ with $\text{Im } \theta$ simple can be extended to some $\phi : M_A \rightarrow L_A$. In case L is L -simple-injective, L is called simple-quasi-injective.

Lemma 3.4. *Let $L \in \text{Mod } A^{\text{op}}$ be a colocal module and put $H = \text{End}_A(L_A)$. Let $f \in A$ be a local idempotent with $\text{soc}(L_A) \cong fA/fJ$. Then the following hold.*

- (1) If L_A is A -simple-injective, then $M = r_{Af}(l_L(M))$ for every submodule M of Af_{fAf} .
- (2) If ${}_H Lf_{fAf}$ is a colocal bimodule and $M = r_{Af}(l_L(M))$ for every submodule M of Af_{fAf} , then L_A is A -simple-injective.

Proof. (1) Let M be a submodule of Af_{fAf} and put $N = r_{Af}(l_L(M))$. We claim $M = N$. Suppose otherwise. Note first that $l_L(N) = l_L(M)$. Since $(NA/MA)f \cong N/M \neq 0$, there exist right ideals K, I of A such that $MA \subset K \subset I \subset NA$ and $I/K \cong fA/fJ \cong \text{soc}(L_A)$. Then we have $\theta : I_A \rightarrow L_A$ with $\text{Im } \theta = \text{soc}(L_A)$ and $\text{Ker } \theta = K$. Let $\mu : I_A \rightarrow A_A$ be the inclusion. There exists $\phi : A_A \rightarrow L_A$ with $\phi \circ \mu = \theta$. Then $\phi(1)I = \phi(I) = \theta(I) \neq 0$ and $\phi(1)K = \phi(K) = \theta(K) = 0$. Thus $\phi(1) \in l_L(K)$ and $\phi(1) \notin l_L(I)$. Since $l_L(N) = l_L(NA) \subset l_L(I) \subset l_L(K) \subset l_L(MA) = l_L(M)$, $l_L(K) \neq l_L(I)$ implies $l_L(M) \neq l_L(N)$, a contradiction.

(2) Let I be a nonzero right ideal of A and $\mu : I_A \rightarrow A_A$ the inclusion. Let $\theta : I_A \rightarrow L_A$ with $\text{Im } \theta = \text{soc}(L_A)$ and put $K = \text{Ker } \theta$. Since by Lemma 1.2(1) $I f / K f f_{fAf} \cong (I/K) f_{fAf}$ is simple, by Lemma 2.3(1) so is ${}_H l_L(Kf) / l_L(I f)$. Let $a \in I f$ with $a \notin K f$. Then, since $l_L(Kf)a \neq 0$ and $l_L(I f)a = 0$, ${}_H l_L(Kf)a$ is simple. Thus by Lemmas 2.1 and 3.1(3) $l_L(Kf)a = \text{soc}(L f_{fAf}) = \text{soc}(L_A)f$, so that $\theta(a) = \theta(a f) = \theta(a)f = xa$ for some $x \in l_L(Kf)$. Define $\phi : A_A \rightarrow L_A$ by $1 \mapsto x$. Then, since by Lemma 3.1(2) $x \in l_L(Kf) = l_L(K)$, and since $I = K + aA$, we have $\phi \circ \mu = \theta$. □

Lemma 3.5. *Let $L \in \text{Mod } A^{\text{op}}$ be a colocal module and put $H = \text{End}_A(L_A)$. Let $f \in A$ be a local idempotent with $\text{soc}(L_A) \cong fA/fJ$. Then the following hold.*

- (1) *If L_A is simple-quasi-injective, then ${}_H L f_{fAf}$ is a colocal bimodule and $l_L(Af) = 0$.*
- (2) *If L_A is A -simple-injective, then $r_{Af}(L) = 0$ and $r_A(L/LJ_A) \subset l_A(\text{soc}(A Af))$.*

Proof. (1) By Lemma 3.2 $L f_{fAf}$ is colocal and $l_L(Af) = 0$. Let $0 \neq x \in \text{soc}(L_A)f$. We claim that $x \in Hy$ for all $0 \neq y \in Lf$. Note that $r_{fA}(x) = fJ$. Let $0 \neq y \in Lf$. Then $r_{fA}(y) \subset fJ = r_{fA}(x)$ and we have $\theta : yA_A \rightarrow xA_A = \text{soc}(L_A)$, $ya \mapsto xa$. Let $\mu : \text{soc}(L_A) \rightarrow L_A$ and $\nu : yA_A \rightarrow L_A$ be inclusions. There exists $h \in H$ with $h \circ \nu = \mu \circ \theta$, so that $x = h(y) \in Hy$. Thus ${}_H L f$ is colocal.

(2) By Lemma 3.4(1) $r_{Af}(L) = r_{Af}(l_L(0)) = 0$. Next, let $a \in r_A(L/LJ)$. Since $La \subset LJ$, $La(\text{soc}(A Af)) \subset LJ(\text{soc}(A Af)) = 0$. Thus $a(\text{soc}(A Af)) \subset r_{Af}(L) = 0$ and $a \in l_A(\text{soc}(A Af))$. □

Lemma 3.6 ([5, Lemma 3.3]). *Let $L \in \text{Mod } A^{\text{op}}$ be a simple-quasi-injective module with $\text{soc}(L_A) \neq 0$. Assume $\text{End}_A(L_A)$ is a local ring. Then $\text{soc}(L_A)$ is simple.*

Proof. Let S be a simple submodule of $\text{soc}(L_A)$. Suppose to the contrary that $S \neq \text{soc}(L_A)$. Let $\pi : \text{soc}(L_A) \rightarrow S_A$ be a projection and $\mu : \text{soc}(L_A) \rightarrow L_A$, $\nu : S_A \rightarrow L_A$ inclusions. There exists $\phi : L_A \rightarrow L_A$ with $\phi \circ \mu = \nu \circ \pi$. Since π is not monic, ϕ is not an isomorphism. Thus $\phi \in \text{rad } \text{End}_A(L_A)$ and $(\text{id}_L - \phi)$ is a unit in $\text{End}_A(L_A)$, so that for $0 \neq x \in S$, since $\phi(x) = \pi(x) = x$, $(\text{id}_L - \phi)(x) = 0$ and $x = 0$, a contradiction. □

4. Injectivity of colocal modules

In this section, by extending the previous results [8, Theorems 2.7, 2.8 and Proposition 2.9], we provide several sufficient conditions for a colocal module over a left or right perfect ring A to be injective.

Lemma 4.1 ([5, Lemma 3.4]). *Let A be a semiperfect ring and $L \in \text{Mod } A^{\text{op}}$ an A -simple-injective colocal module of finite Loewy length. Then L_A is injective.*

Proof. Let I be a right ideal of A and $\mu : I_A \rightarrow A_A$ the inclusion. Let $\theta : I_A \rightarrow L_A$. We make use of induction on the Loewy length of $\theta(I)$ to show the existence of $\phi : A_A \rightarrow L_A$ with $\theta = \phi \circ \mu$. Let $n = \min\{k \geq 0 \mid \theta(I)J^k = 0\}$. We may assume $n > 0$. Since $\text{soc}(L_A)$ is simple, $\text{soc}(L_A) = \theta(I)J^{n-1} = \theta(IJ^{n-1})$. Let μ_1 and θ_1 denote the restrictions of μ and θ to IJ^{n-1} , respectively. Then $\text{Im } \theta_1 = \text{soc}(L_A)$ and there exists $\phi_1 : A_A \rightarrow L_A$ with $\phi_1 \circ \mu_1 = \theta_1$. Since $(\theta - \phi_1 \circ \mu)(I)J^{n-1} = 0$, by induction hypothesis there exists $\phi_2 : A_A \rightarrow L_A$ with $\phi_2 \circ \mu = \theta - \phi_1 \circ \mu$. Thus $\theta = (\phi_1 + \phi_2) \circ \mu$. □

Theorem 4.2. *Let A be a semiperfect ring. Let $L \in \text{Mod } A^{\text{op}}$ be a colocal module of finite Loewy length and put $H = \text{End}_A(L_A)$. Let $f \in A$ be a local idempotent with $\text{soc}(L_A) \cong fA/fJ$. Then the following are equivalent.*

- (1) L_A is injective.
- (2) ${}_H L f_{fA_f}$ is a colocal bimodule and $M = r_{A_f}(l_L(M))$ for every submodule M of $A f_{fA_f}$.

Proof. (1) \Rightarrow (2). By Lemmas 3.5(1) and 3.4(1).

(2) \Rightarrow (1). By Lemmas 3.4(2) and 4.1. □

Corollary 4.3. *Let A be a semiperfect ring. Let $L \in \text{Mod } A^{\text{op}}$ be a colocal module of finite Loewy length and put $H = \text{End}_A(L_A)$. Let $f \in A$ be a local idempotent with $\text{soc}(L_A) \cong fA/fJ$. Assume ${}_H L f_{fA_f}$ is a colocal bimodule and $M = r_{A_f}(l_L(M))$ for every submodule M of $A f_{fA_f}$ with $r_{A_f}(L) \subset M$. Then L_A is quasi-injective.*

Proof. Put $I = r_A(L)$. Then by Theorem 4.2 $L_{A/I}$ is injective, so that L_A is quasi-injective. □

Theorem 4.4. *Let A be a left or right perfect ring. Let $L \in \text{Mod } A^{\text{op}}$ be a colocal module and put $H = \text{End}_A(L_A)$. Let $f \in A$ be a local idempotent with $\text{soc}(L_A) \cong fA/fJ$. Assume $\ell(Af/r_{A_f}(L))_{fA_f} < \infty$. Then the following are equivalent.*

- (1) L_A is injective.
- (2) ${}_H L f_{fA_f}$ is a colocal bimodule and $r_{A_f}(L) = 0$.

(3) ${}_H L f_{f A f}$ is a colocal bimodule and $M = r_{A f}(l_L(M))$ for every submodule M of $A f_{f A f}$.

Proof. (1) \Rightarrow (2). By Lemma 3.5.

(2) \Rightarrow (3). By Lemma 2.4.

(3) \Rightarrow (1). By Lemma 3.4(2) L_A is A -simple-injective. Note that $r_{A f}(L) = r_{A f}(l_L(0)) = 0$. Thus $\ell(A f_{f A f}) < \infty$ and by Lemma 1.1 $J^n f = 0$ for some $n \geq 1$, so that $L J^n A f = L J^n f = 0$ and by Lemma 3.1(1) $L J^n \subset l_L(A f) = 0$. Hence by Lemma 4.1 L_A is injective. \square

Corollary 4.5. *Let A be a left or right perfect ring. Let $L \in \text{Mod } A^{\text{op}}$ be a colocal module and put $H = \text{End}_A(L_A)$. Let $f \in A$ be a local idempotent with $\text{soc}(L_A) \cong f A / f J$. Assume ${}_H L f_{f A f}$ is a colocal bimodule and $\ell(A f / r_{A f}(L))_{f A f} < \infty$. Then L_A is quasi-injective.*

Proof. Put $I = r_A(L)$. Then $r_{A f / I f}(L) = 0$ and by Theorem 4.4 $L_{A / I}$ is injective, so that L_A is quasi-injective. \square

Proposition 4.6. *Let A be a left or right perfect ring. Let $L \in \text{Mod } A^{\text{op}}$ be a colocal module and put $H = \text{End}_A(L_A)$. Let $f \in A$ be a local idempotent with $\text{soc}(L_A) \cong f A / f J$. Then the following are equivalent.*

- (1) L_A is injective and $X = l_L(r_{A f}(X))$ for every submodule X of ${}_H L$.
- (2) ${}_H L f_{f A f}$ is a colocal bimodule, $r_{A f}(L) = 0$ and $\ell(A f_{f A f}) < \infty$.

Proof. (1) \Rightarrow (2). By Lemma 3.5(1) ${}_H L f_{f A f}$ is a colocal bimodule, and by Lemma 3.5(2) $r_{A f}(L) = 0$. It remains to show $\ell(A f_{f A f}) < \infty$. Put $K_n = A f (f J f)^n$ for $n \geq 0$. We claim $\ell(K_n / K_{n+1} f_{f A f}) < \infty$ for all $n \geq 0$. Let $n \geq 0$. Note that by Lemma 3.4(1) the lattice of submodules of $A f_{f A f}$ is anti-isomorphic to the lattice of submodules of ${}_H L$. Thus $\ell(K_n / K_{n+1} f_{f A f}) = \ell({}_H l_L(K_{n+1}) / l_L(K_n))$. Also, since $\text{rad}(K_n / K_{n+1} f_{f A f}) = 0$, ${}_H l_L(K_{n+1}) / l_L(K_n)$ is semisimple. For any submodule X of ${}_H L$, since $r_{A f}(X) = r_A(X) f$, by Lemma 3.1(2) $X = l_L(r_{A f}(X)) = l_L(r_A(X) f) = l_L(r_A(X))$. Thus by Lemma 1.6 ${}_H L \cong \text{Hom}_A(A_A, {}_H L_A)$ is linearly compact, so is ${}_H l_L(K_{n+1}) / l_L(K_n)$ by [10, Proposition 2.2]. Hence by [10, Lemma 2.3] $\ell(K_n / K_{n+1} f_{f A f}) = \ell({}_H l_L(K_{n+1}) / l_L(K_n)) < \infty$. Since $\ell(f J f / (f J f)^2_{f A f}) \leq \ell(K_1 / K_2 f_{f A f}) < \infty$, by [9, Lemma 11] $f A f$ is right artinian. Then $\ell(K_0 / K_1 f_{f A f}) < \infty$ implies $\ell(A f_{f A f}) < \infty$.

(2) \Rightarrow (1). By Theorem 4.4 L_A is injective. Since by Lemma 3.1(1) $l_L(A f) = 0$, by Lemma 2.5 $\ell({}_H L) = \ell(A f_{f A f}) < \infty$ and thus by Lemma 2.4 $X = l_L(r_{A f}(X))$ for every submodule X of ${}_H L$. \square

5. Colocal pairs

We call a pair (eA, Af) of a right ideal eA and a left ideal Af in A a colocal pair if $e, f \in A$ are local idempotents and ${}_{eA}eAf_{fAf}$ is a colocal bimodule. Note that by Lemma 2.5 $\ell({}_{eA}eA/l_{eA}(Af)) = \ell(Af/r_{Af}(eA)_{fAf})$ for every colocal pair (eA, Af) in A . In case $\ell({}_{eA}eA/l_{eA}(Af)) = \ell(Af/r_{Af}(eA)_{fAf}) < \infty$, a colocal pair (eA, Af) in A is called finite.

In [5], a pair (eA, Af) of a right ideal eA and a left ideal Af in A is called an i -pair if $e, f \in A$ are local idempotents, eA_A is colocal with $\text{soc}(eA_A) \cong fA/fJ$ and ${}_AAf$ is colocal with $\text{soc}({}_AAf) \cong Ae/Je$.

Lemma 5.1. *Let $e, f \in A$ be local idempotents. Then the following are equivalent.*

- (1) (eA, Af) is an i -pair in A .
- (2) (eA, Af) is a colocal pair in A with $l_{eA}(Af) = 0$ and $r_{Af}(eA) = 0$.

Proof. (1) \Rightarrow (2). By (1) and (3) of Lemma 3.1.

(2) \Rightarrow (1). By Corollary 3.3. □

The equivalence (1) \Leftrightarrow (2) of the next lemma has been established in [5, Theorem 3.7]. Here we provide another proof of the implication (2) \Rightarrow (1) which does not appeal to Morita duality.

Lemma 5.2 ([5, Theorem 3.7]). *Let (eA, Af) be an i -pair in a left or right perfect ring A . Then the following are equivalent.*

- (1) (eA, Af) is finite.
- (2) Both eA_A and ${}_AAf$ are injective.
- (3) eA_A is injective and ${}_AAf$ is A -simple-injective.

Proof. (1) \Rightarrow (2). By Theorem 4.4.

(2) \Rightarrow (3). Obvious.

(3) \Rightarrow (1). It follows by Lemma 3.4(1) that $X = l_{eA}(r_{Af}(X))$ for every submodule X of ${}_{eA}eA$. Thus by Proposition 4.6 $\ell(Af_{fAf}) < \infty$. □

Lemma 5.3. *Let (eA, Af) be a finite colocal pair in a left or right perfect ring A . Then the following hold.*

- (1) $eA/l_{eA}(Af)_A$ is a quasi-injective colocal module with $\text{soc}(eA/l_{eA}(Af)_A) \cong fA/fJ$.
- (2) If $r_{Af}(eA) = 0$, then $E(fA/fJ_A) \cong eA/l_{eA}(Af)$, so that $E(fA/fJ_A)$ is quasi-projective and $eA/l_{eA}(Af)_A$ is injective.

Proof. Put $I = l_A(Af)$ and $L = eA/eI_A$. Then $l_{eA}(Af) = eI$ and $l_L(Af) = 0$. Note that, since $If = 0$, $Lf_{fAf} \cong eAf_{fAf}$. Thus by Lemma 3.2 L_A is colocal with $\text{soc}(L_A) \cong fA/fJ$. Since $Lf_{fAf} \cong eAf_{fAf}$ and $H = \text{End}_A(L_A) \cong eAe/eIe$, ${}_H Lf_{fAf}$ is a colocal bimodule. Note also that $\ell(Af/r_{Af}(L))_{fAf} = \ell(Af/r_{Af}(eA))_{fAf} < \infty$.

(1) By Corollary 4.5 L_A is quasi-injective.

(2) By Theorem 4.4 L_A is injective. Thus, since $\text{soc}(L_A) \cong fA/fJ$, $E(fA/fJ_A) \cong L$. Since $L_{A/I} \cong e(A/I)_{A/I}$ is projective, L_A is quasi-projective. □

Proposition 5.4. *Let (eA, Af) be a colocal pair with $l_{eA}(Af) = 0$ in a left or right perfect ring A . Put $\bar{A} = A/r_A(eA)$. Let $\pi : A \rightarrow \bar{A}$ be the canonical ring homomorphism and put $\bar{e} = \pi(e)$, $\bar{f} = \pi(f)$. Then the following are equivalent.*

- (1) (eA, Af) is finite.
- (2) eA_A is quasi-injective, ${}_{eAe}eA$ is finitely generated and ${}_A Af/r_{Af}(eA)$ is injective.
- (3) $(\bar{e}\bar{A}, \bar{A}\bar{f})$ is a finite i -pair in \bar{A} .

Proof. Note first that \bar{A} is left or right perfect and $\bar{e}, \bar{f} \in \bar{A}$ are local idempotents. Put $I = r_A(eA)$. Then $eI = 0$ and $If = r_{Af}(eA)$. Thus $\ell(\bar{e}\bar{A})_{\bar{e}\bar{A}} = \ell({}_{eAe}eA)$ and, since ${}_{eAe}e\bar{A}f_{fAf} \cong {}_{eAe}eAf_{fAf}$ is a colocal bimodule, $(\bar{e}\bar{A}, \bar{A}\bar{f})$ is a colocal pair in \bar{A} .

(1) \Rightarrow (2). By Lemma 5.3(1) eA_A is quasi-injective, and by Lemma 5.3(2) ${}_A Af/r_{Af}(eA)$ is injective. Also, since $\ell({}_{eAe}eA) < \infty$, ${}_{eAe}eA$ is finitely generated.

(2) \Rightarrow (3). By [3, Corollary 5.6A] $\bar{e}\bar{A}_{\bar{A}} \cong eA_{\bar{A}}$ is injective. Also, since ${}_A \bar{A}\bar{f} \cong {}_A Af/r_{Af}(eA)$ is injective, so is ${}_{\bar{A}}\bar{A}\bar{f}$. It is obvious that $r_{\bar{A}}(\bar{e}\bar{A}) = 0$. For any $a \in l_{eA}(\bar{A}f)$, since $aAf \subset If$, $aAf = eaAf \subset eIf = 0$ and $a \in l_{eA}(Af) = 0$. It follows that $l_{\bar{e}\bar{A}}(\bar{A}f) = 0$. Thus by Lemmas 5.1 and 5.2 $(\bar{e}\bar{A}, \bar{A}\bar{f})$ is a finite i -pair in \bar{A} .

(3) \Rightarrow (1). Obvious. □

Corollary 5.5. *Let (eA, Af) be an i -pair in a left or right perfect ring A . Then the following are equivalent.*

- (1) (eA, Af) is finite.
- (2) eA_A is quasi-injective, ${}_{eAe}eA$ is finitely generated and ${}_A Af$ is injective.

6. Applications of colocal pairs I

In this section, as applications of colocal pairs, we extend recent results of Baba [1, Theorems 1 and 2] to left perfect rings and provide simple proofs of them.

Lemma 6.1. *Let A be a left perfect ring and $e \in A$ a local idempotent. Assume ${}_A E = E({}_A Ae/Je)$ is quasi-projective. Then ${}_A E/JE$ is simple and for a local idempotent $f \in A$ with ${}_A E/JE \cong Af/Jf$ the following hold:*

- (a) ${}_A E \cong Af/r_{Af}(eA)$;
 (b) ${}_{eAe} eAf \cong {}_{eAe} eE$ is injective; and
 (c) (eA, Af) is a colocal pair in A with $l_{eA}(Af) = 0$.

Proof. Put $I = l_A(E)$. By Lemma 1.4 there exists a local idempotent $f \in A$ such that ${}_A E \cong Af/I$. We claim $If = r_{Af}(eA)$. Since by Lemma 3.5(2) $eAIf = eIf \subset l_{eA}(E) = 0$, $If \subset r_{Af}(eA)$. Conversely, let $a \in r_{Af}(eA)$. Then $eA(a+If) = 0$ and by Lemma 3.1(1) $(a+If) \in r_{Af/If}(eA) = 0$, so that $a \in If$. Next, since $e(r_{Af}(eA)) = 0$, ${}_{eAe} eE \cong {}_{eAe} e(Af/r_{Af}(eA)) \cong {}_{eAe} eAf$. Thus ${}_{eAe} eAf$ is colocal by Lemma 3.1(3) and injective by Lemma 1.2(2). Also, since $\text{End}_A({}_A Af/I) \cong fAf/fIf$, by Lemma 3.5(1) $eAf_f Af$ is colocal. Finally, by Lemma 3.5(2) $l_{eA}(Af) \subset l_{eA}(Af/r_{Af}(eA)) = l_{eA}(E) = 0$. \square

Theorem 6.2 (cf. [1, Theorem 1]). *Let A be a left perfect ring and $e, f \in A$ local idempotents. Put $E = E({}_A Ae/Je)$. Assume $\ell(Af/r_{Af}(eA)_f Af) < \infty$. Then the following are equivalent.*

- (1) eA_A is quasi-injective with $\text{soc}(eA_A) \cong fA/fJ$.
- (2) ${}_A E$ is quasi-projective with ${}_A E/JE \cong Af/Jf$.
- (3) (eA, Af) is a colocal pair in A with $l_{eA}(Af) = 0$.
- (4) ${}_{eAe} eAf$ is colocal and $\text{soc}(eA_A) \cong fA/fJ$.

Proof. (1) \Rightarrow (3). By Lemma 3.5(1).
 (3) \Rightarrow (1). By Lemma 5.3(1).
 (2) \Rightarrow (3). By Lemma 6.1.
 (3) \Rightarrow (2). By Lemma 5.3(2).
 (3) \Rightarrow (4). By Corollary 3.3.
 (4) \Rightarrow (3). By (3) and (1) of Lemma 3.1. \square

Lemma 6.3. *Let (eA, Af) be a colocal pair in a left or right perfect ring A . Put $E = E({}_A Ae/Je)$ and $H = \text{End}_A({}_A E)^{\text{op}}$. Assume $\text{soc}(eA_A)f \neq 0$. Then the following hold.*

- (1) $\text{soc}(eA_A)fA$ is the unique simple submodule of eA_A which is isomorphic to fA/fJ_A .
- (2) If (eA, Af) is finite, then ${}_A E_H$ contains a subbimodule X such that ${}_A X \cong Af/r_A(eA)f$, ${}_{eAe} eX_H$ is a colocal bimodule, $\text{soc}(eA_A)fA \cap l_{eA}(X) = 0$ and $\ell({}_{eAe} eA/l_{eA}(X)) < \infty$.

Proof. (1) Since $\text{soc}(eA_A)f \neq 0$, eA_A contains a simple submodule $K \cong fA/fJ$. On the other hand, by Corollary 3.3 $eA/l_{eA}(Af)_A$ is colocal with $\text{soc}(eA/l_{eA}(Af)_A) \cong fA/fJ$. Thus K is the unique simple submodule of eA_A which is isomorphic to fA/fJ . It follows that $K = \text{soc}(eA_A)fA$.

(2) Put $I = r_A(eA)$. Then $If = r_Af(eA)$ and by Lemma 5.3(1) ${}_A Af/If$ is a quasi-injective colocal module with $\text{soc}({}_A Af/If) \cong Ae/Je$. Thus ${}_A E$ contains a submodule $X \cong {}_A Af/If$. Then by Lemma 1.3 $XH \subset X$. Since by Lemma 3.5(1) ${}_{eAe} eE_H$ is a colocal bimodule, so is ${}_{eAe} eX_H$. Also, since $eI = 0$, $\text{soc}(eA_A)fA(Af/If) \cong \text{soc}(eA_A)fAf \neq 0$. Thus $\text{soc}(eA_A)fA \cap l_{eA}(X) = 0$. Finally, since $l_{eA}(X) = l_{eA}(Af)$, $\ell({}_{eAe} eA/l_{eA}(X)) = \ell({}_{eAe} eA/l_{eA}(Af)) < \infty$. \square

Lemma 6.4. *Let A be a left perfect ring and $e \in A$ a local idempotent. Put $E = E({}_A Ae/Je)$ and $H = \text{End}_A({}_A E)^{\text{op}}$. Assume $\text{soc}(eA_A) \cong \bigoplus_{i=1}^n f_i A/f_i J$ with the (eA, Af_i) finite colocal pairs in A . Then $f_i A/f_i J \not\cong f_j A/f_j J$ for $i \neq j$, $\ell(E_H) = \ell({}_{eAe} eA) < \infty$ and ${}_A E/JE \cong \bigoplus_{i=1}^n Af_i/Jf_i$.*

Proof. By Lemma 6.3(1) $f_i A/f_i J \not\cong f_j A/f_j J$ for $i \neq j$. Also, for each $1 \leq i \leq n$, by Lemma 6.3(2) ${}_A E_H$ contains a subbimodule X_i such that ${}_A X_i \cong Af_i/r_A(eA)f_i$, ${}_{eAe} eX_{iH}$ is a colocal bimodule, $\text{soc}(eA_A)f_i A \cap l_{eA}(X_i) = 0$ and $\ell({}_{eAe} eA/l_{eA}(X_i)) < \infty$. Put ${}_A X_H = \sum_{i=1}^n X_i$. Then, by Lemmas 3.1(1) and 2.5 $\ell(X_{iH}) = \ell({}_{eAe} eA/l_{eA}(X_i)) < \infty$ for all $1 \leq i \leq n$, so that $\ell(X_H) < \infty$. Also, since $\text{soc}(eA_A)f_i A \cap l_{eA}(X) = 0$ for all $1 \leq i \leq n$, by Lemma 6.3(1) $\text{soc}(eA_A) \cap l_{eA}(X) = 0$. Thus, since eA_A has essential socle, $l_{eA}(X) = 0$. Since by Lemma 3.5(1) ${}_{eAe} eE_H$ is a colocal bimodule, so is ${}_{eAe} eX_H$. Thus by Lemma 2.5 $\ell({}_{eAe} eA) = \ell(X_H) < \infty$. Since by Lemma 1.3 we have a surjective ring homomorphism $\rho_X : H \rightarrow \text{End}_A({}_A X)^{\text{op}}$, $h \mapsto h|_X$, it follows by Theorem 4.4 that ${}_A X$ is injective. Thus $X = E$ and we have an epimorphism $\bigoplus_{i=1}^n Af_i/Jf_i \rightarrow {}_A E/JE$. On the other hand, since $f_i A/f_i J \not\cong f_j A/f_j J$ for $i \neq j$, it follows by Lemma 3.5(2) that ${}_A E/JE$ has a direct summand which is isomorphic to $\bigoplus_{i=1}^n Af_i/Jf_i$. Thus ${}_A E/JE \cong \bigoplus_{i=1}^n Af_i/Jf_i$. \square

Theorem 6.5 (cf. [1, Theorem 2]). *Let A be a left perfect ring and $e, f_1, f_2, \dots, f_n \in A$ local idempotents. Put $E = E({}_A Ae/Je)$. Assume (eA, Af_i) is a finite colocal pair in A for all $1 \leq i \leq n$. Then the following are equivalent.*

- (1) $\text{soc}(eA_A) \cong \bigoplus_{i=1}^n f_i A/f_i J$.
- (2) ${}_A E/JE \cong \bigoplus_{i=1}^n Af_i/Jf_i$.

Proof. (1) \Rightarrow (2). By Lemma 6.4.

(2) \Rightarrow (1). It follows by Lemmas 3.5(2) and 6.3(1) that $\text{soc}(eA_A)$ is isomorphic to a direct summand of $\bigoplus_{i=1}^n f_i A/f_i J$. We may assume $\text{soc}(eA_A) \cong \bigoplus_{i=1}^r f_i A/f_i J$ for some $1 \leq r \leq n$. Then by Lemma 6.4 ${}_A E/JE \cong \bigoplus_{i=1}^r Af_i/Jf_i$, so that $r = n$. \square

7. Applications of colocal pairs II

In this section, we provide some other applications of colocal pairs. Recall that a set $\{e_1, \dots, e_n\}$ of orthogonal local idempotents in a semiperfect ring A is called

basic if $(\sum_{i=1}^n e_i)A(\sum_{i=1}^n e_i)$ is a basic ring of A .

Lemma 7.1 ([5, Lemma 3.5]). *Let A be a semiperfect ring and $\{e_1, \dots, e_n\}$ a basic set of orthogonal local idempotents in A . Assume every $e_i A_A$ is A -simple-injective and has essential socle. Then there exists a permutation ν of the set $\{1, \dots, n\}$ such that $(e_i A, A e_{\nu(i)})$ is an i -pair in A for all $1 \leq i \leq n$.*

Proof. By [5, Lemma 3.5] there exists a mapping $\nu : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ such that $(e_i A, A e_{\nu(i)})$ is an i -pair in A for all $1 \leq i \leq n$. Then by the definition of i -pairs ν is injective. \square

Corollary 7.2. *Let A be a left perfect ring. Assume A_A is simple-quasi-injective. Then $E({}_A A)$ and $E(A_A)$ are injective cogenerators in $\text{Mod } A$ and $\text{Mod } A^{\text{op}}$, respectively.*

Lemma 7.3. *Let A be a left perfect ring. Assume $\mathcal{A}_r(A, A)$ satisfies the ACC and $e A_A$ is simple-quasi-injective for every local idempotent $e \in A$. Then A is left artinian.*

Proof. It suffices to show that $\ell({}_{e A e} e A) < \infty$ for every local idempotent $e \in A$. Let $e \in A$ be a local idempotent. Since by Lemma 3.6 $e A_A$ is colocal, there exists a local idempotent $f \in A$ with $\text{soc}(e A_A) \cong f A / f J$. By Lemma 3.5(1) $(e A, A f)$ is a colocal pair in A with $l_{e A}(A f) = 0$. For each $M \in \mathcal{A}_r(e A, A f)$, put $\hat{M} = r_A(l_{e A}(M)) \in \mathcal{A}_r(A, A)$. Then $\hat{M} f = r_{A f}(l_{e A}(M)) = M$ for every $M \in \mathcal{A}_r(e A, A f)$. Thus, for $M, N \in \mathcal{A}_r(e A, A f)$ with $M \subset N$, $\hat{M} \subset \hat{N}$ and $\hat{M} = \hat{N}$ implies $M = \hat{M} f = \hat{N} f = N$. It follows that $\mathcal{A}_r(e A, A f)$ satisfies the ACC. Thus by Lemmas 2.5 and 2.6 $\ell({}_{e A e} e A) = \ell(A f / r_{A f}(e A)_{f A f}) < \infty$. \square

Corollary 7.4. *Let A be a left perfect ring. Assume $\mathcal{A}_r(A, A)$ satisfies the ACC and A_A is simple-quasi-injective. Then A is quasi-Frobenius.*

Proof. By Lemma 7.3 A is left artinian. Then it follows by Lemmas 3.6 and 4.1 that A_A is injective. \square

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