AN INVERSE BOUNDARY VALUE PROBLEM
FOR THE STATIONARY TRANSPORT EQUATION

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1. Introduction

Let $X \subset \mathbf{R}^n$, $n \geq 2$ be an open bounded set with C^1 boundary ∂X and let us assume also that $V \subset \mathbf{R}^n$ is open. Denote $\Gamma_\pm = \{(x,v) \in \partial X \times V; \ \pm n(x) \cdot v > 0\}$, where $n(x)$ is the outer normal to ∂X at $x \in \partial X$. Let us denote by f the solution (if exists) to the following boundary value problem for the stationary linear transport (Boltzmann) equation:

$$(1.1) \quad \begin{cases} -v \cdot \nabla_x f(x,v) - \sigma_a(x,v)f(x,v) + \displaystyle\int_V k(x,v',v)f(x,v')\,dv' = 0 & \text{in } X \times V, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad f|_{\Gamma_-} = f_-. \end{cases}$$

Here f_- is a given function on Γ_-. We assume that the pair (σ_a, k) is *admissible*, i.e.

(i) $0 \leq \sigma_a \in L^\infty(X \times V)$,

(ii) $0 \leq k(x,v',\cdot) \in L^1(V)$ for a.e. $(x,v') \in X \times V$ and $\sigma_p(x,v') := \int_V k(x,v',v)\,dv$ belongs to $L^\infty(X \times V)$.

If the direct problem (1.1) is solvable, one can define the following *albedo* operator

$$\mathcal{A} : f_- \mapsto f|_{\Gamma_+},$$

that maps the incoming flux on the boundary into the outgoing one. We are interested in the following inverse problem:

(IP) Does the albedo operator $\mathcal{A}$ determine uniquely the coefficients $\sigma_a(x,v)$, $k(x,v',v)$?

There are a lot of papers devoted to (IP) both for the stationary transport equation (1.1) and for the time-dependent one (see e.g. [1], [6], [9], [10], [11], [12], [14]). In those papers however there are some restrictive assumptions on the coefficients $k(x,v',v)$ and σ_a, for example k is assumed to be independent of some variables, small enough etc. To our best knowledge the general case has been open until the authors considered in [4] the inverse problem (IP) for the time-dependent transport equation $(\partial_t - T)u = 0$, where T is the differential operator appearing in the left hand side

of the first equation in (1.1). It is shown [4] that the (time-dependent) albedo operator does determine uniquely σ_a and k provided that $\sigma_a = \sigma_a(x, |v|)$. It is easy to see that some restriction on σ_a is unavoidable, because there is a simple example showing that in the general case the uniqueness may fail (see below). In [4] it is also studied the inverse scattering problem for the time-dependent transport equation.

In the stationary case that we consider in this paper we have less data than in [4], because the time variable is missing. Next, the same objection to the uniqueness as in the time-dependent case exists. Namely, let $p = p(x, v)$ be a continuous function such that for $(x, v) \in X \times V$ we have $(x + p(x, v)v, v) \in X \times V$ as well and define $\hat{\sigma}_a(x, v) = \sigma_a(x + p(x, v)v, v)$. Then the pairs $(\sigma_a, 0)$ and $(\hat{\sigma}_a, 0)$ (i.e. $k = \hat{k} = 0$) produce the same albedo operator, while $\sigma_a \neq \hat{\sigma}_a$ in general. In order to avoid this, we will assume further that $\sigma_a = \sigma_a(x, |v|)$.

Let us introduce some notations. For $v \neq 0$, set $\tau_\pm(x, v) = \min\{t \geq 0; \ x \pm tv \in \partial X\}$, $\tau = \tau_- + \tau_+$. Consider the following two measures on $\Gamma_\pm$: $d\xi = |n(x) \cdot v|d\mu(x)dv$ and $d\tilde{\xi} = \min\{\tau(x, v), \lambda\}|n(x) \cdot v|d\mu(x)dv$, where $d\mu(x)$ is the Lebesgue measure on ∂X, $\lambda > 0$ is an arbitrary constant. Using some trace theorems [2], [3] we can show that $\mathcal{A} : L^1(\Gamma_-, d\xi) \to L^1(\Gamma_+, d\xi)$ is a bounded operator if (1.2) holds and $\mathcal{A} : L^1(\Gamma_-, d\tilde{\xi}) \to L^1(\Gamma_+, d\tilde{\xi})$ is bounded if (1.3) holds (see also [7]).

In general, the direct problem (1.1) may not be uniquely solvable, so we consider in this work two physically important situations where (1.1) is well posed. First we will assume that

$$(1.2) \qquad \|\tau\sigma_a\|_{L^\infty} < \infty, \quad \|\tau\sigma_p\|_{L^\infty} < \infty \quad \text{and} \quad \|\tau\sigma_p\|_{L^\infty} < 1.$$

This condition in particular guarantees that the dynamics corresponding to the time-dependent Boltzmann equation is subcritical [13], i.e. the "energy" (the L^1-norm of the solution) is uniformly bounded for $t > 0$. Note that (1.2) holds if in particular $\||v|^{-1}\sigma_a\|_{L^\infty} < \infty$, $\text{diam}(X)\||v|^{-1}\sigma_p\|_{L^\infty} < 1$. The second situation we will consider is when [7]

$$(1.3) \qquad \sigma_a(x, v) - \sigma_p(x, v) \geq \nu > 0 \quad \text{for a.e. } (x, v) \in X \times V$$

with some $\nu > 0$. In other words, (1.3) says that the absorption rate is greater than the production rate. This also implies that the corresponding dynamics is subcritical.

The main result of this paper is the following.

Theorem 1.1. *Let (σ_a, k), $(\hat{\sigma}_a, \hat{k})$ be two admissible pairs with $\sigma_a = \sigma_a(x, |v|)$, $\hat{\sigma}_a = \hat{\sigma}_a(x, |v|)$ and assume that they satisfy either (1.2) or (1.3). Assume that the corresponding albedo operators $\mathcal{A}$ and $\hat{\mathcal{A}}$ coincide. Then*
(a)   *if $n \geq 3$, then $\sigma_a = \hat{\sigma}_a$, $k = \hat{k}$;*
(b)   *if $n = 2$, then $\sigma_a = \hat{\sigma}_a$.*

Note that our proof is constructive and we obtain explicit formulas for σ_a and k

in terms of the distribution kernel α of the albedo operator (see Proposition 4.1 and Proposition 4.2). We study the first two terms in the singular decomposition of α and show in Theorem 3.2 that they are delta functions supported on varieties of different dimension (if $n \geq 3$), while the remainder is a function. We show that the first singular term depends on σ_a only and σ_a can be recovered from it. Next, knowing σ_a, one can recover k from the second term. We followed similar approach in [4] in the time-dependent case. In the two-dimensional case the second term is not a delta function, but a locally L^1-function and cannot be distinguished from the remainder. That is why our approach does not work for recovering k in two dimensions. Notice that the idea of using singular solutions of the transport equation has been used also in [10], [14].

The results we obtain are not restricted to the conditions (1.2) or (1.3) only. If for example $\inf\{|v|;\ v \in V\} > 0$ and if the direct problem $Tf = g$, $f|_{\Gamma_-} = 0$ has unique solution $f \in L^1_{\mathrm{loc}}(X \times V)$ for any $g \in L^1(X \times V)$, then Theorem 1.1 still holds. Theorem 1.1 holds in the presence of a (non-singular) source term as well, because the flux generated by such a source term is not singular and cannot affect the leading singularities of the kernel of $\mathcal{A}$. We would like to mention also that a similar result can be obtained if $V = S^{n-1}$, $n \geq 3$.

The structure of the paper is the following. In section 2. we prove some trace theorems of the type obtained in [2], [3], solve the direct problem and define the albedo operator in suitable L^1 spaces. In section 3. we construct a special solution $f = f(x, v, x', v')$ to (1.1) with $f_- = \delta_{\{x'\}}(x)\delta(v - v')$, where $(x', v') \in \Gamma_-$ are parameters and we study the singularities of f. With the aid of f we obtain a singular decomposition of the distribution kernel α of $\mathcal{A}$. Finally, in section 4. we show how to recover explicitly σ_a, k from α.

The results of this paper have been announced in [5].

2. Preliminaries

We begin this section with a simple lemma.

Lemma 2.1.   *Assume that $f \in L^1(X \times V)$. Then*

$$\int_{X \times V} f(x, v)\, dx\, dv = \int_{\Gamma_\mp} \int_0^{\tau_\pm(x', v)} f(x' \pm tv, v)\, dt\, d\xi(x', v).$$

Proof.   The proof follows by performing a change of variables $X \ni x \mapsto (x', t) \in \Gamma_\mp \times (0, \tau_\pm(x, v))$ given by the formula $x' = x \mp \tau_\mp(x, v)v$, $t = \tau_\mp(x, v)$. This change is smooth except on a closed set of measure zero and $dx = |n(x') \cdot v|\, d\mu(x')\, dt$.
                                                                                                    □

Let us introduce some new notations. Denote

$$T_0 f = -v \cdot \nabla_x f, \qquad A_1 f = -\sigma_a f, \qquad A_2 f = \int_V k(x, v', v) f(x, v') \, dv',$$

$$T_1 = T_0 + A_1, \qquad T = T_0 + A_1 + A_2 = T_1 + A_2.$$

Notice that if (σ_a, k) is admissible (which we will always assume), then A_1 and A_2 are bounded operators in $L^1(X \times V)$. All norms throughout this paper are in $L^1(X \times V)$ except otherwise stated.

We are going to prove next a trace theorem in the spirit of [2], [3].

Theorem 2.1. *We have*

$$(2.1) \qquad \left\| f|_{\Gamma_\pm} \right\|_{L^1(\Gamma_\pm, d\xi)} \leq \|T_0 f\| + \|\tau^{-1} f\|$$

for any function $f(x, v)$, $(x, v) \in X \times V$ for which the right-hand side above is well-defined.

Proof.   Consider first a function $g \in L^1([0, a])$, such that $g' \in L^1([0, a])$, where $a > 0$. Then

$$g(0) = -\int_0^t g'(x) \, dx + g(t),$$

therefore

$$|g(0)| \leq \int_0^a |g'(x)| \, dx + |g(t)| = \|g'\| + |g(t)| \qquad \forall t \in [0, a].$$

Here $\| \cdot \|$ is the norm in $L^1([0, a])$. After integrating that inequality we get

$$(2.2) \qquad |g(0)| = \frac{1}{a} \int_0^a |g(0)| \, dt \leq \|g'\| + \frac{1}{a} \|g\|.$$

Now, let $f(x, v)$ be such that $T_0 f \in L^1(X \times V)$ and $\tau^{-1} f \in L^1(X \times V)$. Set

$$g(t, x', v) = f(x' + tv, v), \quad (x', v) \in \Gamma_-, \quad 0 \leq t \leq \tau_+(x', v).$$

Since $T_0 f \in L^1(X \times V)$, combining Fubini's theorem and Lemma 2.1, we get that for a.e. $(x', v) \in \Gamma_-$, the function $t \mapsto \partial_t g = -(T_0 f)(x' + tv, v)$ belongs to $L^1(0, \tau_+(x', v))$ and so does $g(t, x', v)$. Next, $g(0, x', v) = f(x', v) = f|_{\Gamma_-}$. We therefore get from (2.2)

$$|f(x', v)| \leq \int_0^{\tau_+(x', v)} |(T_0 f)(x' + tv, v)| dt$$

$$+\frac{1}{\tau_+(x',v)}\int_0^{\tau_+(x',v)}|f(x'+tv,v)|\,dt.$$

If we now integrate over Γ_- with respect to $d\xi(x',v)$ and apply Lemma 2.1 to the right-hand side, we complete the proof of the theorem for $f|_{\Gamma_-}$. The proof for $f|_{\Gamma_+}$ is similar. □

Let us set

$$\mathcal{W}=\{f;\ T_0f\in L^1(X\times V),\ \tau^{-1}f\in L^1(X\times V)\},$$
$$\|f\|_{\mathcal{W}}=\|T_0f\|+\|\tau^{-1}f\|.$$

Then Theorem 2.1 says that taking the trace $f\mapsto f|_{\Gamma_\pm}$ is a continuous operator from $\mathcal{W}$ into $L^1(\Gamma_\pm,d\xi)$. As we will see below, the inequality (2.1) cannot be improved, for some functions it turns into equality.

Given $f_-\in L^1(\Gamma_-,d\xi)$, define Jf_- as the following prolongation of f_- inside $X\times V$:

$$Jf_-=e^{-\int_0^{\tau_-(x,v)}\sigma_a(x-sv,v)ds}f_-(x-\tau_-(x,v)v,v),\qquad (x,v)\in X\times V.$$

Note that Jf_- is defined so that $T_1Jf_-=0$, $Jf_-|_{\Gamma_-}=f_-$, therefore J is the solution operator of the problem $T_1f=0$, $f|_{\Gamma_-}=f_-$.

Proposition 2.1. *Assume that $\|\tau\sigma_a\|_{L^\infty}<\infty$. Then*

$$\|Jf_-\|_{\mathcal{W}}\le C\|f_-\|_{L^1(\Gamma_-,d\xi)},$$

with $C=1+\|\tau\sigma_a\|_{L^\infty}$. If $\sigma_a=0$, then we have equality above (and $C=1$).

Proof. Note first that

$$\|Jf_-\|_{\mathcal{W}}\le\|\tau^{-1}Jf_-\|+\|A_1Jf_-\|\le(1+\|\tau\sigma_a\|_{L^\infty})\|\tau^{-1}Jf_-\|,$$

because $T_0Jf_-=-A_1Jf_-$. Next, by Lemma 2.1,

$$(2.3)\quad \int_{X\times V}|\tau^{-1}Jf_-|\,dx\,dv\le\int_{X\times V}\tau^{-1}(x,v)|f_-(x-\tau_-(x,v)v,v)|\,dx\,dv$$
$$=\int_{\Gamma_-}\int_0^{\tau_+(x',v)}\tau_+^{-1}(x',v)|f_-(x',v)|\,dt\,d\xi$$
$$=\|f_-\|_{L^1(\Gamma_-,d\xi)},$$

which completes the proof. □

We are going to use Theorem 2.1 and Proposition 2.1 in the case where (1.2) holds. Then we will work with functions belonging to $L^1(\Gamma_\pm, d\xi)$ on the boundary and to $\mathcal{W}$ inside $X \times V$. In the case where (1.3) holds, we will work with boundary data in $L^1(\Gamma_\pm, d\tilde{\xi})$ and then we will need the following trace theorem due to Cessenat [2], [3]. Recall that $d\tilde{\xi} = \min\{\lambda, \tau(x,v)\}d\xi$, $\lambda > 0$ is an arbitrary fixed constant.

Theorem 2.2 ([2], [3]). *For any $f \in L^1(X \times V)$ such that $T_0 f \in L^1(X \times V)$ we have*

$$\left\| f|_{\Gamma_\pm} \right\|_{L^1(\Gamma_\pm, d\tilde{\xi})} \le \lambda \|T_0 f\| + \|f\|.$$

We refer to [2], [3] for a proof and for other results. The theorem follows also from Theorem 2.1 by setting $f = \min\{\lambda, \tau\}g$.

Denote

$$\tilde{\mathcal{W}} = \{f; \ f \in L^1(X \times V), T_0 f \in L^1(X \times V)\},$$

$$\|f\|_{\tilde{\mathcal{W}}} = \|T_0 f\| + \|f\|.$$

In the case where (1.3) holds, we have the following version of Proposition 2.1.

Proposition 2.2. *For any $f_- \in L^1(\Gamma_-, d\tilde{\xi})$*

$$\|J f_-\|_{\tilde{\mathcal{W}}} \le C \|f_-\|_{L^1(\Gamma_-, d\tilde{\xi})},$$

where $C = (1 + \|\sigma_a\|_{L^\infty}) \max\{1, (\nu\lambda)^{-1}\}$.

Proof.   We obtain as before that $\|J f_-\|_{\tilde{\mathcal{W}}} \le (1 + \|\sigma_a\|_{L^\infty})\|J f_-\|$. Next,

$$\int_{X \times V} |J f_-| \, dx \, dv \le \int_{X \times V} e^{-\nu\tau_-(x,v)} |f_-(x - \tau_-(x,v)v, v)| \, dx \, dv$$

$$= \int_{\Gamma_-} \int_0^{\tau_+(x',v)} e^{-\nu t} |f_-(x', v)| \, dt \, d\xi$$

$$\le \int_{\Gamma_-} \frac{\min\{\tau_+(x',v), 1/\nu\}}{\min\{\tau_+(x',v), \lambda\}} |f_-(x', v)| \, d\tilde{\xi}$$

$$\le \max\left\{1, \frac{1}{\nu\lambda}\right\} \|f_-\|_{L^1(\Gamma_-, d\tilde{\xi})}. \qquad \square$$

We are going now to reduce the boundary value problem (1.1) to an integral equation using standard arguments. Equation (1.1) can be rewritten as $(T_1 + A_2)f = 0$. Let us integrate the identity $\exp\{-\int_0^t \sigma_a(x - sv, v)ds\}[(T_1 + A_2)f](x - tv, v) = 0$ with respect to t from 0 to $\tau_-(x, v)$ and take into account the boundary condition $f|_{\Gamma_-} = f_-$.

We thus get

(2.4)
$$(I + K)f = Jf_-,$$

where I stands for the identity and K is the following integral operator

(2.5)
$$Kf = -\int_0^{\tau_-(x,v)} e^{-\int_0^t \sigma_a(x-sv,v)ds}(A_2 f)(x - tv, v)\, dt.$$

Introduce the following unbounded operators:

$$\mathbf{T_1}f = T_1 f,\ D(\mathbf{T_1}) = \{f \in L^1(X \times V);\ T_1 f \in L^1(X \times V),\ f|_{\Gamma_-} = 0\},$$
$$\mathbf{T}f\ = Tf,\ D(\mathbf{T})\ = \{f \in L^1(X \times V);\ Tf \in L^1(X \times V),\ f|_{\Gamma_-} = 0\}.$$

Notice that formally $K = \mathbf{T_1}^{-1}A_2$ and for $\mathbf{T_1}^{-1}$ we have

$$\mathbf{T_1}^{-1}f = -\int_0^{\tau_-(x,v)} e^{-\int_0^t \sigma_a(x-sv,v)ds} f(x - tv, v)\, dt.$$

In the next two propositions we will show in particular that $\mathbf{T_1}^{-1}$ is well defined.

Proposition 2.3. *Assume* (1.2). *Then*
(a) $\tau^{-1}\mathbf{T_1}^{-1}$, $\tau^{-1}\mathbf{T}^{-1}$ *and* $A_2\tau$ *are bounded operators in* $L^1(X \times V)$ *and therefore* $K = \mathbf{T_1}^{-1}A_2$ *is a bounded operator in* $L^1(X \times V;\ \tau^{-1}dxdv)$. *Moreover, the operator norm of* K *is not greater than* $\|\tau\sigma_p\|_{L^\infty} < 1$ *and therefore* $(I + K)^{-1}$ *exists in this space.*
(b) *The integral equation* (2.4) *and therefore the boundary value problem* (1.1) *are uniquely solvable for any* $f_- \in L^1(\Gamma_-, d\xi)$ *and then* $f \in \mathcal{W}$.
(c) *The albedo operator* $\mathcal{A}$ *is a bounded map* $\mathcal{A}: L^1(\Gamma_-, d\xi) \to L^1(\Gamma_+, d\xi)$.

Proof.   Clearly,

$$\|\tau^{-1}\mathbf{T_1}^{-1}f\| \leq \|f\|,\qquad \forall f \in L^1(X \times V).$$

Next, (1.2) implies

$$\|A_2\tau\|_{\mathcal{L}(L^1(X \times V))} \leq \|\tau\sigma_p\|_{L^\infty} < 1,$$

where τ stands for the operator of multiplication by $\tau(x, v)$. Therefore, we have $K = \mathbf{T_1}^{-1}A_2$ and

$$\|\tau^{-1}Kf\| = \|\tau^{-1}\mathbf{T_1}^{-1}A_2 f\| \leq \|A_2 f\| \leq \|\tau\sigma_p\|_{L^\infty}\|\tau^{-1}f\|,$$

which proves (a). Since $\|\tau\sigma_p\|_{L^\infty} < 1$, the operator $I + K$ is invertible in $L^1(X \times V; \tau^{-1}dxdv)$ and (2.4) has unique solution

$$(2.6) \qquad\qquad f = (I + K)^{-1}Jf_-.$$

By (2.4) we have for f

$$(2.7) \qquad \|\tau^{-1}f\| \le \left(1 - \|\tau\sigma_p\|_{L^\infty}\right)^{-1} \|\tau^{-1}Jf_-\|$$
$$\le \left(1 - \|\tau\sigma_p\|_{L^\infty}\right)^{-1} \|f_-\|_{L^1(\Gamma_-,d\xi)}.$$

Further, since $Tf = 0$, we deduce that $T_0 f = -(A_1 + A_2)f$ and therefore

$$(2.8) \qquad \|T_0 f\| \le \left(\|\tau\sigma_a\|_{L^\infty} + \|\tau\sigma_p\|_{L^\infty}\right) \|\tau^{-1}f\|.$$

Then by (2.7) and (2.8) we deduce that $f \in \mathcal{W}$ and by Theorem 2.1 we conclude that $f|_{\Gamma_+} = \mathcal{A}f_- \in L^1(\Gamma_+, d\xi)$. Moreover, the operator norm of $\mathcal{A}$ does not exceed $(1 + \|\tau\sigma_a\|_{L^\infty} + \|\tau\sigma_p\|_{L^\infty})(1 - \|\tau\sigma_p\|_{L^\infty})^{-1}$. We finally note that $\mathbf{T}$ is invertible, because we can set $\mathbf{T}^{-1} = (I + K)^{-1}\mathbf{T}_1^{-1} : L^1(X \times V) \to L^1(X \times V; \tau^{-1}dxdv)$. $\qquad\square$

Proposition 2.4. *Assume (1.3). Then*

(a) K, $\mathbf{T}_1^{-1}$ *and* $\mathbf{T}^{-1}$ *are bounded operators in* $L^1(X \times V)$ *and* $K = \mathbf{T}_1^{-1}A_2$. *Further,* $I + K$ *is invertible and* $(I + K)^{-1} = I - \mathbf{T}^{-1}A_2$.

(b)   *The integral equation (2.4) and therefore the boundary value problem (1.1) are uniquely solvable for any* $f_- \in L^1(\Gamma_-, d\tilde\xi)$ *and then* $f \in \tilde{\mathcal{W}}$.

(c) $\mathcal{A}$ *is a bounded map* $\mathcal{A} : L^1(\Gamma_-, d\tilde\xi) \to L^1(\Gamma_+, d\tilde\xi)$.

Proof.   Since $\exp\{-\int_0^t \sigma_a(x - sv, v)ds\} \le e^{-\nu t}$, $\|\mathbf{T}_1^{-1}\|_{\mathcal{L}(L^1(X \times V))} \le \nu^{-1}$. The boundedness of $\mathbf{T}^{-1}$ is proven in [7] under the assumption (1.3). Next, it is easy to check directly that $(I + K)(I - \mathbf{T}^{-1}A_2) = (I - \mathbf{T}^{-1}A_2)(I + K) = I$, therefore $(I + K)^{-1} = I - \mathbf{T}^{-1}A_2$. This proves (a). Observe next that if $f_- \in L^1(\Gamma_-, d\tilde\xi)$, then by Proposition 2.4 $Jf_- \in \tilde{\mathcal{W}}$ and the solution f to (2.4) is given by $f = (I - \mathbf{T}^{-1}A_2)Jf_- \in L^1(X \times V)$. Moreover, since $Tf = 0$, we have $T_0 f = -(A_1 + A_2)f \in L^1(X \times V)$. Therefore $T_0 f \in \tilde{\mathcal{W}}$ and by Theorem 2.2, $\mathcal{A}f_- = f|_{\Gamma_+} \in L^1(\Gamma_+, d\tilde\xi)$. $\qquad\square$

3.   The special solution

In this section we study the distribution kernel of $\mathcal{A}$. To this end first we solve (1.1) with

$$f_- = \delta_{\{x'\}}(x)\delta(v - v'),$$

where $(x', v') \in \Gamma_-$ are regarded as parameters and $\delta_{\{x'\}}$ is a distribution on ∂X defined by $(\delta_{\{x'\}}, \varphi) = \int \delta_{\{x'\}}(x) \varphi(x) d\mu(x) = \varphi(x')$. On the other hand, we will denote by δ the ordinary Dirac delta function in $\mathbf{R}^n$. The integral above is to be considered in distribution sense. Let us denote by $f(x, v, x', v')$ the solution (in distribution sense) of

$$(3.1) \qquad \begin{cases} Tf = 0 & \text{in } X \times V, \\ f|_{\Gamma_-} = f_-. \end{cases}$$

Since we assume that the boundary is C^1-smooth and not necessarily C^∞-smooth, we have to work in fact with distributions which are linear functionals on C_0^1. All singular distributions appearing in the analysis of f and the kernel of $\mathcal{A}$ are in fact delta-type distributions supported on some hypersurfaces and can be regarded also as measures.

To solve (3.1), pick $\varphi_- \in C_0^1(\Gamma_-)$ and denote by φ the solution of

$$(3.2) \qquad \begin{cases} T\varphi = 0 & \text{in } X \times V, \\ \varphi|_{\Gamma_-} = \varphi_-. \end{cases}$$

Assume that either (1.2) or (1.3) holds. Then, according to Proposition 2.3 and Proposition 2.4, equation (3.2) has unique solution $\varphi = (I + K)^{-1} J \varphi_-$, which also admits the representation

$$\varphi = J\varphi_- - KJ\varphi_- + (I + K)^{-1} K^2 J \varphi_-.$$

Since $(I + K)^{-1} \mathbf{T}_1^{-1} = \mathbf{T}^{-1}$, $K = \mathbf{T}_1^{-1} A_2$, we get $(I + K)^{-1} K^2 J \varphi_- = (I + K)^{-1} \mathbf{T}_1^{-1} A_2 K J \varphi_- = \mathbf{T}^{-1} A_2 K J \varphi_-$. Therefore,

$$(3.3) \qquad \varphi = J\varphi_- - KJ\varphi_- + \mathbf{T}^{-1} A_2 K J \varphi_-.$$

All terms in (3.3) belong to $L^1(X \times V; \tau^{-1} dx dv)$ or $L^1(X \times V)$, respectively. We proceed with analysis of each term in the right-hand side of (3.3).

For $J\varphi_-$ we have

$$J\varphi_- = E(x, v)\varphi_-(x - \tau_-(x, v)v, v),$$

$$E(x, v) := \exp\left\{ -\int_0^{\tau_-(x, v)} \sigma_a(x - sv, v) ds \right\}.$$

Choose $\psi \in C_0^\infty(X \times V)$ and consider

$$
\begin{aligned}
(J\varphi_-, \psi) &= \int_{X \times V} (J\varphi_-) \psi \, dx \, dv \\
&= \int_{X \times V} E(x, v)\varphi_-(x - \tau_-(x, v)v, v)\psi(x, v) \, dx \, dv.
\end{aligned}
$$

By Lemma 2.1,

$$(J\varphi_-, \psi) = \int_{\Gamma_-} \int_0^{\tau_+(x',v)} E(x' + tv, v)\varphi_-(x', v)\psi(x' + tv, v)\, dt\, d\xi(x', v)$$

$$= \int \ldots \int E(x, v)\varphi_-(x', v')\psi(x, v)\delta(x - x' - tv)\delta(v - v')\, dt\, d\xi(x', v')\, dx\, dv,$$

where the last integral is to be considered in the sense of distributions. Therefore,

$$J\varphi_- = \int_{\Gamma_-} \left(\int_0^{\tau_+(x',v')} E(x, v)\delta(x - x' - tv)\delta(v - v')\, dt \right)$$
$$\times |v' \cdot n(x')| \varphi_-(x', v')\, d\mu(x')\, dv'.$$

Thus,

$$(3.4) \qquad J\varphi_- = \int_{\Gamma_-} f_1(x, v, x', v')\varphi_-(x', v')\, d\mu(x')\, dv'$$

with

$$(3.5) \qquad f_1(x, v, x', v') = |v' \cdot n(x')| \int_0^{\tau_+(x',v')} e^{-\int_0^{\tau_-(x,v)} \sigma_a(x - sv, v)\, ds}$$
$$\times \delta(x - x' - tv)\delta(v - v')\, dt.$$

In other words, $f_1 = Jf_-$ is given by (3.5).

Consider further the second term $-KJ\varphi_-$ in the right-hand side of (3.3).

$$(3.6) \qquad -KJ\varphi_- = \int_0^{\tau_-(x,v)} \int_V e^{-\int_0^s \sigma_a(x - pv, v)\, dp}$$
$$\times k(x - sv, v', v)(J\varphi_-)(x - sv, v')\, dv'\, ds$$
$$= \int_0^{\tau_-(x,v)} \int_V e^{-\int_0^s \sigma_a(x - pv, v)\, dp}$$
$$\times k(x - sv, v', v)E(x - sv, v')$$
$$\times \varphi_-(x - sv - \tau_-(x - sv, v')v', v')\, dv'\, ds.$$

Arguing as above, we get

$$(3.7) \qquad -KJ\varphi_- = \int_{\Gamma_-} f_2(x, v, x', v')\varphi_-(x', v')\, d\mu(x')\, dv',$$

where

$$(3.8) \qquad f_2 = |n(x') \cdot v'| \int_0^{\tau_-(x,v)} \int_0^{\tau_+(x',v')} e^{-\int_0^s \sigma_a(x - pv, v)\, dp}$$

$$\times e^{-\int_0^{\tau_-(x-sv,v')} \sigma_a(x-sv-pv',v')dp}$$

$$\times k(x-sv,v',v)\delta(x-x'-sv-tv')\,dt\,ds.$$

And finally, let us consider the last term in (3.3). We first start with $A_2KJ\varphi_-$.

$$(3.9) \qquad A_2KJ\varphi_- = \int_V \int_V \int_0^{\tau_-(x,v'')} e^{-\int_0^s \sigma_a(x-pv'',v'')dp}$$

$$\times E(x-sv'',v'')k(x,v'',v)k(x-sv'',v',v'')$$

$$\times \varphi_-(x-sv''-\tau_-(x-sv'',v')v',v')\,ds\,dv'\,dv''.$$

We claim that (3.9) is absolutely convergent integral provided that (1.2) or (1.3) holds. Indeed, assume that (1.2) holds. Then by Proposition 2.3 and (2.4),

$$\int_{X\times V} |A_2KJ\varphi_-|\,dx\,dv = \|A_2KJ\varphi_-\|_{L^1(X\times V)}$$

$$\leq \|A_2\tau\|_{\mathcal{L}(L^1(X\times V))}\|\tau^{-1}K\tau\|_{\mathcal{L}(L^1(X\times V))}\|\tau^{-1}J\varphi_-\|$$

$$\leq \|\varphi_-\|_{L^1(\Gamma_-,d\xi)}.$$

Next, in the case where (1.3) holds, by Proposition 2.2 and Proposition 2.4,

$$\int_{X\times V} |A_2KJ\varphi_-|\,dx\,dv = \|A_2KJ\varphi_-\|_{L^1(X\times V)}$$

$$\leq \|A_2\|_{\mathcal{L}(L^1(X\times V))}\|K\|_{\mathcal{L}(L^1(X\times V))}\|J\varphi_-\|$$

$$\leq C\|\varphi_-\|_{L^1(\Gamma_-,d\tilde{\xi})}.$$

In conclusion, we showed that $(A_2KJ\varphi_-)(x,v)$ is absolutely integrable in $X \times V$ and therefore by the Fubini theorem the integral (3.9) is absolutely convergent for a.e. $(x,v) \in X \times V$. This proves our claim.

Let us extend all functions depending on v, v', v'' by zero outside V. Change the order of integration in (3.9) from $ds\,dv'\,dv''$ to $dv''\,dv'\,ds$ and perform first the change of variables $y = x - sv''$, $dv'' = s^{-n}dy$ in (3.9) and next, using Lemma 2.1 let us change the variables again $x' = y - \tau_-(y,v')v'$, $t = \tau_-(y,v')$. Then $(x',v') \in \Gamma_-$ and $dy\,dv' = dt\,d\xi(x',v')$. We thus get

$$A_2KJ\varphi_- = \int_0^\infty \int_{\Gamma_-} \int_0^{\tau_+(x',v')} e^{-\int_0^s \sigma_a(x-pv'',v'')dp}E(x'+tv',v'')k(x,v'',v)$$

$$\times k(x'+tv',v',v'')\chi\varphi_-(x',v')s^{-n}\,dt\,d\xi(x',v')\,ds,$$

with $v'' = (x - x' - tv')/s$. Here $\chi = 1$ if the line segment $[x,x'+tv']$ belongs to X and $\chi = 0$ otherwise. The obtained integral is also absolutely convergent despite the singularity s^{-n} appearing in it. By the Fubini theorem we can change the order of

integration $dt\, d\xi(x', v')\, ds \rightarrow dt\, ds\, d\xi(x', v')$ and after the first two integrations with respect to t and s we obtain a function of (x', v') which is integrable with values in $L^1(\mathbf{R}_x^n \times V_v)$. In other words,

$$(3.10) \qquad A_2 K J \varphi_- = \int_{\Gamma_-} \tilde{f}_3(x, v, x', v') \varphi_-(x', v')\, d\mu(x')\, dv',$$

where

$$\tilde{f}_3 = |n(x') \cdot v'| \int_0^\infty \int_0^{\tau_+(x', v')} e^{-\int_0^s \sigma_a(x - pv'', v'')\, dp} E(x' + tv', v'')$$
$$\times k(x, v'', v) k(x' + tv', v', v'') \chi s^{-n}\, dt\, ds,$$

where $v'' = (x - x' - tv')/s$. Moreover, the arguments above imply that the function $\Gamma_- \ni (x', v') \rightarrow |n(x') \cdot v'|^{-1} \tilde{f}_3(x, v, x', v') \varphi_-(x', v') \in L^1(X \times V)$ belongs to $L^1(\Gamma_-, d\xi)$ for any $\varphi_- \in L^1(\Gamma_-, d\xi)$ provided that (1.2) holds and, respectively, $(\min\{\tau, \lambda\})^{-1} |n(x') \cdot v'|^{-1} \tilde{f}_3 \varphi_-$ belongs to $L^1(\Gamma_-, d\tilde{\xi})$ for any $\varphi_- \in L^1(\Gamma_-, d\tilde{\xi})$ provided that (1.3) holds. Therefore,

$$(3.11) \qquad \begin{aligned} |n(x') \cdot v'|^{-1} \tilde{f}_3 &\in L^\infty(\Gamma_-;\ L^1(X_x \times V_v)),\ \text{if (1.2) holds,} \\ (\min\{\tau, \lambda\})^{-1} |n(x') \cdot v'|^{-1} \tilde{f}_3 &\in L^\infty(\Gamma_-;\ L^1(X_x \times V_v)),\ \text{if (1.3) holds.} \end{aligned}$$

We are ready now to estimate the term $\mathbf{T}^{-1} A_2 K J \varphi_-$ appearing in (3.3). By (3.10),

$$(3.12) \qquad \mathbf{T}^{-1} A_2 K J \varphi_- = \int_{\Gamma_-} f_3(x, v, x', v') \varphi_-(x', v')\, d\mu(x')\, dv',$$

where

$$(3.13) \qquad f_3 := \mathbf{T}^{-1} \tilde{f}_3.$$

Here $f_3 = f_3(x, v, x', v')$ and in the formula above $\mathbf{T}^{-1}$ acts with respect to (x, v) and $(x', v') \in \Gamma_-$ are considered as parameters. Assume first that (1.2) holds. Then by Proposition 2.3 $\mathbf{T}^{-1} : L^1(X \times V) \rightarrow L^1(X \times V;\ \tau^{-1} dx dv)$ is bounded and we get immediately from (3.11) that

$$(3.14) \quad |n(x') \cdot v'|^{-1} f_3 \in L^\infty(\Gamma_-;\ L^1(X_x \times V_v;\ \tau^{-1} dx dv)) \quad \text{when (1.2) holds.}$$

Moreover, since $T_0 f_3 = -(A_1 + A_2) f_3$ and $A_1 + A_2 : L^1(X \times V;\ \tau^{-1} dx dv) \rightarrow L^1(X \times V)$ is bounded, we also get

$$(3.15) \quad |n(x') \cdot v'|^{-1} T_0 f_3 \in L^\infty(\Gamma_-;\ L^1(X_x \times V_v)) \quad \text{when (1.2) holds.}$$

Let us assume next that (1.3) holds. Then by Proposition 2.4, $\mathbf{T}^{-1}$ is bounded in $L^1(X \times V)$ and hence by (3.11),

$$(3.16) \quad (\min\{\tau, \lambda\})^{-1} |n(x') \cdot v'|^{-1} f_3 \in L^\infty(\Gamma_-; L^1(X_x \times V_v)) \quad \text{when (1.3) holds.}$$

We also get as above that

$$(3.17) \quad (\min\{\tau, \lambda\})^{-1} |n(x') \cdot v'|^{-1} T_0 f_3 \in L^\infty(\Gamma_-; L^1(X_x \times V_v)) \quad \text{when (1.3) holds.}$$

Combining (3.3), (3.4), (3.7) and (3.12), we see that the solution to (3.2) is given by

$$(3.18) \qquad \varphi(x, v) = \int_{\Gamma_-} f(x, v, x', v') \varphi_-(x', v') \, d\mu(x') \, dv',$$

where the integral is to be considered in distribution sense and f is given by

$$f = f_1 + f_2 + f_3,$$

with f_1, f_2 defined by (3.5), (3.8) and f_3 satisfying (3.14), (3.15) (respectively (3.16), (3.17)). It is also clear that f solves (3.1) in distribution sense. Let us formulate this in the following theorem.

Theorem 3.1. *Assume that (σ_a, k) is admissible and either (1.2) or (1.3) holds. Then for the solution $f(x, v, x', v')$ of (3.1) we have $f = f_1 + f_2 + f_3$, where*

$$f_1 = |n(x') \cdot v'| \int_0^{\tau_+(x', v')} e^{-\int_0^{\tau_-(x, v)} \sigma_a(x - pv, v) dp} \delta(x - x' - tv) \delta(v - v') \, dt$$

$$f_2 = |n(x') \cdot v'| \int_0^{\tau_-(x, v)} \int_0^{\tau_+(x', v')} e^{-\int_0^s \sigma_a(x - pv, v) dp}$$

$$\times e^{-\int_0^{\tau_-(x - sv, v')} \sigma_a(x - sv - pv', v') dp} k(x - sv, v', v) \delta(x - x' - sv - tv') \, dt \, ds$$

$$|n(x') \cdot v'|^{-1} f_3 \in L^\infty(\Gamma_-; \mathcal{W}), \quad \text{if (1.2) holds,}$$

$$(\min\{\tau, \lambda\})^{-1} |n(x') \cdot v'|^{-1} f_3 \in L^\infty(\Gamma_-; \tilde{\mathcal{W}}), \quad \text{if (1.3) holds.}$$

By (3.18) the so constructed solution $f(x, v, x', v')$ is the distribution kernel of the solution operator $\varphi_- \mapsto \varphi$ of (3.2). In order to find the distribution kernel $\alpha(x, v, x', v')$ $((x, v) \in \Gamma_+, (x', v') \in \Gamma_-)$ of the albedo operator $\mathcal{A}$, it is enough to set

$$\alpha(x, v, x', v') := f(x, v, x', v')|_{(x, v) \in \Gamma_+}, \quad (x', v') \in \Gamma_-.$$

Then Theorem 3.1 yields the following.

Theorem 3.2. *Assume that* (σ_a, k) *is an admissible pair and that either* (1.2) *or* (1.3) *holds. Then the distribution kernel* $\alpha(x, v, x', v')$ *of* $\mathcal{A}$ *satisfies* $\alpha = \alpha_1 + \alpha_2 + \alpha_3$ *with*

$$\alpha_1 = e^{-\int_0^{\tau_-(x,v)} \sigma_a(x-pv,v)dp} \delta_{\{x-\tau_-(x,v)v\}}(x')\delta(v-v')$$

$$\alpha_2 = \int_0^{\tau_-(x,v)} e^{-\int_0^s \sigma_a(x-pv,v)dp} e^{-\int_0^{\tau_-(x-sv,v')} \sigma_a(x-sv-pv',v')dp}$$

$$\times k(x-sv,v',v)\delta_{\{x-sv-\tau_-(x-sv,v')v'\}}(x')\,ds$$

$$|n(x')\cdot v'|^{-1}\alpha_3 \in L^\infty(\Gamma_-; L^1(\Gamma_+, d\xi)), \quad \text{if (1.2) holds and}$$

$$\min\{\tau(x',v'),\lambda\}^{-1}|n(x')\cdot v'|^{-1}\alpha_3 \in L^\infty(\Gamma_-; L^1(\Gamma_+, d\tilde{\xi})), \quad \text{if (1.3) holds.}$$

Proof.   Formally $\alpha_j = f_j|_{(x,v)\in\Gamma_+}$, $j = 1, 2, 3$. To show that α_j are well defined, pick $\varphi \in C_0^1(\Gamma_-)$ as before and consider $\varphi_j = \int_{\Gamma_-} f_j\varphi_-d\mu(x')dv'$, $j = 1, 2, 3$. By (3.4), $\varphi_1 = J\varphi_- = E(x,v)\varphi_-(x - \tau_-(x,v)v,v)$. This proves the formula for α_1 above. Next, from (3.6) we get similarly $\varphi_2 = \int \alpha_2\varphi_-d\mu(x')dv'$ with α_2 as stated above. Finally, combining Theorem 3.1 with Theorem 2.1 and Theorem 2.2, we obtain the properties stated for the trace α_3 of $f_3(\cdot, \cdot, x', v')$ on Γ_+.   □

REMARK.   The first two terms α_1, α_2 above are written as distributions with respect to the variables $(x', v') \in \Gamma_-$ with $(x, v) \in \Gamma_+$ considered as parameters (more precisely as linear functionals on $C^1(\Gamma_-; L^1(\Gamma_+, d\xi))$). One can also write them down as distributions with respect to $(x, v) \in \Gamma_+$ with $(x', v') \in \Gamma_-$ considered as parameters:

$$\alpha_1 = \frac{|n(x')\cdot v'|}{n(x)\cdot v} e^{-\int_0^{\tau_-(x,v)} \sigma_a(x-pv,v)dp} \delta_{\{x'+\tau_+(x',v')v'\}}(x)\delta(v-v'),$$

$$\alpha_2 = \frac{|n(x')\cdot v'|}{n(x)\cdot v} \int_0^{\tau_+(x',v')} e^{-\int_0^{\tau_+(x'+tv',v)} \sigma_a(x-pv,v)dp} e^{-\int_0^t \sigma_a(x+pv',v')dp}$$

$$\times k(x+tv',v',v)\delta_{\{x'+tv'+\tau_+(x'+tv',v)\}}(x)\,dt.$$

4. The inverse problem

Theorem 3.2 suggests the following way for solving (IP). Assume that we are given the albedo operator $\mathcal{A}$, corresponding to some admissible pair (σ_a, k), satisfying either (1.2) or (1.3). Then we also know the distribution $\alpha(x, v, x', v')$. By Theorem 3.2, $\alpha = \alpha_1 + \alpha_2 + \alpha_3$. Here α_1 is a delta-type distribution supported on a $(2n - 1)$-dimensional variety in $\Gamma_+ \times \Gamma_-$. Next, α_2 is also a delta-type distribution (provided

that $n \geq 3$) supported on a $3n$-dimensional variety in $\Gamma_+ \times \Gamma_-$, while α_3 is a (locally L^1) function on the $(4n - 2)$-dimensional $\Gamma_+ \times \Gamma_-$. Notice that if $n = 2$, then α_2 is a function as well. Therefore, if $n \geq 3$, one can distinguish between $\alpha_1 + \alpha_2$ and α_3. Moreover, since α_1 and α_2 have different degrees of singularities, one can recover α_1 and α_2. Now, if $\sigma_a = \sigma_a(x, |v|)$, then α_1 determines the X-ray transform $\int \sigma_a(x + s\omega, |v|)ds$ of σ_a for all x, $|v|$ and ω in an open subset of S^2 (for all $\omega \in S^2$ if V is spherically symmetric). This determines uniquely σ_a (see e.g. [8]). Next, once we know σ_a, from α_2 we can recover k. If $n = 2$, then we can recover α_1 and therefore σ_a, but we cannot (at least using those arguments) distinguish between α_2 and α_3 which are both functions and therefore our approach does not work for reconstructing k in two dimensions. Below we make those arguments precise and moreover we find explicit formulas (see also [4] for the time-dependent case) for σ_a, k in terms of α.

Assume that $\varphi \in C_0^\infty(\mathbf{R}^n)$, $0 \leq \varphi \leq 1$, $\varphi(0) = 1$, $\int \varphi(x)dx = 1$. Given $\varepsilon > 0$, set

$$\phi_\varepsilon(x, v, x', v') = \varphi\left(\frac{x - x' - \tau_-(x, v)}{\varepsilon}\right) \varphi\left(\frac{v - v'}{\varepsilon}\right).$$

Proposition 4.1. *If either (1.2) or (1.3) holds, then*

$$(4.1) \quad \lim_{\varepsilon \to 0} \int_{\Gamma_-} \alpha(x, v, x', v')\phi_\varepsilon(x, v, x', v')\, d\mu(x')\, dv' = e^{-\int_0^{\tau_-(x,v)} \sigma_a(x - pv, v)dp},$$

where the integral is to be considered in distribution sense and the limit holds in $L^1_{\mathrm{loc}}(\Gamma_+, d\xi)$.

Proof.   It should be noted first that $\tau_-(x, v)$ is smooth except on a closed subset of Γ_+ of measure zero, where it may have jumps. Nevertheless, the formal integral above is well-defined as will become clear from the proof.

It is easy to see that the limit (4.1) is trivially satisfied with α replaced by α_1. We will show below that if we replace α by α_2 and α_3, respectively, then the limit in (4.1) vanishes considered in $L^1(\Gamma_+ \cap \{|v| \leq M\}, d\xi)$ for any $M > 0$. To this end, choose $0 \leq \chi \in C_0^\infty(V)$. Then

$$(4.2) \quad 0 \leq \int_{\Gamma_+} \int_{\Gamma_-} \alpha_2(x, v, x', v')\phi_\varepsilon(x, v, x', v')\chi(v)\, d\mu(x')dv'\, d\xi(x, v)$$

$$\leq \int_{\Gamma_+} \int_V \int_0^{\tau_-(x,v)} \varphi\left(\frac{v - v'}{\varepsilon}\right) k(x - sv, v', v)\chi(v)\, ds\, dv'\, d\xi(x, v)$$

$$\leq \int_{X \times V} \int_V \varphi\left(\frac{v - v'}{\varepsilon}\right) k(x, v', v)\chi(v)\, dv'\, dx\, dv$$

$$\leq \int_{W_\varepsilon} \chi(v)k(x, v', v)\, dx\, dv'\, dv$$

$$\to 0, \quad \text{as } \varepsilon \to 0,$$

Here $W_\varepsilon = \{(x, v', v) \in X \times V \times V; \ v \in \operatorname{supp}\chi, \ |v - v'| < c\varepsilon\}$ with $c > 0$ depending on φ. Since $\chi(v)k(x, v', v)$ belongs to $L^1(X \times V \times V)$ and $\operatorname{meas}(W_\varepsilon) \to 0$, as $\varepsilon \to 0$, we get that the limit in (4.2) is zero, as stated.

Finally, assume that (1.2) holds. Then

$$(4.3) \qquad \int_{\Gamma_+} \left| \int_{\Gamma_-} \alpha_3(x, v, x', v') \phi_\varepsilon(x, v, x', v') \chi(v) \, d\mu(x') dv' \right| d\xi(x, v)$$

$$\leq \int_{E_\varepsilon} |n(x') \cdot v'|^{-1} |\alpha_3(x, v, x', v')| \chi(v) \, d\xi(x', v') \, d\xi(x, v)$$

$$\to 0, \quad \text{as } \varepsilon \to 0,$$

where $E_\varepsilon = \{(x, v, x', v') \in \Gamma_+ \times \Gamma_-; \ v \in \operatorname{supp}\chi, \ |v - v'| < c\varepsilon\}$. Since every bounded function on Γ_+, vanishing for large $|v|$, belongs to $L^1(\Gamma_+, d\xi)$, by Theorem 3.2 we conclude that the integrand above belongs to $L^1(\Gamma_+ \times \Gamma_-; d\xi(x,v) d\xi(x',v'))$. The limit in (4.3) is zero as stated, because $\operatorname{meas}(E_\varepsilon) \to 0$, as $\varepsilon \to 0$. Combining (4.2) and (4.3) we complete the proof in the case where (1.2) holds. The proof in the case (1.3) is similar. $\qquad\square$

Next, denote by $\pi_{v,v'}(x)$ the projection of x onto the plane spanned by v, v' provided that v and v' are linearly independent. Pick a vector $m(v, v') \neq 0$ in $\operatorname{span}\{v, v'\}$, such that $m \cdot v' = 0$, for example, $m(v, v') = (v \cdot v'/|v'|^2)v' - v$. Choose $\varphi_1 \in C_0^\infty(\mathbf{R}^1)$ with $0 \leq \varphi_1 \leq 1$, $\int \varphi_1(s) ds = 1$. Consider the function

$$\phi_{\varepsilon_1, \varepsilon_2}(x', v, v') = \frac{1}{\varepsilon_1} \varphi_1\left(\frac{x' \cdot m(v, v')}{\varepsilon_1 v \cdot m(v, v')}\right) \varphi\left(\frac{x' - \pi_{v,v'}(x')}{\varepsilon_2}\right).$$

Denote by $D \subset V^2 := V \times V$ the variety $D = \{(v, v') \in V^2; \ v \text{ and } v' \text{ are linearly dependent}\}$.

Proposition 4.2. *Assume that $n \geq 3$ and either (1.2) or (1.3) holds. Then for $x \in X$ we have*

$$(4.4) \qquad \lim_{\varepsilon_1 \to 0} \lim_{\varepsilon_2 \to 0} \int_{\partial X} \alpha(x + \tau_+(x, v)v, v, x', v')$$

$$\times \phi_{\varepsilon_1, \varepsilon_2}(x' - x + \tau_-(x, v')v', v, v') \, d\mu(x')$$

$$= e^{-\int_0^{\tau_-(x,v')} \sigma_a(x - pv, v') dp} e^{-\int_0^{\tau_+(x,v)} \sigma_a(x + pv, v) dp} k(x, v', v),$$

where the limit holds in $L^1_{\mathrm{loc}}(X \times (V^2 \setminus D))$.

Proof.   Denote

$$E(s, x, v, v') = e^{-\int_0^s \sigma_a(x - pv, v) dp} e^{-\int_0^{\tau_-(x - sv, v')} \sigma_a(x - sv - pv', v') dp}$$

(see the formula for α_2 in Theorem 3.2). Since $\alpha_1 = 0$ for $v \neq v'$, we get that (4.4) vanishes with α replaced by α_1. Next, for α_2 we get

$$\int_{\partial X} \alpha_2(x + \tau_+(x,v)v, v, x', v')\phi_{\varepsilon_1,\varepsilon_2}(x' - x + \tau_-(x,v')v', v, v')\, d\mu(x')$$

$$= \int_0^{\tau(x,v)} E(s, x + \tau_+(x,v)v, v, v')k(x + (\tau_+(x,v) - s)v, v', v)$$

$$\times \phi_{\varepsilon_1,\varepsilon_2}\Big([\tau_+(x,v) - s]v - [\tau_-(x + \tau_+(x,v)v - sv, v') - \tau_-(x,v')]v', v, v'\Big)\, ds$$

$$= \int_0^{\tau(x,v)} \frac{1}{\varepsilon_1}\varphi_1\left(\frac{\tau_+(x,v) - s}{\varepsilon_1}\right) E(s, x + \tau_+(x,v)v, v, v')$$

$$\times k(x + (\tau_+(x,v) - s)v, v', v)\, ds.$$

Since the function $s \to E(s, x + \tau_+(x,v)v, v, v')k(x + (\tau_+(x,v) - s)v, v', v) \in L^1_{\text{loc}}(X \times V \times V)$ is continuous, we get that the limit above as $\varepsilon_1 \to 0$ exists and equals $E(\tau_+(x,v), x + \tau_+(x,v)v, v, v')k(x, v', v)$ which is exactly the right-hand side of (4.4).

In order to complete the proof, we have to show that (4.4) vanishes for $\alpha = \alpha_3$. Fix $\chi \in C_0^\infty(X \times (V^2 \setminus D))$.

$$(4.5) \quad \int_{X \times V} \int_{\Gamma_-} \frac{1}{\varepsilon_1}\varphi_1\left(\frac{(x' - x) \cdot m}{\varepsilon_1 v \cdot m}\right)\varphi\left(\frac{x' - x - \pi_{v,v'}(x' - x)}{\varepsilon_2}\right)$$

$$\times \frac{\alpha_3(x + \tau_+(x,v)v, v, x', v')}{|n(x') \cdot v'|}\chi(x, v, v')\, d\xi(x', v')\, dx\, dv$$

$$\leq \frac{1}{\varepsilon_1}\int_{F_{\varepsilon_2}} \frac{\alpha_3(x + \tau_+(x,v)v, v, x', v')}{|n(x') \cdot v'|}\chi(x, v, v')\, dx\, dv\, d\xi(x', v'),$$

where $F_{\varepsilon_2} = \{(x, v, x', v') \in X \times V \times \Gamma_-;\ (x, v, v') \in \text{supp}\,\chi,\ |x - x' - \pi_{v,v'}(x - x')| < c\varepsilon_2\}$. By Theorem 3.2 and Lemma 2.1,

$$\frac{\alpha_3(x + \tau_+(x,v)v, v, x', v')}{|n(x') \cdot v'|\tau(x,v)} \in L^\infty(\Gamma_-;\ L^1(X_x \times V_v))$$

and clearly, $0 \leq \tau(x,v) \leq C < \infty$ for $(x, v, v') \in \text{supp}\,\chi$. Therefore, the integrand in (4.5) is an L^1-function. On the other hand, $\text{meas}\,(F_{\varepsilon_2}) \to 0$, as $\varepsilon_2 \to 0$, because F_{ε_2} is an ε_2-small neighborhood of a variety of dimension $3n + 1$ in the $4n - 1$ dimensional $X \times V \times \Gamma_-$. Consequently, (4.5) tends to zero, as $\varepsilon_2 \to 0$. $\qquad \square$

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