Kuwae, K.
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FUNCTIONAL CALCULUS FOR DIRICHLET FORMS

Dedicated to Professor Shinzo Watanabe on the occasion of his sixtieth birthday

KAZUHIRO KUWAE

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1. Introduction

In this paper, we investigate the structure of local space (in the broad sense) of (semi-)Dirichlet forms and the functional calculus. In the theory of Dirichlet forms developed by M. Fukushima and M. Silverstein, the local space of Dirichlet forms on a locally compact separable metric space played an important role. In particular, M. Fukushima translated the chain rule of energy measures of continuous part for the functions belonging locally in the Dirichlet space into that of corresponding local martingale additive functionals. Their definition of locality depends on the locally compactness of the state space. The developments can be seen in the textbook M. Fukushima,Y. Oshima and M. Takeda [11].

Meanwhile recent developments of the theory of Dirichlet forms treat the cases of non-locally compact state spaces, for example, topological vector spaces, loop spaces, the space of probability measures, etc. Such examples can be seen in Z.-M. Ma and M. Röckner [17], M. Röckner and B. Schmuland [21], in which the non-symmetric but nearly symmetric Dirichlet forms on a general non-locally compact state space are formulated and necessary and sufficient conditions for the existence of the pairs of Markov processes associated with forms are given. Very recently Z.-M. Ma, L. Overbeck and M. Röckner [16] reformulated the notion of semi-Dirichlet forms and gave the necessary and sufficient conditions for the existence of Markov processes associated with forms on a non-locally compact state space, extending the works of P.A. Ancona [2], S. Carrillo Menendez [4].

The notion of local space in the broad sense employed in §5.5 of [11] is more intrinsic for this general framework than that in the original sense. This broad sense notion is firstly appeared in Y. Oshima and T. Yamada [20] in order to represent the continuous additive functionals locally of zero energy. In this connection, P.J. Fitzsimmons [8] utilized such local spaces to investigate the absolute continuity of symmetric diffusion processes. Our first aim of this paper is to clarify the analytic structure of the local space in the framework of semi-Dirichlet forms on a (not necessarily locally compact) separable metric space. The second purpose is to give the chain rule for the functions in local space which is described in terms of energy measure of continuous part in the framework of symmetric quasi-regular Dirichlet forms. We also propose the stochastic integrals for martingale additive functionals locally of finite energy and the chain rule of stochastic integrals for these functionals. Our result is motivated by (4.1), (4.2) in [8].

The organization of this paper is as follows. In Section 2, we present some basic facts on positivity preserving forms and semi-Dirichlet forms without assuming the quasi-regularity. In Section 3, we collect the $\mathcal{E}$-quasi-notions on a semi-Dirichlet form and investigate its part space on an $\mathcal{E}$-quasi-open set in an analytic way. In Section 4, we define the local space of semi-Dirichlet form and show that our local space is sufficiently large in a sense (Theorem 4.1). We also give an identification of the local space of a part space on an $\mathcal{E}$-quasi-open set (Theorem 4.2). In Section 5, we give the Beurling-Deny and LeJan formulae for symmetric quasi-regular Dirichlet forms. The uniqueness of our Beurling-Deny decomposition is understood in a strict sense. Under some conditions on 1-capacity, the uniqueness of decomposition holds in the ordinary sense as in [1], [23]. The energy measure of continuous part can be extended to the functions in our local space (Lemma 5.2 and Lemma 5.3). In Section 6, we give the chain rule of energy measure of continuous part for symmetric quasi-regular Dirichlet forms (Theorem 6.1). In Section 7, we give the chain rule of stochastic integrals of local square integrable martingales in the framework of symmetric processes. In the last section, we give some examples.

2. Positivity preserving forms and semi-Dirichlet forms

Let X be a separable metric space and m a σ-finite Borel measure on X. For functions u, v on X, we write $u \vee v = \max\{u,v\}$, $u \wedge v = \min\{u,v\}$, $u^+ = u \vee 0$, $u^- = (-u) \vee 0$. Let $\mathcal{E}$ be a bilinear form with domain $\mathcal{F}$ on the real Hilbert space $L^2(X;m)$ with inner product $(\cdot,\cdot)_m$. We set $\mathcal{E}_\alpha(u,v) = \mathcal{E}(u,v) + \alpha(u,v)_m$, $\alpha > 0$, $\hat{\mathcal{E}}(u,v) = \mathcal{E}(v,u)$, $\tilde{\mathcal{E}}(u,v) = 1/2\{\mathcal{E}(u,v) + \hat{\mathcal{E}}(u,v)\}$ and $\check{\mathcal{E}}(u,v) = 1/2\{\mathcal{E}(u,v) - \hat{\mathcal{E}}(u,v)\}$ for u, $v \in \mathcal{F}$. We call $\tilde{\mathcal{E}}$, $\check{\mathcal{E}}$ the symmetric, anti-symmetric part of $\mathcal{E}$, respectively. $(\mathcal{E},\mathcal{F})$ with $\mathcal{F}$ dense in $L^2(X;m)$ is called a coercive closed form if

(i) $(\tilde{\mathcal{E}},\mathcal{F})$ is non-negative definite and closed on $L^2(X;m)$

(ii) (Weak sector condition). There exists a constant $K > 0$ such that

$$|\mathcal{E}_1(u,v)| \leq K\mathcal{E}_1(u,u)^{1/2}\mathcal{E}_1(v,v)^{1/2} \text{ for any } u,v \in \mathcal{F}.$$

We sometimes assume

(ii)′ (Strong sector condition). There exists a constant $K > 0$ such that

$$|\mathcal{E}(u,v)| \leq K\mathcal{E}(u,u)^{1/2}\mathcal{E}(v,v)^{1/2} \text{ for any } u,v \in \mathcal{F}.$$

Proposition 2.1. *Let $(\mathcal{E},\mathcal{F})$ be a coercive closed form on $L^2(X;m)$. Suppose that $\{u_n\}$ is an $\tilde{\mathcal{E}}$-Cauchy sequence in $\mathcal{F}$ and there exists a subsequence $\{n_k\}$ of $\{n\}$*

and $u \in \mathcal{F}$ such that the Cesàro mean $v_k = (1/k)\sum_{i=1}^{k} u_{n_i}$ of u_{n_k} converges to u as $k \to \infty$ in $\tilde{\mathcal{E}}^{1/2}$. Then

$$\lim_{n\to\infty} \tilde{\mathcal{E}}(u_n - u, u_n - u) = 0.$$

Proof. We may assume $u = 0$. It suffices to show $\lim_{k\to\infty} \tilde{\mathcal{E}}(u_{n_k}, u_{n_k}) = 0$. We get easily

$$\begin{aligned}\tilde{\mathcal{E}}(u_{n_k}, u_{n_k})^{1/2} &\le \tilde{\mathcal{E}}(u_{n_k} - v_k, u_{n_k} - v_k)^{1/2} + \tilde{\mathcal{E}}(v_k, v_k)^{1/2} \\ &\le \frac{1}{k}\sum_{i=1}^{k} \tilde{\mathcal{E}}(u_{n_i} - u_{n_k}, u_{n_i} - u_{n_k})^{1/2} + \tilde{\mathcal{E}}(v_k, v_k)^{1/2}.\end{aligned}$$

Since $\{u_{n_k}\}$ is $\tilde{\mathcal{E}}$-Cauchy, the first term of the right hand side converges to 0 as $k \to \infty$. □

A coercive closed form $(\mathcal{E}, \mathcal{F})$ is called a positivity preserving form on $L^2(X; m)$ if in addition

(iii) (Positivity preserving property). For every $u \in \mathcal{F}$, $u^+ \in \mathcal{F}$ and $\mathcal{E}(u, u^+) \ge 0$.

For any positivity preserving form $(\mathcal{E}, \mathcal{F})$ on $L^2(X; m)$, $(\hat{\mathcal{E}}, \mathcal{F})$ and $(\tilde{\mathcal{E}}, \mathcal{F})$ are positivity preserving forms (see Remark 1.4(i) in [18]). In particular, for any positivity preserving form $(\mathcal{E}, \mathcal{F})$ on $L^2(X; m)$, $\mathcal{F}$ is a vector lattice, namely

$$u, v \in \mathcal{F} \Longrightarrow u \wedge v \in \mathcal{F} \text{ and } \tilde{\mathcal{E}}(u \wedge v, u \wedge v) \le \tilde{\mathcal{E}}(u, u) + \tilde{\mathcal{E}}(v, v). \tag{1}$$

Indeed, since $(\tilde{\mathcal{E}}, \mathcal{F})$ is a positivity preserving form on $L^2(X; m)$, we have $|u| \in \mathcal{F}$ and $\tilde{\mathcal{E}}(|u|, |u|) \le \tilde{\mathcal{E}}(u, u)$ for any $u \in \mathcal{F}$ (see Proposition 1.3(ii) in [18]). Hence

$$\begin{aligned}\tilde{\mathcal{E}}(u \wedge v, u \wedge v) &= \tilde{\mathcal{E}}\left(\frac{u + v - |u - v|}{2}, \frac{u + v - |u - v|}{2}\right) \\ &\le \frac{1}{2}\{\tilde{\mathcal{E}}(u + v, u + v) + \tilde{\mathcal{E}}(|u - v|, |u - v|)\} \\ &\le \frac{1}{2}\{\tilde{\mathcal{E}}(u + v, u + v) + \tilde{\mathcal{E}}(u - v, u - v)\} \\ &= \tilde{\mathcal{E}}(u, u) + \tilde{\mathcal{E}}(v, v).\end{aligned}$$

A coercive closed form $(\mathcal{E}, \mathcal{F})$ is called a semi-Dirichlet form on $L^2(X; m)$ if it satisfies

(iv) (Semi-Dirichlet property). For every $u \in \mathcal{F}$, $u^+ \wedge 1 \in \mathcal{F}$ and

$$\mathcal{E}(u + u^+ \wedge 1, u - u^+ \wedge 1) \ge 0,$$

equivalently $\mathcal{E}(u^+ \wedge 1, u - u^+ \wedge 1) \ge 0$.

A coercive closed form $(\mathcal{E},\mathcal{F})$ is called a Dirichlet form on $L^2(X;m)$ if both $(\mathcal{E},\mathcal{F})$ and $(\hat{\mathcal{E}},\mathcal{F})$ are semi-Dirichlet forms.

For any semi-Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(X;m)$, $(\mathcal{E},\mathcal{F})$, $(\hat{\mathcal{E}},\mathcal{F})$ and $(\tilde{\mathcal{E}},\mathcal{F})$ are positivity preserving forms (see Remark 1.4 (iii) in [18]). So $\mathcal{F}$ is a vector lattice.

Next proposition and lemma are not described in [16] explicitely. We show the proofs for readers convenience.

Proposition 2.2. *Let $(\mathcal{E},\mathcal{F})$ be a coercive closed form on $L^2(X;m)$.*

(i) *Let $u \in \mathcal{F}$ and assume that*

(SD) *for every $\varepsilon > 0$ there exists $\varphi_\varepsilon : \boldsymbol{R} \to [-\varepsilon, 1+\varepsilon]$ such that $\varphi_\varepsilon(t) = t$ for all $t \in [0,1]$, $0 \le \varphi_\varepsilon(t) - \varphi_\varepsilon(s) \le t - s$ if $s \le t$, $\varphi_\varepsilon(u) \in \mathcal{F}$, and $\liminf_{\varepsilon\to 0} \mathcal{E}(\varphi_\varepsilon(u), u - \varphi_\varepsilon(u)) \ge 0$.*

Then $u^+ \wedge 1 \in \mathcal{F}$ and $\mathcal{E}(u^+ \wedge 1, u - u^+ \wedge 1) \ge 0$.

(ii) *$(\mathcal{E},\mathcal{F})$ is a semi-Dirichlet form if and only if* (SD) *holds for all u in a dense subset of $\mathcal{F}$.*

Proof. (i) Since $\varphi_\varepsilon(u) \to u^+ \wedge 1$ in $L^2(X;m)$ as $\varepsilon \to 0$, we see

$$\begin{aligned}\liminf_{\varepsilon\to 0} \mathcal{E}_1(\varphi_\varepsilon(u), u - \varphi_\varepsilon(u)) &= \liminf_{\varepsilon\to 0} \mathcal{E}(\varphi_\varepsilon(u), u - \varphi_\varepsilon(u)) + (u^+ \wedge 1, u - u^+ \wedge 1)_m \\ &\ge 0.\end{aligned}$$

Hence we get by the weak sector condition

$$\liminf_{\varepsilon\to 0} \mathcal{E}_1(\varphi_\varepsilon(u), \varphi_\varepsilon(u)) \le K^2 \mathcal{E}_1(u,u).$$

By taking a sequence $\{\varepsilon_n\}$ with $\liminf_{\varepsilon\to 0} \mathcal{E}_1(\varphi_\varepsilon(u), \varphi_\varepsilon(u)) = \lim_{n\to\infty} \mathcal{E}_1(\varphi_{\varepsilon_n}(u), \varphi_{\varepsilon_n}(u))$, we have $\lim_{n\to\infty} \mathcal{E}_1(\varphi_{\varepsilon_n}(u), \varphi_{\varepsilon_n}(u)) \le K^2\mathcal{E}_1(u,u)$, so $\sup_{n\in N} \mathcal{E}_1(\varphi_{\varepsilon_n}(u), \varphi_{\varepsilon_n}(u)) < \infty$. In view of Chapter I Lemma 2.12 in [17], we have $\varphi_{\varepsilon_n}(u) \to u^+ \wedge 1 \in \mathcal{F}$ as $\varepsilon_n \downarrow 0$ weakly in $(\tilde{\mathcal{E}}_1, \mathcal{F})$ and $\mathcal{E}(u^+ \wedge 1, u^+ \wedge 1) \le \liminf_{n\to\infty} \mathcal{E}(\varphi_{\varepsilon_n}(u), \varphi_{\varepsilon_n}(u))$. Hence we get $\lim_{n\to\infty} \mathcal{E}_1(\varphi_{\varepsilon_n}(u), v) = \mathcal{E}_1(u^+ \wedge 1, v)$ for any $v \in \mathcal{F}$ and by (SD)

$$\begin{aligned}\mathcal{E}(u^+ \wedge 1, u - u^+ \wedge 1) &\ge \lim_{n\to\infty} \mathcal{E}(\varphi_{\varepsilon_n}(u), u) - \liminf_{n\to\infty} \mathcal{E}(\varphi_{\varepsilon_n}(u), \varphi_{\varepsilon_n}(u)) \\ &\ge \limsup_{n\to\infty} \mathcal{E}(\varphi_{\varepsilon_n}(u), u - \varphi_{\varepsilon_n}(u)) \\ &\ge \liminf_{n\to\infty} \mathcal{E}(\varphi_{\varepsilon_n}(u), u - \varphi_{\varepsilon_n}(u)) \ge 0.\end{aligned}$$

(ii) Suppose that (SD) holds for all u in a dense subset $\mathcal{D}$ of $\mathcal{F}$. We want to show that it holds for all $u \in \mathcal{F}$. By (i) we have $u^+ \wedge 1 \in \mathcal{F}$ and $\mathcal{E}(u^+ \wedge 1, u - u^+ \wedge 1) \ge 0$ for all $u \in \mathcal{D}$. Assume $u \in \mathcal{F}$ and let $\{u_n\} \subset \mathcal{D}$ be the $\tilde{\mathcal{E}}_1$-approximating sequence for u. Since $0 \le \mathcal{E}(u_n^+ \wedge 1, u_n - u_n^+ \wedge 1)$, we have

$$\mathcal{E}(u_n^+ \wedge 1, u_n^+ \wedge 1) \le \mathcal{E}(u_n^+ \wedge 1, u_n) \le K\mathcal{E}_1(u_n^+ \wedge 1, u_n^+ \wedge 1)^{1/2} \mathcal{E}_1(u_n, u_n)^{1/2}.$$

Hence we obtain

$$\sup_{n\in N} \mathcal{E}(u_n^+ \wedge 1, u_n^+ \wedge 1) \leq K^2 \sup_{n\in N} \mathcal{E}(u_n, u_n) < \infty.$$

Since $u_n^+ \wedge 1 \to u^+ \wedge 1$ in $L^2(X;m)$ as $n \to \infty$, $u_n^+ \wedge 1 \to u^+ \wedge 1$ as $n \to \infty$ weakly in $(\tilde{\mathcal{E}}_1, \mathcal{F})$ and $\mathcal{E}(u^+ \wedge 1, u^+ \wedge 1) \leq \liminf_{n\to\infty} \mathcal{E}(u_n^+ \wedge 1, u_n^+ \wedge 1)$ (Chapter I Lemma 2.12 in [17]). The rest of the proof is similar to (i). □

Lemma 2.1. *Let $(\mathcal{E}, \mathcal{F})$ be a semi-Dirichlet form on $L^2(X;m)$ and $u \in \mathcal{F}$. Then*

(i) $$\lim_{n\to\infty} \mathcal{E}_1(u - (-n) \vee u \wedge n, u - (-n) \vee u \wedge n) = 0.$$

(ii) $$\lim_{\varepsilon\to 0} \mathcal{E}_1((-\varepsilon) \vee u \wedge \varepsilon, (-\varepsilon) \vee u \wedge \varepsilon) = 0.$$

Proof. Since $(-\alpha) \vee u \wedge \alpha = u^+ \wedge \alpha - u^- \wedge \alpha$ for $\alpha > 0$, we may assume $u \geq 0$ m-a.e. Note that $\mathcal{E}(u \wedge \alpha, u \wedge \alpha) \leq \mathcal{E}(u \wedge \alpha, u)$ for any $u \in \mathcal{F}$, $\alpha > 0$ (see Remark 2.2 in [16]). By using the weak sector condition, we have

$$\mathcal{E}_1(u \wedge \alpha, u \wedge \alpha)^{1/2} \leq K \mathcal{E}_1(u, u)^{1/2}.$$

Hence we have $\{u \wedge \alpha\}$ is $\tilde{\mathcal{E}}_1^{1/2}$-bounded and $u \wedge \alpha$ converges $\tilde{\mathcal{E}}_1$-weakly to some element of $\mathcal{F}$. Noting $u \wedge \alpha$ is $L^2(X;m)$-convergent, we get for any $v \in \mathcal{F}$, $\mathcal{E}_1(v, u \wedge \alpha) \to \mathcal{E}_1(v, u)$ as $\alpha \to \infty$ and $\mathcal{E}_1(u \wedge \alpha, v) \to 0$ as $\alpha \to 0$. In particular, $\mathcal{E}_1(u - u \wedge \alpha, u - u \wedge \alpha) \leq \mathcal{E}_1(u, u - u \wedge \alpha) \to 0$ as $\alpha \to \infty$ and $\mathcal{E}_1(u \wedge \alpha, u \wedge \alpha) \leq \mathcal{E}_1(u \wedge \alpha, u) \to 0$ as $\alpha \to 0$. □

Corollary 2.1. *$\mathcal{F}_b = \mathcal{F} \cap L^\infty(X;m)$ is $\tilde{\mathcal{E}}_1^{1/2}$-dense in $\mathcal{F}$.*

For a closed subset F of X, we set $\mathcal{F}_F = \{u \in \mathcal{F} : u = 0\ m\text{-a.e. on } X \setminus F\}$. An increasing sequence $\{F_n\}_{n\in N}$ of closed subset of X is said to be an $\mathcal{E}$-nest or generalized nest if $\bigcup_{n=1}^\infty \mathcal{F}_{F_n}$ is $\tilde{\mathcal{E}}_1^{1/2}$-dense in $\mathcal{F}$. A subset N of X is said to be $\mathcal{E}$-polar or $\mathcal{E}$-exceptional if there exists an $\mathcal{E}$-nest $\{F_n\}_{n\in N}$ such that $N \subset \bigcap_{n=1}^\infty (X \setminus F_n)$. A statement $P = P(x)$ depending $x \in X$ is said to be "P $\mathcal{E}$-q.e." if there exists an $\mathcal{E}$-polar set N such that $P(x)$ holds for $x \in X \setminus N$. A function u is said to be $\mathcal{E}$-quasi-continuous if there exists an $\mathcal{E}$-nest $\{F_n\}_{n\in N}$ such that $u|_{F_n}$ is continuous on F_n for each $n \in \boldsymbol{N}$. A subset E of X is said to be $\mathcal{E}$-quasi-open if there exists an $\mathcal{E}$-nest $\{F_n\}_{n\in N}$ such that $E \cap F_n$ is open with respect to the relative topology on F_n for each $n \in \boldsymbol{N}$. $\mathcal{E}$-quasi-closedness can be defined as the dual notion. For two subsets A, B of X, we write $A \subset B$ $\mathcal{E}$-q.e. if $I_A \leq I_B$ $\mathcal{E}$-q.e. If a function u has an $\mathcal{E}$-quasi-continuous m-version, we denote it by $\tilde{u}$. We introduce three conditions called the conditions of quasi-regularity of $(\mathcal{E}, \mathcal{F})$ as follows:

(QR1) There exists an $\mathcal{E}$-nest of compact sets.

(QR2) There exists an $\mathcal{E}_1^{1/2}$-dense subset of $\mathcal{F}$ whose elements have $\mathcal{E}$-quasi m-continuous-versions.

(QR3) There exist an $\mathcal{E}$-polar set $P \subset X$ and $u_n \in \mathcal{F}$, $n \in \boldsymbol{N}$ having $\mathcal{E}$-quasi m-continuous-versions $\tilde{u}_n$, $n \in \boldsymbol{N}$ such that $\{\tilde{u}_n\}_{n \in N}$ separates the points of $X \setminus P$.

Assume that $(\mathcal{E}, \mathcal{F})$ is a quasi-regular semi-Dirichlet form, namely (QR1), (QR2), (QR3) hold. Then there exists an m-equivalence class of special standard processes $\mathbf{M}/\sim$ properly associated with $(\mathcal{E}, \mathcal{F})$. We consider a special standard process $\mathbf{M} = (\Omega, X_t, \zeta, P_x)$ properly associated with $(\mathcal{E}, \mathcal{F})$. Here properly association means that $x \mapsto \int_\Omega f(X_t(\omega))P_x(d\omega)$ is an $\mathcal{E}$-quasi-continuous m-version of $T_t f$ for $f \in \mathcal{B}_+(X) \cap L^2(X; m)$. If further $(\mathcal{E}, \mathcal{F})$ is a quasi-regular Dirichlet forms, there exists another Dirichlet space $(\hat{\mathcal{E}}, \hat{\mathcal{F}})$ on $L^2(\hat{X}; \hat{m})$ with a locally compact separable metric space $\hat{X}$ and a positive Radon measure $\hat{m}$ with full support on $\hat{X}$, which is C_0-regular and $\mathcal{E}$-quasi-homeomorphic to $(\mathcal{E}, \mathcal{F})$ (cf. [5]). Precisely to say, there exists an $\mathcal{E}$-nest $\{K_n\}$ of compact sets, and a locally compact separable metric space $\hat{X}$, and a map $i : Y = \bigcup_{n=1}^\infty K_n \to \hat{X}$ such that $i|_{K_n}$ is a homeomorphism and the image $(\hat{\mathcal{E}}, \hat{\mathcal{F}})$ of $(\mathcal{E}, \mathcal{F})$ for $\hat{m} = m \circ i^{-1}$ is a C_0-regular Dirichlet form on $L^2(\hat{X}; \hat{m})$ satisfying that $\{i(K_n)\}$ is an $\hat{\mathcal{E}}$-nest. The definition of the image $(\hat{\mathcal{E}}, \hat{\mathcal{F}})$ of $(\mathcal{E}, \mathcal{F})$ is as follows: Define an isometry $i^* : L^2(\hat{X}; \hat{m}) \to L^2(X; m)$ by setting $i^*(u^\sharp)$ to be the m-class represented by $\check{u} \circ i$ for any $\mathcal{B}(\hat{X})$-measurable $\hat{m}$-version $\check{u}$ of $u^\sharp \in L^2(\hat{X}; \hat{m})$. $(\hat{\mathcal{E}}, \hat{\mathcal{F}})$ is defined by $\hat{\mathcal{F}} = \{u^\sharp \in L^2(\hat{X}; \hat{m}) | i^*(u^\sharp) \in \mathcal{F}\}$ and $\hat{\mathcal{E}}(u^\sharp, v^\sharp) = \mathcal{E}(i^*(u^\sharp), i^*(v^\sharp))$ for $u^\sharp, v^\sharp \in \hat{\mathcal{F}}$ (cf. Chapter VI Theorem 1.2 in [17]). For a function u on X, we set $u^\sharp$ by $u^\sharp(y) = u(x)$ if $y = i(x)$ and otherwise $u^\sharp(y) = 0$. Then representations u, v of m-classes in $\mathcal{F}$ satisfy $u^\sharp, v^\sharp \in \hat{\mathcal{F}}$, $i^*(u^\sharp) = u$, $i^*(v^\sharp) = v$ and $\mathcal{E}(u, v) = \hat{\mathcal{E}}(u^\sharp, v^\sharp)$. Hence we can transfer the results of [11] to quasi-regular Dirichlet forms. Such procedure is called the "*transfer method*".

3. $\mathcal{E}$-quasi notions and the part space

Let X, m be as in Section 2. We fix a semi-Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X; m)$. Let $\{T_t\}_{t>0}$, $\{G_\alpha\}_{\alpha>0}$ be the semi-groups, resolvents on $L^2(X; m)$ associated with $(\mathcal{E}, \mathcal{F})$. Semi-Dirichlet property implies the sub-Markov property of T_t, αG_α, namely $0 \le f \le 1$ m-a.e.$\Longrightarrow 0 \le T_t f$, $\alpha G_\alpha f \le 1$ m-a.e. for any t, $\alpha > 0$. Let $p > 0$. $u \in L^2(X; m)$ is called p-excessive if $u \ge 0$ m-a.e. and $e^{-pt}T_t u \le u$ m-a.e. for any $t > 0$, or equivalently, $\alpha G_{\alpha+p} u \le u$ m-a.e. for any $\alpha > 0$. For any $u \in \mathcal{F}$ and $p > 0$, u is p-excessive if and only if $\mathcal{E}_p(u, v) \ge 0$ for any $v \in \mathcal{F}$, $v \ge 0$ m-a.e. (Theorem 2.4 in [16]).

Let h be a function on X. For an open set G, we let

$$\mathcal{L}_{h\,G} = \{u \in \mathcal{F} : u \ge h \ m\text{-a.e. on } G\}.$$

Suppose that $\mathcal{L}_{h\,G} \ne \emptyset$. Then there exists unique $h_G \in \mathcal{L}_{h\,G}$ such that for any $w \in$

$\mathcal{L}_{h\,G}$, $\mathcal{E}_1(h_G, w) \geq \mathcal{E}_1(h_G, h_G)$. h_G satisfies $\mathcal{E}_1(h_G, w) \geq 0$ for any $w \in \mathcal{F}$ with $w \geq 0$ m-a.e. on G. In particular, h_G is 1-excessive and $\mathcal{E}_1(h_G, w) = 0$ for any $w \in \mathcal{F}_{G^c}$.

We consider the class of functions

$$\mathcal{K} = \left\{ g \in L^1(X; m) \cap L^\infty(X; m) : 0 < g \leq 1 \ m\text{-a.e. and } \int_X g dm \leq 1 \right\}.$$

For a $g \in \mathcal{K}$, we let $h = G_1 g$. Then $h \in \mathcal{F}$ is a 1-excessive function: $0 \leq h$, $e^{-t}T_t h \leq h$ m-a.e. for any $t \geq 0$. Since $h \in \mathcal{F}$, $\mathcal{L}_{h\,G} \neq \emptyset$ for all open sets G. We consider the h-weighted capacity denoted by Cap_h defined as follows: for an open set G,

$$\mathrm{Cap}_h(G) = (h_G, g)_m$$

and for any subset B of X,

$$\mathrm{Cap}_h(B) = \inf\{\mathrm{Cap}_h(G) : B \subset G, G \text{ is an open subset of } X\}.$$

Then Cap_h is a Choquet capacity (see Corollary 2.22 in [16]).

It is showed in [16] that $\{F_n\}$ is an $\mathcal{E}$-nest if and only if $\lim_{n\to\infty} \mathrm{Cap}_h(X \setminus F_n) = 0$ (Theorem 2.14 in [16]). In particular the quasi notion defined by Cap_h is independent of the choice of such $g \in \mathcal{K}$. By this property, we get $\tilde{h} > 0$ $\mathcal{E}$-q.e. if h has an $\mathcal{E}$-quasi-continuous m-version $\tilde{h}$ (Proposition 2.18(ii) in [16]). In particular, we can consider the notion of $\mathcal{E}$-quasi-closure $\bar{A}^{\mathcal{E}}$ and $\mathcal{E}$-quasi-interior $A^{\mathcal{E}\text{-int}}$ of a subset A of X. Also for any measure μ which charges no $\mathcal{E}$-polar set, we can consider the notion of $\mathcal{E}$-quasi-support denoted by $\mathcal{E}$-supp$[\mu]$ (see [10]). In what follows, we fix $h = G_1 g$, $g \in \mathcal{K}$. The condition (QR2) implies that every $u \in \mathcal{F}$ has an $\mathcal{E}$-quasi-continuous m-version $\tilde{u}$ (Proposition 2.18(i) in [16]). Under (QR2), we can consider $\mathcal{L}_{h\,A} = \{u \in \mathcal{F} : \tilde{u} \geq \tilde{h} \ \mathcal{E}\text{-q.e. on } A\}$ for a subset A of X.

Lemma 3.1. (i) *Let u be an $\mathcal{E}$-quasi-continuous function on X and E be an $\mathcal{E}$-quasi-open subset of X. If $u \geq 0$ m-a.e. on E, then $u \geq 0$ $\mathcal{E}$-q.e. on E.*

(ii) *Suppose that* (QR2) *holds. Any m-negligible $\mathcal{E}$-quasi-open set is $\mathcal{E}$-polar.*

Proof. (i) The proof is similar to the proof of Proposition 2.18(iii), (iv) in [16].

(ii) Suppose that E is an m-negligible $\mathcal{E}$-quasi-open set. (i) implies $\mathcal{L}_{h\,E} = \{u \in \mathcal{F} : u \geq h \ m\text{-a.e. on } E\} = \mathcal{F}$. So the assertion holds by Theorem 2.10 in [16]. □

Lemma 3.2. *Let $\{\{F_n^k\}_{n\in N}\}_{k\in N}$ be a countable family of $\mathcal{E}$-nests. Then there exists a subsequence $\{n(l,k)\}_{l\geq 1}$ of $\{n\}$ depending on $k \in \boldsymbol{N}$ with $n(l,k) \geq l$ such that $F_l = \bigcap_{k=1}^\infty F_{n(l,k)}^k$ makes an $\mathcal{E}$-nest. In particular, for given $\mathcal{E}$-quasi-continuous functions $\{f_j\}$ (resp. $\mathcal{E}$-quasi-closed sets $\{A_j\}$), we can take common $\mathcal{E}$-nest $\{F_n\}_{n\in N}$ such that $f_j|_{F_n}$ is continuous on F_n (resp. $A_j \cap F_n$ is closed) for all j, $n \in \boldsymbol{N}$. Hence* *a countable intersection (resp. union) of $\mathcal{E}$-quasi-closed (resp. -open) sets is always $\mathcal{E}$-quasi-closed (resp. -open).*

Proof. It suffices to take $\{n(l,k)\}_{l\geq 1}$ with $n(l,k)\geq l$ such that $\mathrm{Cap}_h(X\backslash \hat{F}^k_{n(l,k)}) < 1/(2^k l)$. $\square$

Lemma 3.3. *Suppose that* (QR2) *holds. If $\{F_n\}$ is an $\mathcal{E}$-nest, then $X = \bigcup_{n=1}^{\infty} F_n^{\mathcal{E}\text{-int}}$ $\mathcal{E}$-q.e. if $\{F_n\}$ is an $\mathcal{E}$-nest. Further assume* (QR1). *Then the converse assertion holds.*

Proof. Since $A \subset B$ $\mathcal{E}$-q.e. implies $\bar{A}^{\mathcal{E}} \subset \bar{B}^{\mathcal{E}}$ $\mathcal{E}$-q.e., we see $\mathcal{L}_{h\,B} = \mathcal{L}_{h\,\bar{B}^{\mathcal{E}}}$ for any subset B of X. Here $\mathcal{L}_{h\,B} = \{u \in \mathcal{F} : \tilde{u} \geq \tilde{h}\ \mathcal{E}\text{-q.e. on } B\}$. From Theorem 2.10 in [16], we get $\mathrm{Cap}_h(B) = \mathrm{Cap}_h(\bar{B}^{\mathcal{E}})$ for any subset B of X. So the first assertion holds. Suppose that $(\mathcal{E},\mathcal{F})$ satisfies (QR1). Then any $\mathcal{E}$-quasi-closed set F is $\mathcal{E}$-quasi-compact, namely there exists an increasing sequence of compact sets $\{K_n\}$ with $K_n \subset F$ such that $\lim_{n\to\infty} \mathrm{Cap}_h(F \setminus K_n) = 0$. In particular Cap_h is continuous for decreasing sequence of $\mathcal{E}$-quasi-closed sets (Theorem 2.10 in [10]). Hence the converse assertion holds. $\square$

Consider a $g \in \mathcal{K}$ and set $\mathcal{F}^g = \mathcal{F} \cap L^2(X;gm)(=\mathcal{F})$ and $\mathcal{E}^g(u,v) = \mathcal{E}(u,v) + (u,v)_{gm}$ for $u,v \in \mathcal{F}$. Here $(u,v)_{gm} = \int_X uvg\,dm$. Then $(\mathcal{E}^g,\mathcal{F}^g)$ is a semi-Dirichlet form on $L^2(X;m)$. Let $\bar{\mathcal{F}}^g$ be the $\tilde{\mathcal{E}}^{g\,1/2}$-completion of $\mathcal{F}$. Then $\bar{\mathcal{F}}^g$ is continuously embedded in $L^2(X;gm)$.

Let $\mathcal{F}_e$ be the family of m-measurable functions u on X such that $|u| < \infty$ m-a.e. and there exists an $\tilde{\mathcal{E}}$-Cauchy sequence $\{u_n\}$ of $\mathcal{F}$ such that $\lim_{n\to\infty} u_n = u$ m-a.e. We call $\{u_n\}$ as above an approximating sequence for $u \in \mathcal{F}_e$. The space $\mathcal{F}^g_e$ can be similarly defined by replacing $(\tilde{\mathcal{E}},\mathcal{F})$ with $(\tilde{\mathcal{E}}^g,\mathcal{F}^g)$. We see easily that $\mathcal{F}_e$ is a linear space containing $\mathcal{F}$ and $\bar{\mathcal{F}}^g \subset \mathcal{F}^g_e \subset \mathcal{F}_e \cap L^2(X;gm)$.

Proposition 3.1. (i) *For any $u \in \mathcal{F}_e$ and its approximating sequence $\{u_n\}$, the limit $\tilde{\mathcal{E}}(u,u) = \lim_{n\to\infty} \tilde{\mathcal{E}}(u_n,u_n)$ exists and does not depend on the choice of the approximating sequences for u. $\tilde{\mathcal{E}}^{1/2}$ on $\mathcal{F}_e$ is a semi-norm. If further $(\mathcal{E},\mathcal{F})$ satisfies the strong sector condition, then the limit $\mathcal{E}(u,v) = \lim_{n\to\infty} \mathcal{E}(u_n,v_n)$ exists and does not depend on the choice of the approximating sequences $\{u_n\},\{v_n\}$ for $u,v \in \mathcal{F}_e$, respectively.*

(ii) *$u \in \mathcal{F}_e$ if and only if there exists an $\tilde{\mathcal{E}}^{1/2}$-bounded sequence $\{u_n\}$ of $\mathcal{F}$ such that $u_n \to u$, $n \to \infty$ m-a.e.*

(iii) *$(\tilde{\mathcal{E}},\mathcal{F}_e)$ is a vector lattice, namely*

$$(1)\qquad u,v \in \mathcal{F}_e \Longrightarrow u\wedge v \in \mathcal{F}_e \text{ and } \tilde{\mathcal{E}}(u\wedge v,u\wedge v) \leq \tilde{\mathcal{E}}(u,u) + \tilde{\mathcal{E}}(v,v).$$

If further $(\mathcal{E},\mathcal{F})$ satisfies the strong sector condition, then

$$u \in \mathcal{F}_e \Longrightarrow u^+ \wedge 1 \in \mathcal{F}_e \text{ and } \mathcal{E}(u^+ \wedge 1, u - u^+ \wedge 1) \geq 0.$$

(iv) $\mathcal{F} = \mathcal{F}_e \cap L^2(X;m)$.

Proof. (i) It suffices to show that for any $\tilde{\mathcal{E}}$-Cauchy sequence $\{u_n\}$ with $u_n \to 0$, $n \to \infty$ m-a.e., $\lim_{n\to\infty} \tilde{\mathcal{E}}(u_n, u_n) = 0$. This is a corollary of the results on Fatou property of positivity preserving forms due to B. Schmuland [22]. We will give another proof based on the regular representation. Since $(\tilde{\mathcal{E}}, \mathcal{F})$ is a symmetric positivity preserving form, we may assume the symmetricity of $(\mathcal{E}, \mathcal{F})$. The assertion is well-known when $(\mathcal{E}, \mathcal{F})$ is a C_0-regular symmetric Dirichlet form. We can get rid of the C_0-regularity of $(\mathcal{E}, \mathcal{F})$ under its symmetricity in view of the regular representation of symmetric Dirichlet forms. Indeed, let $(\Phi, X', m', \mathcal{E}', \mathcal{F}')$ be the regular representation of $(X, m, \mathcal{E}, \mathcal{F})$ with respect to some closed subalgebra L of $L^\infty(X;m)$ which satisfies the condition (L) appeared in pp 347 of [11]. Consider a 1-resolvent $G_1\Phi^{-1}g$ for $g \in L^2(X';m')$ with $0 < g \leq 1$ m'-a.e. Then we see that $\Phi(G_1\Phi^{-1}g)$ is the 1-resolvent of g with respect to $(\mathcal{E}', \mathcal{F}')$ and $\Phi(G_1\Phi^{-1}g) > 0$ m'-a.e. Since

$$\int_{X'} \{\Phi(|u_n|) \wedge \Phi(G_1\Phi^{-1}g)\}\Phi(G_1\Phi^{-1}g)dm' = \int_X (|u_n| \wedge G_1\Phi^{-1}g)G_1\Phi^{-1}gdm \to 0 \quad \text{as } n \to \infty,$$

we can take a subsequence $\{n_k\}$ such that $|\Phi(u_{n_k})| = \Phi(|u_{n_k}|) \to 0$, $k \to \infty$ m'-a.e. Hence $\mathcal{E}(u_{n_k}, u_{n_k}) = \mathcal{E}'(\Phi(u_{n_k}), \Phi(u_{n_k})) \to 0$ as $k \to \infty$. The proof is complete if $(\mathcal{E}, \mathcal{F})$ is a symmetric Dirichlet form. We should get rid of the semi-Dirichlet property of the dual form. Suppose that $(\mathcal{E}, \mathcal{F})$ is a symmetric positivity preserving form on $L^2(X;m)$. Take $g \in \mathcal{K}$ and consider $(\mathcal{E}^g, \mathcal{F}^g)$. Then $(\mathcal{E}^g, \mathcal{F}^g)$ is a symmetric positivity preserving form on $L^2(X;m)$. Let $\{T_t^g\}_{t\geq 0}$ be the $L^2(X;m)$-semigroup associated with $(\mathcal{E}^g, \mathcal{F}^g)$. We set $h_g = G^g g = \int_0^\infty T_t^g g dt$. Then we see that

$$\int_X (G^g g)^2 g dm \leq \int_X g dm < \infty \text{ and } h_g \geq G_1^g g > 0 \ m\text{-a.e.}$$

by use of a similar proof of Lemma 1.6.6 in [11] and Lemma 3.6 in [18]. In particular, $h_g^2 m$ is a σ-finite Borel measure on X. We consider another coercive closed form $(\mathcal{E}^{h_g}, \mathcal{F}^{h_g})$ as follows:

$$\begin{cases} \mathcal{F}^{h_g} = \{u \in L^2(X; h_g^2 m) : u h_g \in \mathcal{F}^g\} \\ \mathcal{E}^{h_g}(u, v) = \mathcal{E}^g(u h_g, v h_g), \ u, v \in \mathcal{F}^{h_g}. \end{cases}$$

Then $(\mathcal{E}^{h_g}, \mathcal{F}^{h_g})$ is a symmetric Dirichlet form on $L^2(X; h_g^2 m)$. Indeed the corresponding $L^2(X; h_g^2 m)$-semigroup $\{T_t^{h_g}\}_{t\geq 0}$ is given by $T_t^{h_g} u = h_g^{-1} T_t^g (u h_g)$ $h_g^2 m$-a.e. for $u \in L^2(X; h_g^2 m)$, hence it is Markovian. Recall that $\{u_n\}$ is an $\mathcal{E}$-Cauchy sequence with $u_n \to 0$, $n \to \infty$ m-a.e. Take an $f \in L^1(X;m)$ with $0 < f \leq 1$, m-a.e.

We set $g = f/(\sup_{n\geq 1} u_n^2 \vee 1)$. Then $g \in \mathcal{K}$. We see that $\{u_n\}$ is an $\mathcal{E}^g$-Cauchy sequence. Set $v_n = h_g^{-1} u_n$. Then $\{v_n\}$ is an $\mathcal{E}^{h_g}$-Cauchy sequence in $\mathcal{F}^{h_g}$ with $v_n \to 0$, $n \to \infty$ m-a.e. Owing to the first argument we see $\mathcal{E}^g(u_n, u_n) = \mathcal{E}^{h_g}(v_n, v_n) \to 0$ as $n \to \infty$. Hence $\mathcal{E}(u_n, u_n) \to 0$ as $n \to \infty$. The proof of the first assertion of (i) is complete. The latter assertion of (i) is easy in view of the strong sector condition.

(ii) The "only if"part is clear. Suppose that $\{u_n\}$ is an $\tilde{\mathcal{E}}^{1/2}$-bounded sequence of $\mathcal{F}$ with $u_n \to u$ as $n \to \infty$, m-a.e. Set $g = f/(\sup_{n\geq 1} u_n^2 \vee u^2 \vee 1)$ for $\{u_n\}$, u as in (i). Then $\{u_n\}$ is an $\tilde{\mathcal{E}}^{g\,1/2}$-bounded sequence of $\mathcal{F}^g$. Then by the Banach-Saks theorem, there exists a subsequence $\{n_k\}$ of $\{n\}$ and $v \in \bar{\mathcal{F}}^g \subset \mathcal{F}_e$ such that the Cesàro mean of u_{n_k} converges to v as $k \to \infty$ in $\tilde{\mathcal{E}}^{g\,1/2}$. On the other hand, $\lim_{n\to\infty}(u_n - u, u_n - u)_{gm} = 0$, which tells us $u = v \in \mathcal{F}_e$.

(iii) It suffices to show that

$$u \in \mathcal{F}_e \Longrightarrow |u| \in \mathcal{F}_e \text{ and } \tilde{\mathcal{E}}(|u|, |u|) \leq \tilde{\mathcal{E}}(u, u). \tag{2}$$

Suppose that there exists an $\tilde{\mathcal{E}}$-Cauchy sequence $\{u_n\}$ of $\mathcal{F}$ such that $u_n \to u$ as $n \to \infty$ m-a.e. Then $\{|u_n|\}$ is an $\tilde{\mathcal{E}}^{1/2}$-bounded sequence. Then by the same method as in (ii), there exists a subsequence $\{n_k\}$ and $v \in \bar{\mathcal{F}}^g \subset \mathcal{F}_e$ such that the Cesàro mean $v_k = (|u_{n_1}| + \cdots + |u_{n_k}|)/k$ of $\{|u_{n_k}|\}$ converges to v as $k \to \infty$ in $\tilde{\mathcal{E}}^{g\,1/2}$-norm. In particular, $|u| = v \in \mathcal{F}_e$ and

$$\begin{aligned}
\tilde{\mathcal{E}}(|u|, |u|)^{1/2} &= \lim_{k\to\infty} \tilde{\mathcal{E}}(v_k, v_k)^{1/2} \\
&\leq \lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} \tilde{\mathcal{E}}(|u_{n_i}|, |u_{n_i}|)^{1/2} \\
&\leq \lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} \tilde{\mathcal{E}}(u_{n_i}, u_{n_i})^{1/2} = \tilde{\mathcal{E}}(u, u)^{1/2}.
\end{aligned}$$

$u^+ \wedge 1 \in \mathcal{F}_e$ can be similarly proved. The inequality follows

$$\begin{aligned}
\mathcal{E}(u^+ \wedge 1, u^+ \wedge 1) &\leq \left\{ \liminf_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} \mathcal{E}(u_{n_i}^+ \wedge 1, u_{n_i})^{1/2} \right\}^2 \\
&\leq \liminf_{k\to\infty} \left\{ \frac{1}{k} \left(\sum_{i=1}^{k} \mathcal{E}(u_{n_i}^+ \wedge 1, u_{n_i}) \right)^{1/2} k^{1/2} \right\}^2 \\
&= \liminf_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} \mathcal{E}(u_{n_i}^+ \wedge 1, u_{n_i}) = \mathcal{E}(u^+ \wedge 1, u).
\end{aligned}$$

(iv) First we show that

$$u \in \mathcal{F}_e,\ v \in \mathcal{F} \Longrightarrow (-|u|) \vee v \wedge |u| \in \mathcal{F}. \tag{3}$$

Suppose that $\{u_n\}$ be an $\tilde{\mathcal{E}}$-Cauchy sequence of $\mathcal{F}$ such that $u_n \to u$ as $n \to \infty$ m-a.e. Then by (2), $\{(-|u_n|) \vee v \wedge |u_n|\}$ is an $\tilde{\mathcal{E}}^{1/2}$-bounded sequence of $\mathcal{F}$ such that $(-|u_n|) \vee v \wedge |u_n| \to (-|u|) \vee v \wedge |u|$ $n \to \infty$ m-a.e. On the other hand, $(-|u_n|) \vee v \wedge |u_n|$ is $L^2(X;m)$-bounded. So the Banach-Saks theorem tells us $(-|u|) \vee v \wedge |u| \in \mathcal{F}$. Next we show the assertion. Suppose that $u \in \mathcal{F}_e \cap L^2(X;m)$. Then there exists an $\tilde{\mathcal{E}}$-Cauchy sequence $\{u_n\}$ of $\mathcal{F}$ such that $u_n \to u$ as $n \to \infty$ m-a.e. Owing to (4), we have $(-|u|) \vee u_n \wedge |u| \in \mathcal{F}$ and $\{(-|u|) \vee u_n \wedge |u|\}$ is $\tilde{\mathcal{E}}^{1/2}$-bounded by (2) and $L^2(X;m)$-bounded. On the other hand, $(-|u|) \vee u_n \wedge |u| \to u$ as $n \to \infty$ m-a.e. Hence by the Banach-Saks theorem, we have $u \in \mathcal{F}$. The converse assertion is trivial. □

By virtue of Proposition 3.1, $\tilde{\mathcal{E}}$ can be well extended to $\mathcal{F}_e$ as a non-negative definite symmetric bilinear form. If $(\mathcal{E}, \mathcal{F})$ satisfies the strong sector condition, $\mathcal{E}$ can be well extended to $\mathcal{F}_e$ as a non-symmetric bilinear form satisfying

$$|\mathcal{E}(u,v)| \leq K \tilde{\mathcal{E}}(u,u)^{1/2} \tilde{\mathcal{E}}(v,v)^{1/2} \text{ for any } u, v \in \mathcal{F}_e,$$

for some $K > 0$. We call $\mathcal{F}_e$ the extended Dirichlet space of $(\mathcal{E}, \mathcal{F})$.

Proposition 3.2. *Assume that* (QR2) *and the strong sector condition hold. Then every $u \in \mathcal{F}_e$ has an $\mathcal{E}$-quasi-continuous m-version $\tilde{u}$ which is $\mathcal{E}$-q.e. finite.*

Proof. Let $(\mathcal{E}^g, \mathcal{F}^g)$ be as in the above. We see easily that $\{F_n\}_{n \in N}$ is an $\mathcal{E}$-nest if and only if $\{F_n\}$ is an $\mathcal{E}^g$-nest. An increasing sequence of closed sets $\{F_n\}$ is said to be an $\mathcal{E}_0^g$-nest if $\bigcup_{n=1}^{\infty} \mathcal{F}^g_{e\,F_n}$ is $\tilde{\mathcal{E}}^{g\,1/2}$-dense in $\mathcal{F}_e^g$. Here $\mathcal{F}^g_{e\,F_n} = \{u \in \mathcal{F}_e^g \colon u = 0$ m-a.e. on $F_n^c\}$ and $\mathcal{F}_e^g$ is the extended Dirichlet space of $(\mathcal{E}^g, \mathcal{F}^g)$. We see easily that an $\mathcal{E}^g$-nest is an $\mathcal{E}_0^g$-nest by the $\tilde{\mathcal{E}}^{g\,1/2}$-completeness of $\mathcal{F}_e^g$. Also the converse holds. Indeed, suppose that $\{F_l\}_{l \in N}$ is an $\mathcal{E}_0^g$-nest. Take $u \in \mathcal{F}^g$. Then there exists $u_n \in \bigcup_{l=1}^{\infty} \mathcal{F}^g_{e\,F_l}$ such that u_n converges to u in $\tilde{\mathcal{E}}^{g\,1/2}$-norm as $n \to \infty$. Put $\hat{u}_n = (-|u|) \vee u_n \wedge |u|$. Then $\hat{u}_n \in \bigcup_{l=1}^{\infty} \mathcal{F}^g_{F_l}$ by $\mathcal{F}^g = \mathcal{F}_e^g \cap L^2(X;m)$. Since $\mathcal{E}^g(\hat{u}_n, \hat{u}_n) \leq \mathcal{E}^g(u_n, u_n) + 2\mathcal{E}^g(u,u)$, $\{\hat{u}_n\}$ is an $\tilde{\mathcal{E}}_1^{g\,1/2}$-bounded sequence. So by the Banach-Saks theorem, there exists a subsequence $\{n_k\}$ such that the Cesàro mean of u_{n_k} belongs to $\bigcup_{l=1}^{\infty} \mathcal{F}^g_{F_l}$ and converges to u in $\tilde{\mathcal{E}}_1^{g\,1/2}$-norm as $k \to \infty$. Consequently the above three quasi notions are equivalent to each other. Suppose that $u \in \mathcal{F}_e$ and $\{u_n\} \subset \mathcal{F}$ be the approximating sequence of u. Then there exists $g \in \mathcal{K}$ with $u_n \in \mathcal{F}^g$, $u \in \mathcal{F}_e^g$ such that $\lim_{n \to \infty} \tilde{\mathcal{E}}^g(u_n - u, u_n - u) = 0$ as in the proof of Proposition 3.1. Owing to the above argument, every $\mathcal{E}$-nest is an $\mathcal{E}_0^g$-nest. So u_n has an $\mathcal{E}_0^g$-quasi-continuous m-version. Note that the resolvents αG_α^g associated with $(\mathcal{E}^g, \mathcal{F}^g)$ are sub-Markovian. So the proof of Lemma 1.6.6 in [11] holds in our situation. Hence we have that the 0-order resolvent G^g satisfies $h_0^g = G^g g \leq 1$ m-a.e. Owing to the strong sector condition, we can consider the notion of h_0^g-weighted capacity. Therefore u has an $\mathcal{E}_0^g$-quasi-continuous m-version which is $\mathcal{E}_0^g$-q.e. finite by modifications of Proposition 2.17 and Proposition 2.18 in [16] for $(\mathcal{E}^g, \mathcal{F}_e^g)$. □

Let E be an $\mathcal{E}$-quasi-open set. Assume that any $u \in \mathcal{F}$ has an $\mathcal{E}$-quasi-continuous m-version $\tilde{u}$. Then we can set $\mathcal{F}_E = \{u \in \mathcal{F} : \tilde{u} = 0\ \mathcal{E}\text{-q.e. on } X \setminus E\}$ and $\mathcal{E}_E(u,v) = \mathcal{E}(u,v)$ for u, $v \in \mathcal{F}_E$. The set $\mathcal{F}_E$ is a subspace of $L^2_E(X;m) = \{u \in L^2(X;m) : u = 0\ m\text{-a.e. on } X \setminus E\}$ which can be identified with $L^2(E;m)$. $(\mathcal{E}_E, \mathcal{F}_E)$ is a semi-Dirichlet form on $L^2(E;m)$ in the wide sense.

Lemma 3.4. *Assume that $(\mathcal{E}, \mathcal{F})$ is quasi-regular. Fix an $\mathcal{E}$-quasi-open set E.*

(i) *$\mathcal{F}_E$ is dense in $L^2(E;m)$. In particular, $(\mathcal{E}_E, \mathcal{F}_E)$ is a semi-Dirichlet form on $L^2(E;m)$.*

(ii) *$\mathcal{E}$-polar subset of E is $\mathcal{E}_E$-polar and $\mathcal{E}$-quasi-open subset of E is $\mathcal{E}_E$-quasi-open, and the restrictions on E of $\mathcal{E}$-quasi-continuous functions are $\mathcal{E}_E$-quasi-continuous. In particular, $(\mathcal{E}_E, \mathcal{F}_E)$ is quasi-regular.*

Proof. (i), (ii): Owing to Lemma 2.1(ii), the method of the proof of Lemma 2.7, Remark 2.8, Lemma 2.9 and Corollary 2.10 in [21] remains valid in the framework of semi-Dirichlet forms. In particular, there exists an $\mathcal{E}$-nest $\{\hat{K}_n\}$ of compact sets in X such that for each $n \in \boldsymbol{N}$ there exists $u \in \mathcal{F}$ with $u \geq 0$ m-a.e. so that $0 \leq \tilde{u} \leq 1$ $\mathcal{E}$-q.e. and $\tilde{u} = 1$ on $\hat{K}_n$. Hence the proof of Lemma 2.12 in [21] remains valid. In particular, (2.17) in [21] holds. Consequently, we have that for any $\mathcal{E}$-quasi-closed sets F_1, F_2 with $F_1 \cap F_2 = \emptyset$ $\mathcal{E}$-q.e., there exist a common $\mathcal{E}$-nest $\{K_n\}$ of compact sets for F_1, F_2 and $f_n \in \mathcal{F}$ with $\tilde{f}_n|_{K_n} \in C(K_n)$, $0 < \tilde{f}_n \leq 1$, $\tilde{f}_n = 1$ on $F_1 \cap K_n$ and $\tilde{f}_n = 0$ on $F_2 \cap K_n$. However $\mathcal{F}_b$ is not necessarily an algebra, we can use the Stone-Weierstrass theorem for lattice version (cf. 4C.Lemma in [15]). The function $w = uv$ appeared in (2.18) of [21] is replaced by $w = u \wedge v$ in our setting. □

Lemma 3.5. *Assume that $(\mathcal{E}, \mathcal{F})$ is quasi-regular. Fix an $\mathcal{E}$-quasi-open set E.*

(i) *Let F_1, F_2 be $\mathcal{E}$-quasi-closed sets with $F_1 \cap F_2 = \emptyset$ $\mathcal{E}$-q.e. Then there exists $u_n \in \mathcal{F}$ with $0 \leq \tilde{u}_n \leq \tilde{u}_{n+1} \leq \tilde{h}$ such that $\tilde{u}_n = 0$ $\mathcal{E}$-q.e. on F_1 and $\tilde{u}_n \to \tilde{h} (n \to \infty) \mathcal{E}$-q.e. on F_2.*

(ii) *For a subset P of E, P is $\mathcal{E}_E$-polar if and only if P is $\mathcal{E}$-polar.*

(iii) *There exists an increasing sequence of $\mathcal{E}$-quasi-open sets $\{G_k\}$ such that $\bar{G}_k^{\mathcal{E}} \subset G_{k+1}$ $\mathcal{E}$-q.e. for each $k \in \boldsymbol{N}$ and $E = \bigcup_{k=1}^{\infty} G_k$ $\mathcal{E}$-q.e. Further there exists a sequence $\{e_{G_k}\}$ of $\mathcal{F}$ such that $0 \leq \tilde{e}_{G_k} \leq 1$ $\mathcal{E}$-q.e. and $\tilde{e}_{G_k} = 1$ $\mathcal{E}$-q.e. on G_k.*

(iv) *For a subset G of E, G is is $\mathcal{E}_E$-quasi-open if and only if P is $\mathcal{E}$-quasi-open.*

To prove Lemma 3.5, we need the notion "*of finite energy integrals*"(see Definition 2 in [6]). Let S_0 (resp. $S_0(E)$) be the totality of measures of finite energy integrals for $(\mathcal{E}, \mathcal{F})$ (resp. $(\mathcal{E}_E, \mathcal{F}_E)$). We can consider the α-potential for $\mu \in S_0$ denoted by $U_\alpha \mu \in \mathcal{F} (\alpha > 0)$. The form $(\mathcal{E}, \mathcal{F})$ in [6] is assumed to satisfy the semi-Dirichlet property of the dual form. However the results in [6] remain valid in our setting by use of Lemma 2.1(ii). Under the quasi-regularity of $(\mathcal{E},\mathcal{F})$, $\mu \in S_0$ charges no $\mathcal{E}$-polar set. We see $S_0|_{\mathcal{B}(E)} \subset S_0(E)$, so $\mu \in S_0$ charges no $\mathcal{E}_E$-polar set. Let G_1^E be the 1-resolvent on $L^2(E;m)$ for $(\mathcal{E}_E,\mathcal{F}_E)$. We set $h^E = G_1^E g$, $g \in \mathcal{K}$.

Proof of Lemma 3.5. (i) Let $\{K_n\}$ be the $\mathcal{E}$-nest and $f_n \in \mathcal{F}$ the function specified in the proof of Lemma 3.4. Put $u_n = \bigvee_{i=1}^n (\tilde{h} - (\tilde{h}f_i) \vee \tilde{h}_{X\setminus K_i})$. Then u_n satisfies the desired assertion.

(ii) The "if"part is showed in Lemma 3.4(i). We know $\mu \in S_0$ charges no $\mathcal{E}_E$-polar set. We may assume that E is a Borel set. So it suffices to show that for each $P \in \mathcal{B}(X)$, $\mu(P) = 0$ for any $\mu \in S_0$ implies P is $\mathcal{E}$-polar. To prove this, we should show the implication (a)$\Rightarrow$(b) for $u \in \mathcal{F}$, $\alpha > 0$ and an $\mathcal{E}$-quasi-closed set F:

(a) $\mathcal{E}_\alpha(u,v) \geq 0$ for any $v \in \mathcal{F}$ with $\tilde{v} \geq 0$ $\mathcal{E}$-q.e. on F.

(b) $u = U_\alpha\mu$ for some $\mu \in S_0$ with $\mathcal{E}$-supp$[\mu] \subset F$ $\mathcal{E}$-q.e.

Then we get the $\mathcal{E}$-polarity of P by a similar argument of the proof of Theorem 2.2.3(ii)$\Rightarrow$(i) in [11]. Suppose that (a) holds. Owing to Theorem 3 in [6], we get $u = U_\alpha\mu$ for some $\mu \in S_0$. Let $\hat{F}$ be another $\mathcal{E}$-quasi-closed set with $F\cap\hat{F} = \emptyset$ $\mathcal{E}$-q.e. By use of (i), we can take $u_n \in \mathcal{F}$ with $0 \leq \tilde{u}_n \leq \tilde{u}_{n+1} \leq \tilde{h}$ such that $\tilde{u}_n = 0$ $\mathcal{E}$-q.e. on F and $\tilde{u}_n \to \tilde{h}(n \to \infty)\mathcal{E}$-q.e. on $\hat{F}$. Then we have

$$\int_{\hat{F}} \tilde{h}(x)\mu(dx) \leq \liminf_{n\to\infty} \int_X \tilde{u}_n(x)\mu(dx) \leq \liminf_{n\to\infty} \mathcal{E}_\alpha(u,u_n) = 0.$$

Since $\tilde{h} > 0$ $\mathcal{E}$-q.e., we get $\mu(\hat{F}) = 0$. Put $A_k = \{\tilde{h}^{F^c} > 1/k\}$. Then $\bar{A}_k^{\mathcal{E}}$ satisfies the condition for $\hat{F}$. We have $\mu(\bar{A}_k^{\mathcal{E}}) = 0$, hence $\mu(F^c) = 0$, which implies (a)$\Rightarrow$(b). Here we use that μ charges no $\mathcal{E}_{F^c}$-polar and $F^c = \bigcup_{k=1}^\infty A_k$ $\mathcal{E}_{F^c}$-q.e.

(iii) It suffices to set $G_k = \{\tilde{h}^E > 1/k\}$ by Lemma 3.3, Lemma 3.4 and (ii). The latter is easy.

(iv) The "if"part is showed in Lemma 3.4(ii). Consider $\{G_k\}_{k\in N}$ constructed in (iii). We may assume $E = \bigcup_{k=1}^\infty G_k$ and $\bar{G}_k^{\mathcal{E}} \subset G_{k+1}$ for any $k \in \boldsymbol{N}$. Suppose that $G(\subset E)$ is $\mathcal{E}_E$-quasi-open. Since $G \cap G_k$ is also $\mathcal{E}_E$-quasi-open by Lemma 3.4(ii), there exists a common $\mathcal{E}_E$-nest $\{F_n\}_{n\in N}$ of compact sets such that $(G\cap G_k)^c \cap F_n$ is closed for each $k,n \in \boldsymbol{N}$ by Lemma 3.2. It suffices to show that $G \cap G_k$ is $\mathcal{E}$-quasi-open for each $k \in \boldsymbol{N}$, because a countable union of $\mathcal{E}$-quasi-open sets is $\mathcal{E}$-quasi-open. Let $\{\check{F}_n\}_{n\in N}$ be the common $\mathcal{E}$-nest such that $G_k \cap \check{F}_n$ is open in $\check{F}_n$ for $k,n \in \boldsymbol{N}$ by Lemma 3.2. We set $H_n^k = F_n \cup (G_k^c \cap \check{F}_n)$. Then by the property of $\mathcal{E}$-quasi-interior, we have $H_n^{k\,\mathcal{E}\text{-int}} \supset F_n^{\mathcal{E}\text{-int}} \cup ((\bar{G}_k^{\mathcal{E}})^c \cap \check{F}_n^{\mathcal{E}\text{-int}}) \supset (E^c \cup F_n^{\mathcal{E}\text{-int}}) \cap \check{F}_n^{\mathcal{E}\text{-int}}$ $\mathcal{E}$-q.e. So Lemma 3.3 and (ii) tell us that $\{H_n^k\}_{n\in N}$ is an $\mathcal{E}$-nest for each $k \in \boldsymbol{N}$. On the other hand, $H_n^k \cap (G\cap G_k)^c = \{F_n \cap (G\cap G_k)^c\} \cup (G_k^c \cap \check{F}_n)$ is closed for each $n \in \boldsymbol{N}$. Thus $G \cap G_k$ is $\mathcal{E}$-quasi-open. □

REMARK 3.1. (i) Let $\mathcal{M}_0$ be the totality of measures which charge no $\mathcal{E}$-polar set. As in [12], we can define the notion of permanent sets of $\mu \in \mathcal{M}_0$ denoted by $P_{(\mu)}$ in our setting. Owing to Lemma 3.5, we can deduce several similar results as in Section 4 of [12] in the framework of semi-Dirichlet forms. In particular, analogous results of Lemma 4.1, Theorem 4.2, Proposition 4.3, Proposition 4.4, Lemma 4.5, Lemma 4.6 and Corollary 4.7 in [12] hold in the present setting. See also Proposition 2.13 in [21], which discuss the quasi-regularity on $P_{(\mu)}$ of the perturbation $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ by $\mu \in \mathcal{M}_0$ of (non-symmetric) Dirichlet form $(\mathcal{E}, \mathcal{F})$.

(ii) (Errata for [13]). The condition (QR1) is indispensable to show that $\mathcal{F}_E$ is dense in $L^2(E; m)$. In [13], we missed this condition. All results in [13] should be read under (QR1) or assumption (A) in [13]. Otherwise, (Γ6) and Proposition 2.5 in [13] fail. But Corollary 1.3 and 1.3' in [13] remain valid without using Proposition 2.5 in [13]. So (QR1) holds in Example 3.2 in [13]. The statement of Theorem 1.2, Theorem 1.2', Theorem 4.2 in [13] contain mistypes: $\rho(F_1, F_2) > 0$ should be replaced by $\rho(F_1, F_2) \geq 1$. The metric of Example 3.2 in [13] is precisely $\|x - y\|/c$. In the case of strongly local Dirichlet forms, the results of [13] are recovered in [14] under the quasi-regularity.

4. Local spaces for semi-Dirichlet forms

Let X, m be as in Section 2. Throughout this section, we fix a semi-Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X; m)$ and assume that $(\mathcal{E}, \mathcal{F})$ satisfies (QR2). Let $L^0(X; m)$ be the totality of m-a.e. defined finite functions on X. We denote $\mathcal{A}_b$ by $\mathcal{A} \cap L^\infty(X; m)$ for $\mathcal{A}(\subset L^0(X; m))$. For $\mathcal{A}(\subset L^0(X; m))$ and an $\mathcal{E}$-quasi-open set E, we define the local space $\dot{\mathcal{A}}_{E\,loc}(\subset L^0(E; m))$ of $\mathcal{A}$ on E. Let Ξ_E be the family of sequences of $\mathcal{E}$-quasi-open sets defined by

$$\Xi_E = \left\{ \{G_n\}_{n\in N} : \ G_n \text{ is } \mathcal{E}\text{-quasi-open for all } n, \right.$$
$$\left. G_n \subset G_{n+1} \ \mathcal{E}\text{-q.e. and } E = \bigcup_{n=1}^{\infty} G_n \mathcal{E}\text{-q.e.} \right\}.$$

Then we let

$$\dot{\mathcal{A}}_{E\,loc} = \left\{ u \in L^0(E; m) : \exists \{E_n\}_{n\in N} \in \Xi_E \text{ and } \exists u_n \in \mathcal{A} \text{ such that} \right.$$
$$\left. u = u_n \ m\text{-a.e. on } E_n \right\}.$$

When $X = E$ $\mathcal{E}$-q.e., we simply write $\Xi = \Xi_E$ and $\dot{\mathcal{A}}_{loc} = \dot{\mathcal{A}}_{E\,loc}$.

Lemma 4.1. *Assume that $(\mathcal{E}, \mathcal{F})$ is quasi-regular. Fix an $\mathcal{E}$-quasi-open set E. Then $u \in \dot{\mathcal{F}}_{E\,loc}$ has an $\mathcal{E}_E$-quasi-continuous m-version $\tilde{u}$ on E, which is $\mathcal{E}_E$-q.e. finite on E.*

Proof. Let $\{E_n\}_{n\in N} \in \Xi_E$ and $\{u_n\} \subset \mathcal{F}$ be associated with $u \in \dot{\mathcal{F}}_{E\,loc}$. Since

$\{E_n\}_{n\in N}$ are $\mathcal{E}_E$-quasi-open by Lemma 3.4(ii), u has an m-version $\tilde{u}$ on E such that $\tilde{u}|_{E_n} = \tilde{u}_n|_{E_n}$ $\mathcal{E}_E$-q.e. $n \in \boldsymbol{N}$ in view of Lemma 3.1(i). We get easily $\tilde{u}$ is $\mathcal{E}_E$-quasi-continuous. □

Theorem 4.1. (i) $(\dot{\mathcal{F}}_b)_{loc} = \dot{\mathcal{F}}_{loc}$. *Further assume* $(\mathcal{E},\mathcal{F})$ *satisfies the strong sector condition. Then* $(\dot{\mathcal{F}}_b)_{loc} = (\dot{\mathcal{F}}_{e\,b})_{loc} = (\dot{\mathcal{F}}_e)_{loc} = \dot{\mathcal{F}}_{loc}$.

(ii) *Assume that* $(\mathcal{E},\mathcal{F})$ *satisfies* (QR1). *Let* E *be an* $\mathcal{E}$*-quasi-open set. Then the local space of* $\dot{\mathcal{F}}_{loc}$ *on* E *is* $\dot{\mathcal{F}}_{E\,loc}$.

(iii) $1 \in \dot{\mathcal{F}}_{loc}$.

Proof. (i) We first prove the latter assertion. Strong sector condition is only used the existence of $\mathcal{E}$-quasi-continuous m-version of $u \in \mathcal{F}_e$. It suffices to show that $(\dot{\mathcal{F}}_e)_{loc} \subset (\dot{\mathcal{F}}_b)_{loc}$. Then $(\dot{\mathcal{F}}_e)_{loc} \subset (\dot{\mathcal{F}}_b)_{loc} \subset (\dot{\mathcal{F}}_{e\,b})_{loc} \subset (\dot{\mathcal{F}}_e)_{loc}$ and $(\dot{\mathcal{F}}_e)_{loc} \subset (\dot{\mathcal{F}}_b)_{loc} \subset \dot{\mathcal{F}}_{loc} \subset (\dot{\mathcal{F}}_e)_{loc}$. Suppose that $u \in (\dot{\mathcal{F}}_e)_{loc}$. Then u has an $\mathcal{E}$-quasi-continuous m-version $\tilde{u}$ which is $\mathcal{E}$-q.e. finite by the same method of the proof of Lemma 5.1 and Proposition 3.2. For $v \in \mathcal{F}_e$, we know by Proposition 3.1(iii), (iv), $v^{(nh)} = (-nh)\vee v \wedge nh \in \mathcal{F}_b$, $n \in \boldsymbol{N}$. Let $u_n \in \mathcal{F}_e$ and $\{E_n\}_{n\in N} \in \Xi$ such that $u = u_n$ m-a.e. on E_n. Then we have $u = u_n = u_n^{(nh)}$ m-a.e. on $E_n \cap \{|\tilde{u}_n| < n\tilde{h}\} = E_n \cap \{|\tilde{u}| < n\tilde{h}\}$. Since $\tilde{h} > 0$, $\mathcal{E}$-q.e. and $|\tilde{u}| < \infty$ $\mathcal{E}$-q.e., $u \in (\dot{\mathcal{F}}_b)_{loc}$. The first assertion is similar.

(ii) Suppose that u is an element of the local space of $\dot{\mathcal{F}}_{loc}$ on E. Then there exist $\{G_n\}_{n\in N} \in \Xi_E$ and $u_n \in \dot{\mathcal{F}}_{loc}$ such that $u = u_n$ m-a.e. on G_n. Since $u_n \in \dot{\mathcal{F}}_{loc}$, there exist $\{G_n^k\}_{k\in N} \in \Xi$ and $u_n^k \in \mathcal{F}$ such that $u_n = u_n^k$ m-a.e. on G_n^k. Let $F_k^n = \overline{G_n^k}^{\mathcal{E}}$ and $\{\check{F}_l\}_{l\in N}$ be the common $\mathcal{E}$-nest such that $F_k^n \cap \check{F}_l$ is closed in $\check{F}_l$ for each k, l, $n \in \boldsymbol{N}$ by Lemma 3.2. Put $\hat{F}_k^n = F_k^n \cap \check{F}_k$. Then $\{\hat{F}_k^n\}_{k\in N}$ is an $\mathcal{E}$-nest by Lemma 3.3. Here we use (QR1). By use of Lemma 3.2, there exists a subsequence $\{k(l,n)\}_{l\geq 1}$ of $\{k\}$ with $k(l,n) \geq l$ such that $F_l = \bigcap_{n=1}^{\infty} \hat{F}_{k(l,n)}^n$ makes an $\mathcal{E}$-nest. Since F_k^n is the $\mathcal{E}$-quasi-closure of G_n^k, we have $\tilde{u}_n = \tilde{u}_n^k$ $\mathcal{E}$-q.e. on F_k^n for each k, $n \in \boldsymbol{N}$. So we have $\tilde{u}_n = \tilde{u}_n^{k(l,n)}$ $\mathcal{E}$-q.e. on F_l for each $l, n \in \boldsymbol{N}$. Hence by setting $E_l = F_l^{\mathcal{E}\text{-int}}$, we have $u_n = u_n^{k(l,n)}$ m-a.e. on E_l for each l, $n \in \boldsymbol{N}$. So $u = u_n = u_n^{k(l,n)}$ m-a.e. on $E_l \cap G_n$. Thus we have $u = u_n = u_n^{k(n,n)}$ m-a.e. on $E_n \cap G_n$. Applying Lemma 3.3 again, we have $\{E_n\}_{n\in N} \in \Xi$, hence $\{E_n \cap G_n\}_{n\in N} \in \Xi_E$, which implies $u \in \dot{\mathcal{F}}_{E\,loc}$.

(iii) Owing to the semi-Dirichlet property, we have $nh \wedge 1 \in \mathcal{F}$. Then $1 = nh \wedge 1$ m-a.e. on $\{\tilde{h} > 1/n\}$ implies $1 \in \dot{\mathcal{F}}_{loc}$ by $\tilde{h} > 0$ $\mathcal{E}$-q.e. □

Theorem 4.2. *Assume that* $(\mathcal{E},\mathcal{F})$ *is quasi-regular. Then* $\dot{\mathcal{F}}_{E\,loc} = (\dot{\mathcal{F}}_E)_{loc}$. *In particular,* $\mathcal{F}|_E \subset (\dot{\mathcal{F}}_E)_{loc}$. *Here* $\mathcal{F}|_E$ *is the totality of restrictions of elements in* $\mathcal{F}$ *to* E.

Proof. Owing to Lemma 3.5, it is clear that $(\dot{\mathcal{F}}_E)_{loc} \subset \dot{\mathcal{F}}_{E\,loc}$. It suffices to show that $\dot{\mathcal{F}}_{E\,loc} \subset (\dot{\mathcal{F}}_E)_{loc}$. Suppose that $u \in \dot{\mathcal{F}}_{E\,loc}$. Then by Lemma 4.1, there exists an $\mathcal{E}_E$-quasi-continuous m-version $\tilde{u}$ of u on E, and there exist $\{E_n\}_{n\in N} \in \Xi_E$ and $u_n \in \mathcal{F}$ such that $u = u_n$ m-a.e. on E_n. Recall $h^E = G_1^E g$ be the 1-order $L^2(E;m)$-resolvent associated with $(\mathcal{E}_E, \mathcal{F}_E)$ for $g \in \mathcal{K}$. Then we have $u_n^{(nh^E)} = (-nh^E) \vee u_n \wedge nh^E \in \mathcal{F}_E$ by (1) and $u = u_n = u_n^{(nh^E)}$ m-a.e. on $E_n \cap \{|\tilde{u}_n| < n\tilde{h}^E\} = E_n \cap \{|\tilde{u}| < n\tilde{h}^E\}$. Since $|\tilde{u}| < \infty$ $\mathcal{E}_E$-q.e. on E and $\tilde{h}^E > 0$ $\mathcal{E}_E$-q.e. on E, we have $u \in (\mathcal{F}_E)_{loc}$. □

5. Formulae of Beurling Deny and LeJan

Let X, m, $(\mathcal{E}, \mathcal{F})$ be as in Section 2. Throughout this section, we assume that $(\mathcal{E}, \mathcal{F})$ is a symmetric quasi-regular Dirichlet form. For a given $\mathcal{E}$-quasi-closed set F, an $\mathcal{E}$-quasi-open set E is said to be an $\mathcal{E}$-neighbourhood of F if $F \subset E$ $\mathcal{E}$-q.e. For a measure μ charging no $\mathcal{E}$-polar set, we denote the $\mathcal{E}$-quasi-support of μ by $\mathcal{E}$-supp$[\mu]$.

Theorem 5.1. *Let $(\mathcal{E}, \mathcal{F})$ be a symmetric Dirichlet form on $L^2(X;m)$ which is quasi-regular. Then there exist unique $\mathcal{E}^{(c)}$, J, k satisfying*

(i) *$(\mathcal{E}^{(c)}, \mathcal{F})$ is a positive definite symmetric bilinear form which is strongly local in the general sense that*

$$\mathcal{E}^{(c)}(u,v) = 0 \text{ if } u, v \in \mathcal{F} \text{ with } u = \text{const. } m\text{-a.e. on an } \mathcal{E}\text{-neighbourhood of } \mathcal{E}\text{-supp}[|v|m].$$

(ii) *J is a σ-finite symmetric positive measure on $\mathcal{B}(X \times X \setminus d)$ (where d is diagonal) such that J does not charge any subset of $X \times X \setminus d$ whose projection on the factor X is $\mathcal{E}$-polar.*

(iii) *k is a σ-finite positive measure on $\mathcal{B}(X)$ which charges no $\mathcal{E}$-polar set.*

(iv) *For any $u, v \in \mathcal{F}$, $[\tilde{u}] \in L^2(X \times X \setminus d\ ;\ J)$ and $\tilde{u} \in L^2(X;k)$ and*

$$\mathcal{E}(u,v) = \mathcal{E}^{(c)}(u,v) + \mathcal{E}^{(j)}(u,v) + \mathcal{E}^{(k)}(u,v).$$

Here $\mathcal{E}^{(j)}(u,v) = \int_{X\times X\setminus d} [\tilde{u}](x,y)[\tilde{v}](x,y)J(dxdy)$, $\mathcal{E}^{(k)}(u,v) = \int_X \tilde{u}(x)\tilde{v}(x)k(dx)$ and $[\tilde{u}](x,y) = \tilde{u}(x) - \tilde{u}(y)$. Further assume that there exists an increasing sequence of open sets $\{G_l\}$ each of whose component has finite 1-capacity and $X = \bigcup_{l=1}^\infty G_l$. Then $\mathcal{E}^{(c)}$ is characterized as the unique bilinear form on $\mathcal{F}$ which satisfies the strong local property in the ordinary sense that

$$\mathcal{E}^{(c)}(u,v) = 0 \text{ if } u, v \in \mathcal{F} \text{ with } u = \text{const. } m\text{-a.e. on a neighbourhood of supp}[|v|m].$$

Corollary 5.1. *Let E be an $\mathcal{E}$-quasi-open set. Let J be the jumping measure appeared in Theorem 5.1. We set $J^E(A) = J(A \times E^c)$ for $A \in \mathcal{B}(E)$. Then every $u \in \mathcal{F}_E$ satisfies $\tilde{u} \in L^2(E; J^E)$. In particular, in the framework of C_0-regular Dirichlet forms,*

J is a symmetric Radon measure on $X \times X \setminus d$ which satisfies $J(K \times G^c) < \infty$ for any compact set K and its open neighbourhood G.

Theorem 5.2. *Let $(\mathcal{E}, \mathcal{F})$ be a symmetric Dirichlet forms on $L^2(X;m)$ which is quasi-regular. Let $(\mathcal{E}^{(c)}, \mathcal{F})$ be the bilinear form constructed in the preceding theorem. Then for each $u \in \mathcal{F}_b$, there exists unique finite positive Borel measure $\mu^{(c)}_{\langle u\rangle}$ on X which charges no $\mathcal{E}$-polar set such that*

$$\int_X \tilde{f}(x)\mu^{(c)}_{\langle u\rangle}(dx) = 2\mathcal{E}^{(c)}(uf, u) - \mathcal{E}^{(c)}(u^2, f), \ f \in \mathcal{F}_b.$$

Proof of Theorem 5.1 and Theorem 5.2.

Existence of $\mathcal{E}^{(c)}$, J, k, $\mu^{(c)}_{\langle u\rangle}$, $u \in \mathcal{F}_b$. The existence and construction of $\mathcal{E}^{(c)}$, J, k, $\mu^{(c)}_{\langle u\rangle}, u \in \mathcal{F}_b$ in Theorem 5.1 and Theorem 5.2 follow from the transfer method. We should show the strong local property of $\mathcal{E}^{(c)}$ in our sense. First we consider the regular representation $(\hat{\mathcal{E}}, \hat{\mathcal{F}})$ of $(\mathcal{E}, \mathcal{F})$ on $L^2(\hat{X};\hat{m})$ discussed in the last part of Section 2. Then the Beurling-Deny decomposition holds for $(\hat{\mathcal{E}}, \hat{\mathcal{F}})$ (Theorem 3.2.1, Lemma 4.5.4 and Theorem 5.3.1 in [11]): there exist $\hat{\mathcal{E}}^{(c)}$, $\hat{J}$, $\hat{k}$ satisfying

(i) $(\hat{\mathcal{E}}^{(c)}, \hat{\mathcal{F}})$ is a positive definite symmetric bilinear form which is strongly local in the ordinary sense of

$\hat{\mathcal{E}}^{(c)}(u,v)=0$ if $u,v \in \hat{\mathcal{F}}$ with $u=$const. $\hat{m}$-a.e. on a neighbourhood of supp$[|v|\hat{m}]$.

(ii) $\hat{J}$ is a σ-finite symmetric positive measure on $\mathcal{B}(\hat{X}\times\hat{X}\setminus d)$ (where d is diagonal) such that $\hat{J}$ does not charge any subset of $\hat{X} \times \hat{X} \setminus d$ whose projection on the factor $\hat{X}$ is $\hat{\mathcal{E}}$-polar.

(iii) $\hat{k}$ is a σ-finite positive measure on $\mathcal{B}(\hat{X})$ which charges no $\hat{\mathcal{E}}$-polar set.

(iv) For any u, $v \in \hat{\mathcal{F}}$, $[\tilde{u}] \in L^2(\hat{X} \times \hat{X} \setminus d\ ;\ \hat{J})$ and $\tilde{u} \in L^2(\hat{X};\hat{k})$ and

$$\hat{\mathcal{E}}(u,v) = \hat{\mathcal{E}}^{(c)}(u,v) + \int_{\hat{X}\times\hat{X}\setminus d}[\tilde{u}](x,y)[\tilde{v}](x,y)\hat{J}(dxdy) + \int_{\hat{X}}\tilde{u}(x)\tilde{v}(x)\hat{k}(dx).$$

Further for $u \in \hat{\mathcal{F}}_b$ there exists a positive finite measure $\hat{\mu}^{(c)}_{\langle u\rangle}$ on $\mathcal{B}(\hat{X})$ which charges no $\hat{\mathcal{E}}$-polar set such that

$$\int_{\hat{X}} \tilde{f}(x)\hat{\mu}^{(c)}_{\langle u\rangle}(dx) = 2\hat{\mathcal{E}}^{(c)}(uf,u) - \hat{\mathcal{E}}^{(c)}(u^2,f), \ f \in \hat{\mathcal{F}}_b$$

(cf. Lemma 3.2.3, Lemma 3.2.4 and Lemma 3.2.5 in [11]). We set $\mathcal{E}^{(c)}(u,v) = \hat{\mathcal{E}}^{(c)}(u^\sharp, v^\sharp)$ for u, $v \in \mathcal{F}$ and $J(A \times B) = \hat{J}(i(A \cap Y) \times i(B \cap Y))$ for A, $B \in \mathcal{B}(X)$ with $A \cap B = \emptyset$, and $k(A) = \hat{k}(i(A \cap Y))$, $\mu_{\langle u,v\rangle}(A) = \hat{\mu}_{\langle u^\sharp, v^\sharp\rangle}(i(A \cap Y))$ for $A \in \mathcal{B}(X)$, u, $v \in \mathcal{F}$. Since $(\mathcal{E}, \mathcal{F})$ and $(\hat{\mathcal{E}}, \hat{\mathcal{F}})$ are $\mathcal{E}$-quasi-homeomorphic, $\mathcal{E}^{(c)}$, J, k satisfy (ii), (iii), (iv). To prove (i) for $\mathcal{E}^{(c)}$, it suffices to show (i) for $\hat{\mathcal{E}}^{(c)}$, namely,

$\hat{\mathcal{E}}^{(c)}(u,v)=0$ if $u,v\in\hat{\mathcal{F}}$ with $u=$ const. $\hat{m}$-a.e.

on an $\hat{\mathcal{E}}$-neighbourhood of $\hat{\mathcal{E}}$-supp$[|v|\hat{m}]$.

Since $\hat{\mu}^{(c)}_{\langle u,v\rangle}$ satisfies the derivation property (Lemma 3.2.5 in [11]): u, v, $w\in\hat{\mathcal{F}}_b$

$$\hat{\mu}^{(c)}_{\langle uv,w\rangle}=\tilde{u}\hat{\mu}^{(c)}_{\langle v,w\rangle}+\tilde{v}\hat{\mu}^{(c)}_{\langle u,w\rangle},$$

we have that for u, $v\in\mathcal{F}$ and an $\hat{\mathcal{E}}$-quasi-open set $\hat{E}$, $u=$const. $\hat{m}$-a.e. on $\hat{E}$ implies $I_{\hat{E}}\hat{\mu}^{(c)}_{\langle u,v\rangle}=0$ by using Lemma 3.5(i) (cf. the proof of Corollary 3.2.1 in [11] and the proof of Proposition 2.4(Γ6) in [13]). Suppose that $u=$const. $\hat{m}$-a.e. on an $\hat{\mathcal{E}}$-neighbourhood $\hat{E}$ of $\hat{\mathcal{E}}$-supp$[|v|\hat{m}]$. Then $I_{\hat{E}}\hat{\mu}^{(c)}_{\langle u,v\rangle}=0$. On the other hand, $v=0$ $\hat{m}$-a.e. on $\hat{X}\backslash\hat{\mathcal{E}}$-supp$[|v|\hat{m}]$ implies $I_{\hat{X}\backslash\hat{\mathcal{E}}\text{-supp}[|v|\hat{m}]}\hat{\mu}^{(c)}_{\langle u,v\rangle}=0$, hence $I_{\hat{X}\backslash\hat{E}}\hat{\mu}^{(c)}_{\langle u,v\rangle}=0$. Thus we have $\hat{\mu}^{(c)}_{\langle u,v\rangle}=I_{\hat{E}}\hat{\mu}^{(c)}_{\langle u,v\rangle}+I_{\hat{X}\backslash\hat{E}}\hat{\mu}^{(c)}_{\langle u,v\rangle}=0$, which implies $\hat{\mathcal{E}}^{(c)}(u,v)=0$. Note that for $v\in\mathcal{F}$, $\hat{\mathcal{E}}$-supp$[|v^\sharp|\hat{m}]\subset i(\mathcal{E}\text{-supp}[|v|m]\cap Y)$ $\hat{\mathcal{E}}$-q.e. Hence we have the assertion.

Next we should show the uniqueness. Let ρ be the metric which is compatible with the given topology on X. Suppose that $(\mathcal{E},\mathcal{F})$ has another decomposition with $\bar{\mathcal{E}}^{(c)}$, $\bar{J}$, $\bar{k}$

Uniqueness of J. Let G_1, G_2 be open sets with $\rho(G_1,G_2)>0$. First we show $J(G_1\times G_2)=\bar{J}(G_1\times G_2)$. Set $\hat{G}_i=\{x\in X:\rho(x,G_i)<(1/3)\rho(G_1,G_2)\}(i=1,2)$. Then $\rho(\hat{G}_1,\hat{G}_2)\geq(1/3)\rho(G_1,G_2)>0$ and the closure F_i of G_i satisfies $F_i\subset\hat{G}_i$ for each $i=1$, 2. Let h^{G_i} be the function constructed in Lemma 3.5(i) for $(\mathcal{E}_{G_i},\mathcal{F}_{G_i})$. For simplicity, we assume its $\mathcal{E}$-quasi-continuity. We let $h_n^{G_i}=nh^{G_i}\wedge 1$. Then we see supp$[h_n^{G_1}m]\subset F_1\subset\hat{G}_1$ and supp$[h_n^{G_2}m]\subset F_2\subset\hat{G}_2$. Hence we have $h_n^{G_1}=0$ m-a.e. on $\hat{G}_2$ and $h_n^{G_2}=0$ m-a.e. on $\hat{G}_1$, consequently $h_n^{G_1}=0$ $\mathcal{E}$-q.e. on $\hat{G}_2$ and $h_n^{G_2}=0$ $\mathcal{E}$-q.e. on $\hat{G}_1$. Since k and $\bar{k}$ charge no $\mathcal{E}$-polar set and the strong local property of $\mathcal{E}^{(c)}$, $\bar{\mathcal{E}}^{(c)}$, we get $\mathcal{E}^{(c)}(h_n^{G_1},h_n^{G_2})=\bar{\mathcal{E}}^{(c)}(h_n^{G_1},h_n^{G_2})=\int_X h_n^{G_1}h_n^{G_2}dk=\int_X h_n^{G_1}h_n^{G_2}d\bar{k}=0$. Hence we have

$$\mathcal{E}(h_n^{G_1},h_n^{G_2})=\int_{X\times X\backslash d}[h_n^{G_1}][h_n^{G_2}]dJ=\int_{X\times X\backslash d}[h_n^{G_1}][h_n^{G_2}]d\bar{J}.$$

On the other hand, J and $\bar{J}$ charge no subset of $X\times X\setminus d$ whose projection on the factor X is $\mathcal{E}$-polar. So we have

$$\mathcal{E}(h_n^{G_1},h_n^{G_2})=-2\int_{X\times X\backslash d}h_n^{G_1}(x)h_n^{G_2}(y)J(dxdy)=-2\int_{X\times X\backslash d}h_n^{G_1}(x)h_n^{G_2}(y)\bar{J}(dxdy).$$

Thus we have

$$\int_{X\times X\backslash d}h_n^{G_1}(x)h_n^{G_2}(y)J(dxdy)=\int_{X\times X\backslash d}h_n^{G_1}(x)h_n^{G_2}(y)\bar{J}(dxdy).$$

Letting $n \to \infty$, we get $J(G_1 \times G_2) = \bar{J}(G_1 \times G_2)$. By a standard argument of the Dynkin class theorem, we get $I_{G_1 \times G_2} J = I_{G_1 \times G_2} \bar{J}$ for the above G_1, G_2. So we have $J(A_1 \times A_2) = \bar{J}(A_1 \times A_2)$ for any Borel sets A_1, A_2 with $\rho(A_1, A_2) > 0$. By taking a redecomposition, we have

$$J\Big(\bigcup_{i=1}^{n}(A_1^i \times A_2^i)\Big) = \bar{J}\Big(\bigcup_{i=1}^{n}(A_1^i \times A_2^i)\Big)$$

for any Borel sets A_1^i, A_2^i with $\rho(A_1^i, A_2^i) > 0 (1 \leq i \leq n)$. Let $\mathcal{O}_0(X)$ be the countable topological base of X and put $\mathcal{O}_0(X \times X \setminus d) = \{G_1 \times G_2 : G_1, G_2 \in \mathcal{O}_0(X),\ \rho(G_1, G_2) > 0\}$. Then $\mathcal{O}_0(X \times X \setminus d)$ is a countable topological base of $X \times X \setminus d$. In particular, we have that $X \times X \setminus d = \bigcup_{i=1}^{\infty}(G_1^i \times G_2^i)$ for some G_1^i, $G_2^i \in \mathcal{O}_0(X)$ with $\rho(G_1^i, G_2^i) > 0 (i \in N)$. So

$$J(X \times X \setminus d) = \lim_{n\to\infty} J\Big(\bigcup_{i=1}^{n}(G_1^i \times G_2^i)\Big) = \lim_{n\to\infty} \bar{J}\Big(\bigcup_{i=1}^{n}(G_1^i \times G_2^i)\Big) = \bar{J}(X \times X \setminus d).$$

Applying the Dynkin class theorem again, we have $J = \bar{J}$.

Uniqueness of k, $\mathcal{E}^{(c)}$ in the general sense. Take an open set G and let $\{G_l\}_{l\in N} \in \Xi_G$ (resp. e_{G_l}) be the sequence of $\mathcal{E}$-quasi-open sets (resp. of $\mathcal{F}$) constructed in Lemma 3.5(iii) for $(\mathcal{E}_G, \mathcal{F}_G)$. Then $e_{G_{l+1}} = 1$ m-a.e. on $G_{l+1} \supset \bar{G}_l^{\mathcal{E}} \supset \mathcal{E}\text{-supp}[nh^{G_l} \wedge 1]$. Here $h^{G_l} = G_1^{G_l} g$ for $g \in \mathcal{K}$. We assume its $\mathcal{E}$-quasi-continuity. We let $h_n^{G_l} = nh^{G_l} \wedge 1$. Then $\mathcal{E}^{(c)}(e_{G_{l+1}}, h_n^{G_l}) = \bar{\mathcal{E}}^{(c)}(e_{G_{l+1}}, h_n^{G_l}) = 0$. On the other hand, we know $J = \bar{J}$, hence we have

$$\int_X \tilde{e}_{G_{l+1}} h_n^{G_l} dk = \int_X \tilde{e}_{G_{l+1}} h_n^{G_l} d\bar{k}, \text{ namely } \int_X h_n^{G_l} dk = \int_X h_n^{G_l} d\bar{k}.$$

Letting $n \to \infty$ and $l \to \infty$, we have $k(G) = \bar{k}(G)$. So the Dynkin class theorem tells us $k = \bar{k}$. Hence we have $\mathcal{E}^{(c)} = \bar{\mathcal{E}}^{(c)}$.

Next we show the last assertion in Theorem 4.1.

Uniqueness of k, $\mathcal{E}^{(c)}$ in the ordinary sense. Suppose that there exists an increasing sequence $\{G_k\}$ of open sets whose component has finite 1-capacity and $X = \bigcup_{k=1}^{\infty} G_k$. When $1 \in \mathcal{F}$, the assertion is well-known, hence we may assume $G_k \neq X$ for all $k \in \boldsymbol{N}$. Then for any open set G, we can construct an increasing sequence $\{O_k\} \in \Xi_G$ of open subsets of G with $O_k \neq X$. We can also construct sequence $\{A_l\}_{l\in N} \in \Xi_{O_k}$ of open sets of finite 1-capacity with $\bar{A}_l \subset A_{l+1}$, $l \in \boldsymbol{N}$. Indeed, it suffices to put $A_l = \{x \in X : \rho(x, O_k^c) > 1/l\}$. Then we have $e_{A_{l+1}} = 1$ m-a.e. on $A_{l+1} \supset \bar{A}_l \supset \text{supp}[nh^{A_l} \wedge 1]$. We let $h_n^{A_l} = nh^{A_l} \wedge 1$. Then $\mathcal{E}^{(c)}(e_{A_{l+1}}, h_n^{A_l}) = \bar{\mathcal{E}}^{(c)}(e_{A_{l+1}}, h_n^{A_l}) = 0$. Thus we have $\int_X h_n^{A_l} dk = \int_X h_n^{A_l} d\bar{k}$. Hence $k(A_l) = \bar{k}(A_l)$, which implies $k(O_k) = \bar{k}(O_k)$, so $k(G) = \bar{k}(G)$. Therefore $k = \bar{k}$.

Uniqueness of $\mu_{\langle u \rangle}^{(c)}$. Consider another finite measure $\bar{\mu}_{\langle u \rangle}^{(c)}$ charging no $\mathcal{E}$-polar set such that

$$\int_X \tilde{f}(x)\bar{\mu}^{(c)}_{\langle u\rangle}(dx) = 2\mathcal{E}^{(c)}(uf,u) - \mathcal{E}^{(c)}(u^2,f), \ f \in \mathcal{F}_b.$$

Then

$$\int_X \tilde{f}(x)\mu^{(c)}_{\langle u\rangle}(dx) = \int_X \tilde{f}(x)\bar{\mu}^{(c)}_{\langle u\rangle}(dx), \ f \in \mathcal{F}_b.$$

As in the uniqueness proof of k, we get

$$\int_X h_n^{G_l}(x)\mu^{(c)}_{\langle u\rangle}(dx) = \int_X h_n^{G_l}(x)\bar{\mu}^{(c)}_{\langle u\rangle}(dx).$$

So we have $\mu^{(c)}_{\langle u\rangle} = \bar{\mu}^{(c)}_{\langle u\rangle}$. □

Proof of Corollary 6.1. Since $(\mathcal{E}_E, \mathcal{F}_E)$ on $L^2(E;m)$ is quasi-regular, there exist unique $\mathcal{E}^{(c)}_E$, J_E, k_E such that $\mathcal{E}^{(c)}_E$ satisfies the strong local property in the sense that

$$\mathcal{E}^{(c)}_E(u,v) = 0 \text{ if } u, v \in \mathcal{F} \text{ with } u = \text{const. } m\text{-a.e.}$$
$$\text{on an } \mathcal{E}_E\text{-neighbourhood of } \mathcal{E}_E\text{-supp}[|v|m],$$

J_E is a symmetric σ-finite Borel measure on $E \times E \setminus d$ such that J_E does not charge any subset of $E \times E \setminus d$ whose projection on the factor E is $\mathcal{E}_E$-polar, k_E is a σ-finite positive measure on $\mathcal{B}(E)$ which charges no $\mathcal{E}_E$-polar set, and for any $u, v \in \mathcal{F}_E$, $[\tilde{u}] \in L^2(X \times X \setminus d;\, J_E)$ and $\tilde{u} \in L^2(E; k_E)$ and

$$\mathcal{E}(u,v) = \mathcal{E}^{(c)}_E(u,v) + \iint_{E\times E\setminus d} [\tilde{u}][\tilde{v}]dJ_E + \int_E \tilde{u}\tilde{v}dk_E.$$

By the uniqueness of the decomposition, Lemma 3.4 and Lemma 3.5, we have $\mathcal{E}^{(c)}_E = \mathcal{E}^{(c)}$ on $\mathcal{F}_E \times \mathcal{F}_E$, $J_E = J|_{E\times E\setminus d}$ and $k_E = 2J^E + k|_E$. Hence the assertion holds by $\tilde{u} \in L^2(E;k_E)$ for $u \in \mathcal{F}_E$. □

Let $\mu^{(c)}_{\langle u\rangle}$, $u \in \mathcal{F}_b$ be the measure in Theorem 5.2. Put $\mu^{(c)}_{\langle u,v\rangle} = (1/2)\{\mu^{(c)}_{\langle u+v\rangle} - \mu^{(c)}_{\langle u\rangle} - \mu^{(c)}_{\langle v\rangle}\}$, u, $v \in \mathcal{F}_b$. We then have

$$\int_X \tilde{f}(x)\mu^{(c)}_{\langle u,v\rangle}(dx) = \mathcal{E}^{(c)}(uf,v) + \mathcal{E}^{(c)}(vf,u) - \mathcal{E}^{(c)}(uv,f), \ u, v \in \mathcal{F}_b.$$

Hence we see, by using $h_n^{G_l}$ as in the uniqueness proof of k, for any open set G, $I_G\mu^{(c)}_{\langle u,v\rangle}$ is bilinear, hence $\mu^{(c)}_{\langle u,v\rangle}$ is bilinear. So we have the Cauchy-Schwarz inequality

$$\left|\int_X f(x)\mu^{(c)}_{\langle u,v\rangle}(dx)\right| \le \left(\int_X f(x)\mu^{(c)}_{\langle u\rangle}(dx)\right)^{1/2}\left(\int_X f(x)\mu^{(c)}_{\langle v\rangle}(dx)\right)^{1/2}$$

and

$$\left|\left(\int_X f(x)\mu^{(c)}_{\langle u\rangle}(dx)\right)^{1/2}-\left(\int_X f(x)\mu^{(c)}_{\langle v\rangle}(dx)\right)^{1/2}\right|\le\left(\int_X f(x)\mu^{(c)}_{\langle u-v\rangle}(dx)\right)^{1/2}$$

for u, $v \in \mathcal{F}_b$ and non-negative bounded Borel function f. By this inequality, $\mu^{(c)}_{\langle u\rangle}$ can be uniquely defined for $u \in \mathcal{F}$ and the above inequalities hold for $u, v \in \mathcal{F}$ and non-negative bounded Borel function f. Hence $\mu^{(c)}_{\langle u\rangle}$ can be uniquely defined for $u \in \mathcal{F}_e$ and the above inequalities hold for $u, v \in \mathcal{F}_e$. Indeed, $\mu^{(c)}_{\langle u\rangle}$, $u \in \mathcal{F}$ is defined by the limit $\langle\mu^{(c)}_{\langle u\rangle}, f\rangle = \lim_{n\to\infty}\langle\mu^{(c)}_{\langle u_n\rangle}, f\rangle$ for an $\mathcal{E}_1$-approximating sequence $\{u_n\} \subset \mathcal{F}_b$ to u and a bounded Borel function f, which can not depend on the choice of $\{u_n\}$. Also $\mu^{(c)}_{\langle u\rangle}$, $u \in \mathcal{F}_e$ is defined by a similar method. Hence we see $\mu^{(c)}_{\langle u\rangle}$ $u \in \mathcal{F}_e$ is a finite Borel measure which charges no $\mathcal{E}$-polar set. On the other hand, $\mathcal{E}^{(c)}$ can be extended uniquely to $\mathcal{F}_e$ by

$$\mathcal{E}^{(c)}(u,v)=\lim_{n\to\infty}\mathcal{E}^{(c)}(u_n,v_n)$$

for approximating sequences $\{u_n\}$ to u and $\{v_n\}$ to v.

Lemma 5.1. *For u, $v \in \mathcal{F}_e$,*

$$\mathcal{E}^{(c)}(u,v)=\frac{1}{2}\mu^{(c)}_{\langle u,v\rangle}(X).$$

Proof. Let $\{G_l\}_{l\in N} \in \Xi$ be the sequence of $\mathcal{E}$-quasi-open sets constructed in Lemma 3.5(iii) for $(\mathcal{E},\mathcal{F})$. We put $F_l = \bar{G}_l$, the closure of G_l. Then $\{F_l\}_{l\in N}$ is an $\mathcal{E}$-nest by Lemma 3.3. Hence $\bigcup_{l=1}^{\infty}\mathcal{F}_{G_l\,b}$ is dense in $\mathcal{F}$ in $\mathcal{E}_1^{1/2}$-norm. So it suffices to show the case $u \in \mathcal{F}_{G_l\,b}$. Take a sequence $\{e_{G_l}\}$ of $\mathcal{F}$ for $\{G_l\}_{l\in N}$ in Lemma 3.5(iii). Then $e_{G_j} = 1$ m-a.e. on $G_j \supset \bar{G}_l^{\mathcal{E}} \supset \mathcal{E}\text{-supp}[|u|m]$ for any $j \ge l+1$. Thus we have

$$\int_X \tilde{e}_{G_j}(x)\mu^{(c)}_{\langle u\rangle}(dx)=2\mathcal{E}^{(c)}(ue_{G_j},u)-\mathcal{E}^{(c)}(u^2,e_{G_j})=2\mathcal{E}^{(c)}(u,u).$$

Noting that $0 \le \tilde{e}_{G_j} \le 1$, $\tilde{e}_{G_j} \to 1$, $j \to \infty$ $\mathcal{E}$-q.e., we obtain our assertion. □

We collect several properties of the energy measure of continuous part $\mu^{(c)}_{\langle u,v\rangle}$ for u, $v \in \mathcal{F}$.

Lemma 5.2. *The energy measure of continuous part $\mu^{(c)}_{\langle u,v\rangle}$ satisfies the following properties.*

(Γ1) (*Cauchy-Schwarz inequality*) *For u, $v \in \mathcal{F}$,*

$$\left|\int_X f(x)g(x)\mu^{(c)}_{\langle u,v\rangle}(dx)\right| \le \sqrt{\int_X f^2(x)\mu^{(c)}_{\langle u\rangle}(dx)}\sqrt{\int_X g^2(x)\mu^{(c)}_{\langle v\rangle}(dx)}.$$

(Γ2) (*Markovian property*) *For any* $u \in \mathcal{F}$ *and* $r > 0$, $\mu^{(c)}_{\langle 0\vee u\wedge r\rangle} \le \mu^{(c)}_{\langle u\rangle}$.

(Γ3) (*Chain rule*) *For any* $\mathbf{u} = (u_1, \cdots, u_k)$, $\mathbf{v} = (v_1, \cdots, v_l)$ *with* u_i, $v_j \in \mathcal{F}_b$ $(1 \le i \le k, 1 \le j \le l)$ *and* $F \in C^1(\boldsymbol{R}^k)$, $G \in C^1(\boldsymbol{R}^l)$ *with* $F(0) = G(0) = 0$,

$$\mu^{(c)}_{\langle F(\mathbf{u}),G(\mathbf{v})\rangle} = \sum_{i=1}^{k}\sum_{j=1}^{l}\frac{\partial F}{\partial x_i}(\tilde{\mathbf{u}})\frac{\partial G}{\partial x_j}(\tilde{\mathbf{v}})\mu^{(c)}_{\langle u_i,v_j\rangle}.$$

(Γ4) (*Derivation property*) *For* u, v, $w \in \mathcal{F}_b$, $\mu^{(c)}_{\langle uv,w\rangle} = \tilde{u}\mu^{(c)}_{\langle v,w\rangle} + \tilde{v}\mu^{(c)}_{\langle u,w\rangle}$.

(Γ5) *For* u, $v \in \mathcal{F}$, $\mu^{(c)}_{\langle u\rangle} \le m$, $\mu^{(c)}_{\langle v\rangle} \le m$ *implies* $\mu^{(c)}_{\langle u\vee v\rangle} \le m$.

(Γ6) (*Strong local property*) *For any* $\mathcal{E}$*-quasi-open set* E *and* $u \in \mathcal{F}$ *with* u *=constant* m*-a.e. on* E, $I_E\mu^{(c)}_{\langle u\rangle} = 0$. *In particular* $I_{X\setminus E}\mu^{(c)}_{\langle u\rangle} = 0$ *for* $u \in \mathcal{F}_E$.

Proof. Owing to the transfer method, we may consider the case of regular Dirichlet space setting. The properties (Γ1), (Γ3), (Γ4) are clear (see Lemma 5.6.1, Theorem 3.2.2, and Lemma 3.2.5 in [11]). (Γ2) follows from 4 iv) in [24]. Owing to 4 iv) in [24] and Lemma 2.1(ii) in [19], we see $\mu^{(c)}_{\langle u\vee v\rangle} = I_{\{\tilde{u}>\tilde{v}\}}\mu^{(c)}_{\langle u\rangle} + I_{\{\tilde{u}\le\tilde{v}\}}\mu^{(c)}_{\langle v\rangle}$ for u, $v \in \mathcal{F}_b$. Hence (Γ5) holds. The first assertion of (Γ6) is clear from Lemma 5.3.1 in [11] in view of the transfer method, because $\mathcal{E}$-quasi-open set has a finely open Borel $\mathcal{E}$-q.e. version. The latter assertion follows the first assertion, Lemma 3.3, Lemma 3.5(ii) and the fact that there always exists an $\mathcal{E}_E$-nest of closed sets in X. □

Lemma 5.3. *Fix an* $\mathcal{E}$*-quasi-open set* E.

(i) *For* $u \in \dot{\mathcal{F}}_{E\,loc}$, *we can define a unique* σ*-finite Borel measure* $\mu^{(c)}_{\langle u\rangle}$ *on* E *such that* $I_{E_n}\mu^{(c)}_{\langle u\rangle} = I_{E_n}\mu^{(c)}_{\langle u_n\rangle}$ *for* $\{E_n\}_{n\in N} \in \Xi_E$, $u_n \in \mathcal{F}$ *satisfying* $u = u_n$ m*-a.e. on* E_n. *In particular,* $\mu^{(c)}_{\langle 1\rangle} = 0$.

(ii) *For* $u \in \dot{\mathcal{F}}_{E\,loc}$, $\mu^{(c)}_{\langle u\rangle}$ *charges no* $\mathcal{E}$*-polar subset of* E.

(iii) *All assertions in Lemma* 5.2 *hold with the functions in* $\dot{\mathcal{F}}_{loc}$ *by replacing the functions in* $\mathcal{F}$ *except the latter assertion in* (Γ6). *The latter assertion is replaced by that* $u \in \dot{\mathcal{F}}_{loc}$ *with* $\tilde{u} = 0$ $\mathcal{E}$*-q.e. on* E^c *satisfies* $I_{E^c}\mu^{(c)}_{\langle u\rangle} = 0$.

Proof. First we show (i). Fix an $\{E_n\}_{n\in N} \in \Xi_E$ and $\{u_n\} \subset \mathcal{F}$ appeared in the definition of $u \in \dot{\mathcal{F}}_{E\,loc}$. Put $\mu_n = I_{E_n}\mu^{(c)}_{\langle u_n\rangle}$. Then by (Γ6), $I_{E_n}\mu_m = \mu_n$ for $m > n$. Define $\mu = \lim_{n\to\infty}\mu_n$. Then $I_{E_n}\mu = \mu_n$ and μ charges no $\mathcal{E}$-polar set. Consider another $\{\hat{E}_n\}_{n\in N} \in \Xi_E$ and $\{\hat{u}_n\} \subset \mathcal{F}$ represent $u \in \dot{\mathcal{F}}_{E\,loc}$ and put $\hat{\mu}_n = I_{\hat{E}_n}\mu^{(c)}_{\langle\hat{u}_n\rangle}$ and define $\hat{\mu} = \lim_{n\to\infty}\hat{\mu}_n$. It suffices to show $\mu = \hat{\mu}$. Taking $E_n \cap \hat{E}_n$, we may assume $\hat{E}_n \subset E_n\, \mathcal{E}$-q.e. and $u_n = \hat{u}_n\, m$-a.e. on $\hat{E}_n$ for all $n \in N$. Then by (Γ6), $I_{\hat{E}_n}\mu_n = \hat{\mu}_n$. Hence $\hat{\mu} = I_{\cup_{n=1}^{\infty}\hat{E}_n}\mu$. Since μ charges no $\mathcal{E}$-polar set, we have $\hat{\mu} = \mu$. Thus we can define $\mu^{(c)}_{\langle u\rangle}$ by this μ. Hence (i) holds and (ii) is clear. By polarization we define $\mu^{(c)}_{\langle u,v\rangle} = 1/2\{\mu^{(c)}_{\langle u+v\rangle} - \mu^{(c)}_{\langle u\rangle} - \mu^{(c)}_{\langle v\rangle}\}$ for $u, v \in \dot{\mathcal{F}}_{E\,loc}$. Note that $1 = nh\wedge 1\, m$-a.e. on $E_n = \{\tilde{h} > 1/n\}$. We have $I_{E_n}\mu^{(c)}_{\langle 1\rangle} = I_{E_n}\mu^{(c)}_{\langle nh\wedge 1\rangle} = 0$ by (Γ6). Hence $\mu^{(c)}_{\langle 1\rangle} = 0$. (iii) is easy in view of Theorem 4.1(i). We only prove the last assertion. Suppose that $u \in \dot{\mathcal{F}}_{loc}$ satisfies $\tilde{u} = 0\, \mathcal{E}$-q.e. on E^c. Take $u_n \in \mathcal{F}$ and $\{E_n\}_{n\in N} \in \Xi$ for $u \in \dot{\mathcal{F}}_{loc}$. Then $\tilde{u} = \tilde{u}_n\, \mathcal{E}$-q.e. on $\bar{E}^{\mathcal{E}}_n$. Hence $\tilde{u}_n = \tilde{u} = 0\, \mathcal{E}$-q.e. on $\bar{E}^{\mathcal{E}}_n \setminus E$, which implies $u_n \in \mathcal{F}_{E\cup(X\setminus \bar{E}^{\mathcal{E}}_n)}$ and $I_{\bar{E}^{\mathcal{E}}_n\setminus E}\mu^{(c)}_{\langle u_n\rangle} = 0$. Thus $I_{E_n\setminus E}\mu^{(c)}_{\langle u_n\rangle} = 0$. On the other hand, from $I_{E_n}\mu^{(c)}_{\langle u\rangle} = I_{E_n}\mu^{(c)}_{\langle u_n\rangle}$, we get $I_{E_n\setminus E}\mu^{(c)}_{\langle u\rangle} = I_{E_n\setminus E}\mu^{(c)}_{\langle u_n\rangle} = 0$. Therefore we have $I_{X\setminus E}\mu^{(c)}_{\langle u\rangle} = 0$. □

6. Functional Calculus

Throughout this section we assume the semi-Dirichlet property of the dual form $(\hat{\mathcal{E}}, \mathcal{F})$, namely $(\mathcal{E}, \mathcal{F})$ is a (non-symmetric) Dirichlet form on $L^2(X; m)$. In particular, we get that $\mathcal{F}_b$ is an algebra and every normal contraction operates on $\mathcal{F}$.

Theorem 6.1. *Assume that $(\mathcal{E}, \mathcal{F})$ satisfies* (QR1), (QR2). *Let F be a C^1-class function on $\boldsymbol{R}^d$. Suppose that $u_1, u_2, \cdots, u_d \in \dot{\mathcal{F}}_{loc}$ and denote $\mathbf{u} = (u_1, u_2, \cdots, u_d)$. Then $F(\mathbf{u}) \in \dot{\mathcal{F}}_{loc}$. Further assume $(\mathcal{E}, \mathcal{F})$ is a symmetric quasi-regular Dirichlet form and let $\mu^{(c)}_{\langle u\rangle}$, $u \in \dot{\mathcal{F}}_{loc}$ be the energy measure of continuous part constructed in Lemma* 5.3. *Then* (Γ3) *holds for functions in $\dot{\mathcal{F}}_{loc}$, namely for $w \in \dot{\mathcal{F}}_{loc}$ and $\mathbf{u} = (u_1, u_2, \cdots, u_d)$ with $u_i \in \dot{\mathcal{F}}_{loc}(1 \le i \le d)$,*

$$\mu^{(c)}_{\langle F(\mathbf{u}),w\rangle} = \sum_{i=1}^{d} \frac{\partial F}{\partial x_i}(\tilde{\mathbf{u}})\mu^{(c)}_{\langle u_i,w\rangle}. \tag{1}$$

Proof. We set $F_0(x) = F(x) - F(0)$. Suppose that $\mathbf{u} = (u_1, u_2, \cdots, u_d)$ with $u_i \in \dot{\mathcal{F}}_{loc}(1 \le i \le d)$. Then there exists $\{G_k\}_{k\in N} \in \Xi$ and $u_i^k \in \mathcal{F}_b$ such that $\mathbf{u} = \mathbf{u}_k$ m-a.e. on G_k by Theorem 4.1(i). Here $\mathbf{u}_k = (u_1^k, u_2^k, \cdots, u_d^k)$. We set $K = \prod_{i=1}^{d}[-\|u_i^k\|_\infty, \|u_i^k\|_\infty]$ and $M = \max_{1\le i\le d}\|\partial F_0/\partial x_i\|_{K\,\infty} < \infty$. Then $|F_0(x) - F_0(y)| \le M\sum_{i=1}^{d}|x_i - y_i|$ and $|F_0(x)| \le M\sum_{i=1}^{d}|x_i|$ for $x, y \in K$. Hence $F_0(\mathbf{u}_k) \in \mathcal{F}_b$ by Chapter I Proposition 4.11 in [17]. So $F_0(\mathbf{u}) = F_0(\mathbf{u}_k)$ m-a.e. on G_k implies $F_0(\mathbf{u}) \in \dot{\mathcal{F}}_{loc}$, hence $F(\mathbf{u}) \in \dot{\mathcal{F}}_{loc}$ by $1 \in \dot{\mathcal{F}}_{loc}$. Suppose that $(\mathcal{E}, \mathcal{F})$ is symmetric and quasi-regular. Then we see easily (5) holds for $u_i^k \in \mathcal{F}_b$ by $1 \in \dot{\mathcal{F}}_{loc}$, $\mu^{(c)}_{\langle 1\rangle} = 0$. Hence we get

$$\mu^{(c)}_{\langle F_0(\mathbf{u}_k),w\rangle}=\sum_{i=1}^{d}\frac{\partial F_0}{\partial x_i}(\tilde{\mathbf{u}}_k)\mu^{(c)}_{\langle u_i^k,w\rangle}.$$

Thus we have

$$\begin{aligned}\mu^{(c)}_{\langle F(\mathbf{u}),w\rangle}&=\mu^{(c)}_{\langle F_0(\mathbf{u}),w\rangle}=\lim_{k\to\infty}I_{G_k}\mu^{(c)}_{\langle F_0(\mathbf{u}_k),w\rangle}\\&=\lim_{k\to\infty}I_{G_k}\sum_{i=1}^{d}\frac{\partial F_0}{\partial x_i}(\tilde{\mathbf{u}}_k)\mu^{(c)}_{\langle u_i^k,w\rangle}\\&=\lim_{k\to\infty}I_{G_k}\sum_{i=1}^{d}\frac{\partial F_0}{\partial x_i}(\tilde{\mathbf{u}})\mu^{(c)}_{\langle u_i,w\rangle}\\&=\sum_{i=1}^{d}\frac{\partial F_0}{\partial x_i}(\tilde{\mathbf{u}})\mu^{(c)}_{\langle u_i,w\rangle}.\end{aligned}$$

□

Corollary 6.1. *Assume that $(\mathcal{E},\mathcal{F})$ is quasi-regular. Let D be an open domain of $\boldsymbol{R}^d(d\geq 1)$ and F a C^1-class function on D. Suppose that $u_1,\ u_2,\ \cdots,\ u_d\in\dot{\mathcal{F}}_{loc}$. Denote $\mathbf{u}=(u_1,u_2,\cdots,u_d)$ and $\tilde{\mathbf{u}}=(\tilde{u}_1,\tilde{u}_2,\cdots,\tilde{u}_d)$. Then $F(\mathbf{u})\in\dot{\mathcal{F}}_{\tilde{\mathbf{u}}^{-1}(D)\,loc}$. Further assume that $(\mathcal{E},\mathcal{F})$ is symmetric quasi-regular and let $\mu^{(c)}_{\langle\cdot\rangle}$ be the measure constructed in Lemma 6.3. Then for $w\in\dot{\mathcal{F}}_{loc}$ and $\mathbf{u}=(u_1,u_2,\cdots,u_d)$ with $u_i\in\dot{\mathcal{F}}_{loc}(1\leq i\leq d)$,*

$$\mu^{(c)}_{\langle F(\mathbf{u}),w\rangle}=I_{\tilde{\mathbf{u}}^{-1}(D)}\sum_{i=1}^{d}\frac{\partial F}{\partial x_i}(\tilde{\mathbf{u}})\mu^{(c)}_{\langle u_i,w\rangle}.$$

Proof. It is clear in view of Theorem 4.2. □

Corollary 6.2. *Assume that $(\mathcal{E},\mathcal{F})$ is quasi-regular. Let $u,\ v\in\dot{\mathcal{F}}_{loc}$ and $n\in\boldsymbol{N}$, $p<0$. Then $1/u^n$, $|u|^p$, $\log|u|\in\dot{\mathcal{F}}_{\{\tilde{u}\neq 0\}\,loc}$ and $uv\in\dot{\mathcal{F}}_{loc}$. If further $(\mathcal{E},\mathcal{F})$ is symmetric quasi-regular, then for $w\in\dot{\mathcal{F}}_{loc}$*

$$\begin{aligned}I_{\{\tilde{u}\neq 0\}}\mu^{(c)}_{\langle 1/u^n,w\rangle}&=-I_{\{\tilde{u}\neq 0\}}\frac{n}{\tilde{u}^{n+1}}\mu^{(c)}_{\langle u,w\rangle}.\\I_{\{\tilde{u}\neq 0\}}\mu^{(c)}_{\langle |u|^p,w\rangle}&=I_{\{\tilde{u}\neq 0\}}p|\tilde{u}|^{p-1}\mu^{(c)}_{\langle u,w\rangle}.\\I_{\{\tilde{u}\neq 0\}}\mu^{(c)}_{\langle \log|u|,w\rangle}&=I_{\{\tilde{u}\neq 0\}}\frac{1}{|\tilde{u}|}\mu^{(c)}_{\langle u,w\rangle}.\\\mu^{(c)}_{\langle uv,w\rangle}&=\tilde{u}\mu^{(c)}_{\langle v,w\rangle}+\tilde{v}\mu^{(c)}_{\langle u,w\rangle}.\end{aligned}$$

Next we assume that $(\mathcal{E},\mathcal{F})$ be a symmetric quasi-regular Dirichlet form on $L^2(X;m)$ and let $\mathbf{M}{=}(X_t,P_x)_{\{x\in X\}}$ be an m-symmetric special standard process $\mathcal{E}$-properly associated with $(\mathcal{E},\mathcal{F})$. Then every finely open Borel set with respect to $\mathbf{M}$ is $\mathcal{E}$-quasi-open and every $\mathcal{E}$-quasi-open has an $\mathcal{E}$-q.e. version of finely open Borel set by Theorem 4.6.1 in [11] and the transfer method. In this connection, we use the equivalence of the $\mathcal{E}$-quasi-notion and the quasi-notion with respect to 1-capacity in the framework of C_0-regular Dirichlet forms (cf. Proposition 2.5 (ii) $\Longleftrightarrow$ (iv) in [12] or Chapter IV Lemma 4.5 in [17]). Hence we may assume

$$\Xi = \{\{G_n\} : G_n \text{ is finely open Borel}, G_n \subset G_{n+1}\ \mathcal{E}\text{-q.e. } \forall n \text{ and } \lim_{n\to\infty} \tau_{G_n} = \zeta,\ P_x\text{-a.s. } \mathcal{E}\text{-q.e.}\}.$$

Corollary 6.3. *For a finely closed Borel set F, $1/(1 - E_x[e^{-\sigma_F}])$, $\log(1 - E_x[e^{-\sigma_F}]) \in \dot{\mathcal{F}}_{F^c\,loc}$. Here $\sigma_F = \inf\{t > 0 : X_t \in F\}$.*

Proof. Owing to the transfer method, we get $e_F(x) = E_x[e^{-\sigma_F}]$ is $\mathcal{E}$-quasi-continuous, so F is $\mathcal{E}$-quasi-closed and $F = \{e_F = 1\}$ $\mathcal{E}$-q.e. by Theorem 4.6.1(i) and Theorem 4.1.3 in [11]. It suffices to show that $e_F(x) = E_x[e^{-\sigma_F}] \in \dot{\mathcal{F}}_{loc}$. By using the symmetry of $(\mathcal{E},\mathcal{F})$, we get $e_F \wedge (nh \wedge 1) \in \mathcal{F}$ and $e_F = e_F \wedge (nh \wedge 1)$ m-a.e. on $\{\tilde{h} > 1/n\}$. □

7. Stochastic Integrals

In this section, we give an extension of the definition of stochastic integrals for local functions and local martingales and give a stochastic version of formula (5). As in the last part of the preceding section, we assume that $(\mathcal{E},\mathcal{F})$ be a symmetric quasi-regular Dirichlet form on $L^2(X;m)$ and let $\mathbf{M}=(\Omega,\mathcal{F}_\infty,\mathcal{F}_t,\theta_t,X_t,\zeta,\{P_x\}_{x\in X})$ be an m-symmetric special standard process $\mathcal{E}$-properly associated with $(\mathcal{E},\mathcal{F})$. A family $(A_t)_{t\geq 0}$ of functions on Ω is said to be an *additive functional* (abbreviated in AF) of $\mathbf{M}$ if:

(i) $A_t(\cdot)$ is $\mathcal{F}_t$-measurable for all $t \geq 0$.

(ii) There exists a *defining set* $\Lambda \in \mathcal{F}_\infty$ and an *exceptional set* $N \subset X$ which is $\mathcal{E}$-polar such that $P_x(\Lambda) = 1$ for all $x \in X \setminus N$, $\theta_t(\Lambda) \subset \Lambda$ for all $t > 0$ and for each $\omega \in \Lambda$, $t \mapsto A_t(\omega)$ is right continuous on $[0,\infty)$ and has left limits on $[0,\zeta(\omega))$, $A_0(\omega) = 0$, $|A_t(\omega)| < \infty$ for $t < \zeta(\omega)$, $A_t(\omega) = A_{\zeta(\omega)}(\omega)$ for $t \geq \zeta(\omega)$, and $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t\omega)$ for t, $s \geq 0$.

Two AF's A, B are called *equivalent* and we write $A = B$ if they have a common defining set Λ and a common exceptional set N such that $A_t(\omega) = B_t(\omega)$ for all $t \geq 0$, $\omega \in \Lambda$. An additive functional is called a *continuous additive functional* (abbreviated in CAF) if $A_t(\omega) \geq 0$ for all $t \geq 0$, $\omega \in \Lambda$ and a *positive continuous additive functional*(abbreviated in PCAF) if $t \mapsto A_t(\omega)$ is continuous on $[0,\infty)$ for each $\omega \in \Lambda$.

Recall that S is the totality of $\mathcal{E}$-smooth measures. Then there exists one to one correspondence between S and the equivalence class of PCAF's which is specified by

$$\text{(1)} \qquad \lim_{t\downarrow 0}\frac{1}{t}E_m\Big[\int_0^t f(X_s)dA_s\Big] = \int_X f(x)\mu(dx) \text{ for any } f\in\mathcal{B}^+(X).$$

For an AF A, we define its energy

$$e(A)=\lim_{t\downarrow 0}\frac{1}{2t}E_m[A_t^2]$$

if this limits exists in $[0,\infty]$. Define

$$\mathcal{M}=\{M : M \text{ is an AF of } \mathbf{M},\ E_x[M_t^2]<\infty,$$
$$E_x[M_t]=0 \text{ for } \mathcal{E}\text{-q.e. } x\in X \text{ and all } t\geq 0\}.$$

Then every element M of $\mathcal{M}$ is a version of square integrable martingale which has left limits on a defining set. Furthermore, $e(M)$ exists in $[0,\infty]$. Define

$$\overset{\circ}{\mathcal{M}}=\{M\in\mathcal{M} : e(M)<\infty\}.$$

For A, $B\in\overset{\circ}{\mathcal{M}}$, we can define the mutual energy

$$e(A,B)=\lim_{t\downarrow 0}\frac{1}{2t}E_m[A_tB_t].$$

Owing to Theorem 5.2.1 in [11], $(\overset{\circ}{\mathcal{M}},e)$ has a Hilbertian structure. $M\in\overset{\circ}{\mathcal{M}}$ is called a *martingale additive functionals of finite energy* (abbreviated MAF). $M\in\mathcal{M}$ admits a PCAF $\langle M\rangle$ such that $E_x[\langle M\rangle_t]=E_x[M_t^2]$ $\mathcal{E}$-q.e. $x\in X$, $t>0$. $\langle M\rangle$ is called its *quadratic variation* associated with M. Denote by $\mu_{\langle M\rangle}$ the Revuz measure of $\langle M\rangle$ according to (6). For M, $L\in\overset{\circ}{\mathcal{M}}$, we see easily that $\langle M+L\rangle+\langle M-L\rangle=2\langle M\rangle+2\langle L\rangle$ and $\langle aM\rangle=a^2\langle M\rangle$, hence by (6), $\mu_{\langle M+L\rangle}+\mu_{\langle M-L\rangle}=2\mu_{\langle M\rangle}+2\mu_{\langle L\rangle}$ and $\mu_{\langle aM\rangle}=a^2\mu_{\langle M\rangle}$. We let $\langle M,L\rangle=(1/2)\{\langle M+L\rangle-\langle M\rangle-\langle L\rangle\}$ and $\mu_{\langle M,L\rangle}=(1/2)\{\mu_{\langle M+L\rangle}-\mu_{\langle M\rangle}-\mu_{\langle L\rangle}\}$. Then $\langle M,L\rangle$ is a CAF and $\mu_{\langle M,L\rangle}$ is a signed finite measure on $\mathcal{B}(X)$. Next lemma is a variant of Lemma 5.6.1 in [11]. We omit its proof.

Lemma 7.1. *If M, $L\in\overset{\circ}{\mathcal{M}}$, $f\in L^2(X;\mu_{\langle M\rangle})$ and $g\in L^2(X;\mu_{\langle L\rangle})$, then fg is integrable with respect to the absolute variation $|\mu_{\langle M,L\rangle}|$ of $\mu_{\langle M,L\rangle}$ and*

$$\Big(\int_X |fg|d|\mu_{\langle M,L\rangle}|\Big)^2\leq\int_X f^2d\mu_{\langle M\rangle}\int_X g^2d\mu_{\langle L\rangle}.$$

Hence we have next theorem.

Theorem 7.1. *Given $M\in\overset{\circ}{\mathcal{M}}$ and $f\in L^2(X;\mu_{\langle M\rangle})$, there exists a unique element $f\bullet M\in\overset{\circ}{\mathcal{M}}$ such that*

(2) $$e(f \bullet M, L) = \frac{1}{2} \int_X f(x) \mu_{\langle M,L \rangle}(dx)$$

for any $L \in \overset{\circ}{\mathcal{M}}$. *Further*

(3) $$d\mu_{\langle f \bullet M, L \rangle} = f d\mu_{\langle M,L \rangle}(dx), \ L \in \overset{\circ}{\mathcal{M}}.$$

Corollary 7.1. (i) *For* $M \in \overset{\circ}{\mathcal{M}}$, $f \in L^2(X; \mu_{\langle M \rangle})$ *and* $g \in L^2(X; f^2 \mu_{\langle M \rangle})$,

(4) $$g \bullet (f \bullet M) = (gf) \bullet M.$$

(ii) *For* $M, L \in \overset{\circ}{\mathcal{M}}$, $f \in L^2(X; \mu_{\langle M \rangle})$ *and* $g \in L^2(X; \mu_{\langle L \rangle})$,

(5) $$e(f \bullet M, g \bullet L) = \frac{1}{2} \int_X f(x) g(x) \mu_{\langle M,L \rangle}(dx).$$

The proof of Theorem 7.1 and Corollary 7.1 is similar to Theorem 5.6.1, Lemma 5.6.1 and Corollary 5.6.1 in [11] in view of the transfer method. So we omit its proof. Furthermore we define

$$\mathcal{N}_c = \{N : N \text{ is a CAF}, e(N) = 0, \ E_x[|N_t|] < \infty \text{ for } \mathcal{E}\text{-q.e. } x \in X \text{ and all } t \geq 0\}.$$

$N \in \mathcal{N}_c$ is called a *continuous additive functionals of zero energy*. Every $u \in \mathcal{F}$ has $\mathcal{E}$-quasi-continuous m-version $\tilde{u}$ which is finely continuous $\mathcal{E}$-q.e. So $t \mapsto \tilde{u}(X_t) - \tilde{u}(X_0)$ is an AF of $\mathbf{M}$. Note that one obtains an equivalent AF if one choose a different $\mathcal{E}$-quasi-continuous m-version of u. Therefore we may set $A^{[u]} = (\tilde{u}(X_t) - \tilde{u}(X_0))_{t \geq 0}$. The following decomposition holds: for any $u \in \mathcal{F}$ there exists unique $M^{[u]} \in \overset{\circ}{\mathcal{M}}$ and $N^{[u]} \in \mathcal{N}_c$ such that

(6) $$A^{[u]} = M^{[u]} + N^{[u]}.$$

The decomposition (11) is called the Fukushima decomposition. The MAF $M^{[u]}$ in (11) has an e-orthogonal decomposition

(7) $$M^{[u]} = M^{[u],c} + M^{[u],j} + M^{[u],k}, \ u \in \mathcal{F}$$

such that for $u, v \in \mathcal{F}$,

$$\begin{aligned} e(M^{[u],c}, M^{[v],c}) &= \mathcal{E}^{(c)}(u,v), \ e(M^{[u],j}, M^{[v],j}) = \mathcal{E}^{(j)}(u,v), \\ e(M^{[u],k}, M^{[v],k}) &= \frac{1}{2}\mathcal{E}^{(k)}(u,v). \end{aligned}$$

$M_t^{[u],c}$, $M_t^{[u],j}$, $M_t^{[u],k}$ is called the square integrable martingale additive functional of continuous part, of jumping part, of killing part, respectively (see §5.3. in [11]). As in the last part of the preceding section, we may set

$$\Xi = \{\{G_n\} : G_n \text{ is finely open Borel, } G_n \subset G_{n+1}\ \mathcal{E}\text{-q.e. } \forall n \text{ and } \lim_{n\to\infty} \tau_{G_n} = \zeta,\ P_x\text{-a.s. } \mathcal{E}\text{-q.e.}\}.$$

According to Lemma 5.6.4 in [11], $M_t^{[u],c}$ is extended to a additive functional for $u \in \dot{\mathcal{F}}_{loc}$:

$$M_t^{[u],c} = M_t^{[u_n],c},\ t < \tau_{E_n},$$

where $\{E_n\} \in \Xi$ and $\{u_n\} \subset \mathcal{F}$ such that $u = u_n$ m-a.e. on E_n. In particular, we obtain

$$u, v \in \dot{\mathcal{F}}_{loc},\ u - v = \text{constant} \Longrightarrow M^{[u],c} = M^{[v],c}.$$

We say that a local AF M of $\mathbf{M}$ is locally in $\overset{\circ}{\mathcal{M}}$ ($M \in \overset{\circ}{\mathcal{M}}_{loc}$ in notation) if there exists $\{E_n\} \in \Xi$ and a sequence $\{M^{(n)}\}$ of MAF's in $\overset{\circ}{\mathcal{M}}$ such that $M_t = M_t^{(n)}$, $t < \tau_{E_n}$ and its quadratic variation $\langle M\rangle$ as a PCAF is well-defined by $\langle M\rangle_t = \langle M^{(n)}\rangle_t$, $t < \tau_{E_n}$, $n = 1, 2, \cdots$ by choosing an appropriate defining set and exceptional sets of $\langle M\rangle$. $\langle M\rangle$ does not depend (up to an equivalence) on the special choice of $\{G_n\}_{n\in N} \in \Xi$ and $\{M^{(n)}\}$ for M. We see easily $M^{[u],c} \in \overset{\circ}{\mathcal{M}}_{loc}$ for $u \in \dot{\mathcal{F}}_{loc}$. We can still define the energy measure of $M \in \overset{\circ}{\mathcal{M}}_{loc}$ as the Revuz measure $\mu_{\langle M\rangle}$ of the PCAF $\langle M\rangle$. Owing to a version of Lemma 5.1.4 for smooth measures in [11], we have

$$\int_X f(x)\mu_{\langle M\rangle}(dx) = \int_X f(x)\mu_{\langle M^{(n)}\rangle}(dx),\ f \in \mathcal{B}_b(X),\ \mathrm{supp}[f] \subset G_n.$$

In particular, Lemma 7.1 extends to M, $L \in \overset{\circ}{\mathcal{M}}_{loc}$ and Theorem 7.1 extends to $M \in \overset{\circ}{\mathcal{M}}_{loc}$ and $f \in L^2(X; \mu_{\langle M\rangle})$, so that there exists $f \bullet M \in \overset{\circ}{\mathcal{M}}$ such that (7) holds. For $M \in \overset{\circ}{\mathcal{M}}_{loc}$, we then define the local $L^2(X; \mu_{\langle M\rangle})$-space in the broad sense denoted by $\dot{L}^2_{loc}(X; \mu_{\langle M\rangle})$ as follows:

$$\dot{L}^2_{loc}(X; \mu_{\langle M\rangle}) = \{f\ :\ f|_{G_n} \in L^2(G_n; \mu_{\langle M\rangle}) \text{ for some } \{G_n\}_{n\in N} \in \Xi \text{ and MAF's } \{M^{(n)}\} \text{ which represent } M\}.$$

For $M \in \overset{\circ}{\mathcal{M}}_{loc}$ and $f \in \dot{L}^2_{loc}(X; \mu_{\langle M\rangle})$, we can finally define the stochastic integral $f \bullet M \in \overset{\circ}{\mathcal{M}}_{loc}$ by

$$(f \bullet M)_t = ((I_{G_n} f) \bullet M)_t,\ t < \tau_{G_n},$$

$\{G_n\}_{n\in N} \in \Xi$ being the sequence in the definition of $f \in \dot{L}^2_{loc}(X; \mu_{\langle M\rangle})$. The above definition is well-defined. Indeed, Let $\{\hat{G}_n\}_{n\in N} \in \Xi$ be another sequence such that

$f \in L^2(\hat{G}_n; \mu_{\langle M \rangle})$ and $M_t = \hat{M}_t^{(n)}$, $t < \tau_{\hat{G}_n}$ for some MAF's $\{\hat{M}^{(n)}\}$. It suffices to show that $\lim_{n\to\infty} I_{\{t<\tau_{G_n}\}}((I_{G_n} f) \bullet M)_t = \lim_{n\to\infty} I_{\{t<\tau_{\hat{G}_n}\}}((I_{\hat{G}_n} f) \bullet M)_t$. We may assume $G_n \subset \hat{G}_n$ for each $n \in \boldsymbol{N}$. Hence we only to show

$$(I_{G_n} f \bullet M)_t = (I_{\hat{G}_n} f \bullet M)_t, \ t < \tau_{G_n}. \tag{8}$$

To establish (13), we need next lemma.

Lemma 7.2. *For $M \in \overset{\circ}{\mathcal{M}}$, a finely open Borel set G and $f \in L^2(X; \mu_{\langle M \rangle})$ with $f = 0$ on G, then $(f \bullet M)_t = 0$, $t < \tau_G$ P_x-a.s. $\mathcal{E}$-q.e.*

Proof. It suffices to show that $\lim_{t\downarrow 0}(1/t)E_m[(f \bullet M)^2_{t\wedge\tau_G}] = \int_G f(x)\mu_{\langle M \rangle}(dx)$. The left hand side is $\lim_{t\downarrow 0}(1/t)E_m[\langle f \bullet M \rangle_{t\wedge\tau_G}]$, which coincides with the right hand side by a version of Lemma 5.1.4 and Lemma 5.5.2(ii) in [11]. □

Lemma 7.2 extends to $M \in \overset{\circ}{\mathcal{M}}_{loc}$ and $f \in L^2(X; \mu_{\langle M \rangle})$ with $f = 0$ on a finely open Borel set G. So (13) holds. Next lemma holds in view of the transfer method (see Lemma 5.6.1 in [11]).

Lemma 7.3. *Let $\mathcal{D}$ be an $\mathcal{E}_1^{1/2}$-dense subfamily of $(\mathcal{E}, \mathcal{F})$. Then the family $\{M^{[u]} : u \in \mathcal{D}\}$ is dense in $(\overset{\circ}{\mathcal{M}}, e)$.*

Next theorem is an extension of Theorem 5.6.2 in [11].

Theorem 7.2. *Let F be a C^1-class function on $\boldsymbol{R}^d$. Suppose that u_1, u_2, $\cdots$, $u_d \in \dot{\mathcal{F}}_{loc}$ and denote $\mathbf{u} = (u_1, u_2, \cdots, u_d)$. Then $(\partial F/\partial x_i)(\tilde{\mathbf{u}}) \in \dot{L}^2_{loc}(X; \mu_{\langle M^{[u_i],c} \rangle})$ for each $i = 1, 2, \cdots, d$ and*

$$M^{[F(\mathbf{u})],c} = \sum_{i=1}^{d} \frac{\partial F}{\partial x_i}(\tilde{\mathbf{u}}) \bullet M^{[u_i],c}$$

and

$$\langle M^{[F(\mathbf{u})],c}, L \rangle = \sum_{i=1}^{d} \int_0^{\bullet} \frac{\partial F}{\partial x_i}(\tilde{\mathbf{u}})(X_s) d\langle M^{[u_i],c}, L \rangle_s, \ L \in \overset{\circ}{\mathcal{M}}_{loc}.$$

Proof. The first assertion is easily checked by using Theorem 4.1(i). We can take common $\{G_n\}_{n\in N} \in \Xi$ such that $I_{G_n} \bullet M^{[F(\mathbf{u})],c}$, $I_{G_n}(\partial F/\partial x_i)(\tilde{\mathbf{u}}) \bullet M^{[u_i],c} \in \overset{\circ}{\mathcal{M}}$ for each i, $n \in \boldsymbol{N}$. The isometry (10), (5) and the orthogonality of the decomposition (12) give

$$e(I_{G_n} \bullet M^{[F(\mathbf{u})],c}, M^{[v]}) = e\Big(\sum_{i=1}^{d} I_{G_n} \frac{\partial F}{\partial x_i}(\tilde{\mathbf{u}}) \bullet M^{[u_i],c}, M^{[v]}\Big),\ v \in \mathcal{F}.$$

On account of Lemma 7.3, we have the following identity in $\overset{\circ}{\mathcal{M}}$

$$I_{G_n} \bullet M^{[F(\mathbf{u})],c} = \sum_{i=1}^{d} I_{G_n} \frac{\partial F}{\partial x_i}(\tilde{\mathbf{u}}) \bullet M^{[u_i],c}.$$

Thus we obtain our assertion. □

8. Examples

We follow the formulations in [13]. Let X be a separable metric space and m be a σ-finite Borel measure on X with full topological support. We consider a subalgebra $\mathcal{C}$ of $C_b(X) \cap L^2(X;m)$ which is dense in $L^2(X;m)$. We assume that $\mathcal{C}$ is closed under composition of C_1-class function F on $\boldsymbol{R}$ with $F(0) = 0$, namely $F(u) \in \mathcal{C}$ if $u \in \mathcal{C}$. We take an $L^1(X;m)$-valued symmetric bilinear form $\Gamma(\cdot,\cdot)$ on $\mathcal{C} \times \mathcal{C}$ which satisfies $\Gamma(F(u),v) = F'(u)\Gamma(u,v)$ for the above function F. We consider next bilinear form:

$$\mathcal{E}(u,v) = \frac{1}{2}\int_X \Gamma(u,v)dm,\ u,v \in \mathcal{C}.$$

We assume the closability of $(\mathcal{E},\mathcal{C})$ on $L^2(X;m)$ and denote by $(\mathcal{E},\mathcal{F})$ its closure on $L^2(X;m)$. We then see $(\mathcal{E},\mathcal{F})$ is a Dirichlet form on $L^2(X;m)$. We assume the quasi-regularity of $(\mathcal{E},\mathcal{F})$. In view of Proposition 2.4(Γ6) in [13](see Remark 3.1(ii)), $(\mathcal{E},\mathcal{F})$ satisfies the strong local property in the general sense

$$\mathcal{E}(u,v) = 0 \text{ if } u = \text{const. } m\text{-a.e. on an } \mathcal{E}\text{-neighbourhood of } \mathcal{E}\text{-supp}[|v|m].$$

We can extends Γ on $\dot{\mathcal{F}}_{loc} \times \dot{\mathcal{F}}_{loc}$. Then the chain rule for $\dot{\mathcal{F}}_{loc}$ holds in the next style: Let F be a C^1-class function on $\boldsymbol{R}^d$ and $\mathbf{u} = (u_1,u_2,\cdots,u_d)$ with $u_i \in \dot{\mathcal{F}}_{loc}(1 \le i \le d)$. Then for $w \in \dot{\mathcal{F}}_{loc}$,

$$\Gamma(F(\mathbf{u}),w) = \sum_{i=1}^{d} \frac{\partial F}{\partial x_i}(\tilde{\mathbf{u}})\Gamma(u_i,w).$$

EXAMPLE 8.1. Let (B,H,μ) be the abstract Wiener space. B is a real Banach space. μ is a mean 0 Gaussian Borel measure on B. H is a real Hilbert space such that H is continuously densely embedded in B and

$$\int_B e^{\sqrt{-1}l(z)}\mu(dz) = e^{-(1/2)|l|^2_{H^*}},\ l \in B^* \subset H^*$$

Here H^*, B^* is the dual space of H, B, respectively. We take FC_b^∞ as $\mathcal{C}$, the cylindrical functions with bounded derivative on B:

$$FC_b^\infty = \{u : B \to \boldsymbol{R} | \exists n \in \boldsymbol{N}, \quad \exists f \in C_b^\infty(\boldsymbol{R}^n), \exists l_1, l_2, \cdots, l_n \in B^* \text{ such that } u(z) = f(l_1(z), l_2(z), \cdots, l_n(z))\}.$$

Here $C_b^\infty(\boldsymbol{R}^n)$ is the totality of smooth functions on $\boldsymbol{R}^n$ such that all derivatives of elements in $C_b^\infty(\boldsymbol{R}^n)$ are bounded. Also we let FC^∞ the family of smooth cylindrical functions on B:

$$FC^\infty = \{u : B \to \boldsymbol{R} | \exists n \in \boldsymbol{N} \exists f \in C^\infty(\boldsymbol{R}^n), \exists l_1, l_2, \cdots, l_n \in B^* \text{ such that } u(z) = f(l_1(z), l_2(z), \cdots, l_n(z))\}.$$

The derivative of h-direction $D_h u$ of $u \in FC_b^\infty$ for $h \in H$ is given by

$$D_h u(z) = \lim_{t \to 0} \frac{u(z + th) - u(z)}{t}.$$

$Du(z)$ is defined by $(Du(z), i(h))_{H^*} = D_h u(z)$. Here $i : H \to H^*$ is the identification map. We set $\Gamma(u, v)(z) = (Du(z), Dv(z))_{H^*}$ and

$$\mathcal{E}(u, v) = \frac{1}{2} \int_B (Du(z), Dv(z))_{H^*} \mu(dz), \; u, v \in FC_b^\infty.$$

It is well-known $(\mathcal{E}, FC_b^\infty)$ is closable on $L^2(B; \mu)$ and the closure $(\mathcal{E}, \mathcal{F})$ on $L^2(B; \mu)$ is associated with the Ornstein-Uhlenbeck process on B. We see $FC^\infty \subset \dot{\mathcal{F}}_{loc}$. Fix $l \in B^*$ with $l \neq 0$. Then we see $F(l) \notin \mathcal{F}_e$ for $F(t) = e^{t^4}$, but $F(l) \in \dot{\mathcal{F}}_{loc}$. The chain rule is $(DF(l)(z), Dv(z))_{H^*} = 4l(z)^3 e^{l(z)^4} (l, Dv(z))_{H^*}$.

EXAMPLE 8.2. We follow the notations in [21]. Let $X = \mathcal{M}(S)$ be the probability measures on a Polish space S. Define Γ by

$$\Gamma(u, v)(\mu) = \int_S \nabla u(\mu) \nabla v(\mu) d\mu - \int_S \nabla u(\mu) d\mu \int_S \nabla v(\mu) d\mu, \; u, v \in FC_b^\infty, \; \mu \in \mathcal{M}(S),$$

where $\nabla u(\mu) = (\partial u / \delta_x)(\mu) = (d/ds) u(\mu + s\delta_x)|_{s=0}$, $x \in S$, $\mu \in \mathcal{M}(S)$ and

$$FC_b^\infty = \{u : \mathcal{M}(S) \to \boldsymbol{R} | \exists n \in \boldsymbol{N}, \; \exists f \in C_b^\infty(\boldsymbol{R}^n), \; \exists \psi_1, \psi_2, \cdots, \psi_n \in C_b(S) \text{ such that } u(z) = f(\langle \cdot, \psi_1 \rangle, \langle \cdot, \psi_2 \rangle, \cdots, \langle \cdot, \psi_n \rangle)\}.$$

Let m be the reversible invariant probability measure for Fleming-Viot processes. We let $\mathcal{E}(u, v) = (1/2) \int_{\mathcal{M}(S)} \Gamma(u, v) m(d\mu)$ for u, $v \in FC_b^\infty$. Then $(\mathcal{E}, FC_b^\infty)$ is closable on $L^2(\mathcal{M}(S);m)$. Denote by $(\mathcal{E},\mathcal{F})$ its closure on $L^2(\mathcal{M}(S);m)$. Then $(\mathcal{E},\mathcal{F})$ is a quasi-regular Dirichlet form. We let

$$FC^\infty = \{u : \mathcal{M}(S) \to \boldsymbol{R} | \exists n \in \boldsymbol{N},\ \ \exists f \in C^\infty(\boldsymbol{R}^n),\ \exists \psi_1, \psi_2, \cdots, \psi_n \in C_b(S)$$
$$\text{such that } u(z) = f(\langle\cdot,\psi_1\rangle, \langle\cdot,\psi_2\rangle, \cdots, \langle\cdot,\psi_n\rangle)\}.$$

Then we see $FC^\infty \subset \dot{\mathcal{F}}_{loc}$.

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After finishing this paper, the author knows an independent result on Beurling-Deny type decomposition for quasi-regular Dirichlet forms in terms of extended Dirichlet space by Z. Dong, Z.-M. Ma and W. Sun [7]. Their decomposition is given for elements in $\mathcal{F}_e$ and the uniqueness is formulated in the general sense. They also note that the uniqueness of decomposition in the ordinary sense holds if $1 \in \mathcal{F}$. But the present condition for the decomposition in the ordinary sense is much milder. He also knows another result on the chain rule for local martingale additive functionals in the framework of (not necessarily strong) local non-symmetric Dirichlet forms by G. Trutnau [25], in which however the chain rule of energy measure of continuous part for local space is not presented even if the form is symmetric. He thanks Professor M. Röckner, who gives the imformation of [7], [25].

References

[1] S. Albeverio, Z.-M. Ma and M. Röckner: *A Beurling-Deny type structure theorem for Dirichlet forms on general state spaces*, Ideas and Methods in Mathematical Analysis, S-tochastics, and Applications. Edit. S. Albeverio, J.E. Fenstad, H. Holden, T. Lindstrϕm, Cambridge, (1991), 115–123.

[2] P.A. Ancona: *Sur les espaces de Dirichlet: Principes, fonction de Green*, J. Math. Pures et Appl. **54** (1975), 75–124.

[3] N. Bouleau and F. Hirsch: Dirichlet Forms and Analysis on Wiener Space, Walter de Gruyter, 1991.

[4] S. Carrillo Menendez: *Processus de Markov associé à une forme de Dirichlet non symmétrique*, Z. Wahrsch. Verw. Gebiete, **33** (1975), 139–154.

[5] Z.Q. Chen, Z.-M. Ma and M. Röckner: *Quasi-homeomorphism of Dirichlet forms*, Nagoya Math. J. **136** (1994), 1–15.

[6] Z. Dong and Z.-M. Ma: *An integral representation theorem for quasi-regular Dirichlet s-paces*, Chinese Sci. Bull. **38** (1993), 1355–1358.

[7] Z. Dong, Z.-M. Ma and W. Sun: *A note on Beurling-Deny formulae in infinite dimensional spaces*, preprint (1997).

[8] P.J. Fitzsimmons: *Absolute continuity of symmetric diffusions*, Ann. of Prob. **25** (1997), 230–258.

[9] P.J. Fitzsimmons: *On the quasi-regularity of semi-Dirichlet forms*, preprint (1996).

[10] B. Fuglede: *The quasi topology associated with a countably subadditive set function*, Ann. Inst. Fourier, **21** (1971), 123–169.

[11] M. Fukushima, Y. Oshima and M. Takeda: Dirichlet Forms and Symmetric Markov Processes, Walter de Gruyter, 1994.

[12] K. Kuwae: *Permanent sets of measures charging no exceptional sets and the Feynman-Kac Formula*, Forum Math. **7** (1995), 575–605.

[13] K. Kuwae: *On pseudo metrics of Dirichlet forms on separable metric spaces*, Proceedings of the Seventh Japan-Russia Symposium on Probability Theory and Mathematical Statistics, Edit. S. Watanabe, M. Fukushima, Yu.V. Prohorov, A.N. Shiryaev, World Scientific, (1996), 256–265.

[14] K. Kuwae: *Recurrence and conservativeness of symmetric diffusion processes by Girsanov transformations*, J. Math. Kyoto Univ. **37** (1997), 735–756.

[15] L.H. Loomis: An Introduction to Abstract Harmonic Analysis, D. Van Nostrand Company, 1953.

[16] Z.-M. Ma, L. Overbeck and M. Röckner: *Markov processes associated with semi-Dirichlet forms*, Osaka J. Math. **32** (1994), 97–119.

[17] Z.-M. Ma and M. Röckner: Introduction to the Theory of (Non-Symmetric) Dirichlet Forms, Springer Universitext, 1992.

[18] Z.-M. Ma and M. Röckner: *Markov processes associated with positivity preserving coercive forms*, Can. J. Math. **47** (1995), 817–840.

[19] H. Ôkura: *Capacitary inequalities and global properties of symmetric Dirichlet forms*, Proceedings of Dirichlet Forms and Stochastic Processes, Edit. Z.-M. Ma, M. Röckner, J.-A. Yan, Walter de Gruyter (1995), 291–303.

[20] Y. Oshima and T. Yamada: *On some representations of continuous additive functionals locally of zero energy*, J. Math. Soc. Japan, **36** (1984), 315–339.

[21] M. Röckner and B. Schmuland: *Quasi-regular Dirichlet forms: Examples and counterexamples*, Can. J. Math. **47** (1995), 165–200.

[22] B. Schmuland: *Positivity preserving forms have the Fatou property*, To appear in Potential Analysis (1996).

[23] I. Shigekawa and S. Taniguchi: *Dirichlet forms on separable metric spaces*, Proceedings of the Sixth USSR-Japan Symposium on Probability Theory and Mathematical Statistics, Edit. A.N. Shiryaev, V.S. Korolyuk, S. Watanabe, M. Fukushima, World Scientific, (1992), 324–353.

[24] K.Th. Sturm: *Analysis on local Dirichlet spaces I,-Recurrence, conservativeness and L^p-Liouville properties-*, J. Reine. Angew. Math. **456** (1994), 173–196.

[25] G. Trutnau: *Die Ito-Formel im (nicht-symmetrischen) quasi-regulären Dirichletraum*, Dip- lomarbeit (1996).

Department of Information Science
Saga University
Saga, 840-8502, Japan
e-mail: kuwae@is.saga-u.ac.jp