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ARTINIAN RINGS RELATED TO RELATIVE ALMOST PROJECTIVITY II

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Let R be an artinian ring. In [10], we have studied R on which the following condition holds: for R-modules M and N, if M is N-projective, then M' is always almost N-projective for every submodule M' of M. If M' is always N-projective in the above, then this property characterizes hereditary rings with $J^2=0$ [2] and [6], where J is the Jacobson radical of R.

We have investigated the above condition in [10], when i); M and N are local and ii): M is local and N is a direct sum of local modules. In this paper we give a characterization of R over which the above condition is satisfied for any R-modules M and N.

1. Preliminaries

In this paper R is always an artinian ring with identity, and every module is a finitely generated R-module. We shall use the same notations given in [10].

We have studied rings R over which the following condition is satisfied in [10]:

For any R-modules M and N

(#) if M is N-projective, then M' is always almost N-projective for every submodule M' of M.

We denote primitive idempotents in R by e, f, g, and so on. Assume that (#) holds whenever M and N are local. Then we have shown in [10] that R has the following structure:

$J^3=0$ and for a primitive idempotent e with $eJ^2\neq 0$

(0) $eR\supset eJ\simeq\sum_K\oplus(f_kR)^{(n_k)}\oplus\sum_J\oplus S_j$,

where the f_iR is a uniserial and projective module with $f_iJ^2=0$, $f_iJ\neq 0$ and the S_j is simple.

(If necessary, we use the following decomposition:

$e_iR\supset e_iJ\approx\sum_{K(1)}\oplus(f_{ik}R)^{(n_{ik})}\oplus\sum_{J(1)}\oplus S_{ij}$.)

We shall use frequently the following theorem: [10], Theorem 1.

Theorem 0. *Let R be artinian. Then (#) holds whenever M and N are local* *if and only if* i) $J^3=0$ *and* $(eR\supset)$ eJ *has the decomposition* (0), ii) **if** $e_1R \not\approx e_2R$, **then** $f_{1i}R \not\approx f_{2j}$ **for any** i **and** j, iii) fR/fJ *is never isomorphic to any simple component of* $\mathrm{Soc}(R)$ *and* iv) *for any simple submodule* S *in* $\sum_K \oplus (f_kR)^{(n_k)}$, $eReS=\sum_{K'}\oplus(f_kJ)^{(n_k)}$ *for some* $K'\subset K$, *where* e *and* f *are in* (0).

2. Condition(#)

In this section we study (#) where M and N are any finitely generated R-modules. In this case gJ is almost projective for any primitive idempotent g, and hence R is a right almost hereditary ring [7], i.e. J is almost projective as a right R-module.

First we give

Proposition 1. Let R be right almost hereditary. Then the following are equvalent:

1) (#) *holds whenever* M *and* N *are local.*

2) (#) *holds whenever* M *is local and* N *is a finite direct sum of local modules.*

Proof. 1)→2). Since R is almost hereditarty, $\mathrm{Soc}(R)$ is almost projective by definition and Theorem 0. If gR/gJ is monomorphic to $\mathrm{Soc}(R)$, then gR is uniserial by [9], Theorem 1 and we have 2) by [10], Theorem 2.

Next we study (#) when M is local and N is any R-module. We use the decomposition (0) of eJ. Put $f_1R=fR$ and $S=\mathrm{Soc}(fR)=fJ$.

Lemma 1. *Assume that* (#) *holds whenever* M *is local and let* $eR\supset fR\supset S$ *be as above. Then* 1) *every element in* $\mathrm{Hom}_R(S, S)$ *is extensible to an element in* $\mathrm{Hom}_R(fR, fR)$, *and* 2) S *is neither isomorphic to any simple component of* $\mathrm{Soc}(gR)$ *nor any* S_j *in* (0), *where* $gR\not\approx eR$, $gR\not\approx fR$ *and* $gJ\neq 0$.

Proof. Assume that $\mathrm{Soc}(gR)$ contains a simple component isomorphic to S via θ for some primitive idempotent g. Take $fR\oplus gR$ and its submodule $\tilde{S}=\{s+\theta(s)|s\in S\}$ and put $N=(fR\oplus gR)/\tilde{S}$. If $gR\not\approx eR$, eR/S is N-projective by [1], p. 22, Exercise 4 and [10], Lemma 6. Hence fR/S is almost N-projective by (#). However fR/S is not N-projective by [1], p. 22, Exercise 4. Therefore N is decomposable by [3], Theorem 1. Let $N=N_1\oplus N_2$. Suppose $gR\not\approx fR$. Then we can assume $N_1/J(N_1)\approx fR/fJ$ and $N_2/J(M_2)\approx gR/gJ$. Further $N=\tilde{f}R+\tilde{g}R$, where $\tilde{f}R\approx fR$ and $\tilde{g}R\approx gR$. Since $|N|=|\tilde{f}R|+|\tilde{g}R|-1$, we obtain a) $N_1\approx \tilde{f}R$ or b) $N_2\approx \tilde{g}R$ via the projections. In a) $N=\tilde{f}R\oplus N_2$, and hence θ^{-1} is extensible to an element in $\mathrm{Hom}_R(gR, fR)$, and in b) $N=N_1\oplus \tilde{g}R$, and hence θ is extensible to an element in $\mathrm{Hom}_R(fR, gR)$. We obtain the similar result even if $gR\approx fR$. Hence from the above observation we obtain 1) and that S is never isomorphic to any simple component of $\mathrm{Soc}(e'R)$ and $\mathrm{Soc}(f'R)$ form Theorem 0, where $e'R \not\approx eR$, $f'R \not\approx fR$ and $e'J^2 \neq 0$, $f'J \neq 0$. Finally assume $S \approx S_1 \subset eR$, where S_1 is a simple module in (0). Take $(fR \oplus eR/(fJ)^{(n_1)})/\tilde{S}$. Since $eRe(fJ) \subset (fJ)^{(n_1)}$, eR/S is $(fR \oplus eR)/(\tilde{S} \oplus (fJ)^{(n_1)})$-projective. Similarly to the above we obtain an extension of θ(or θ^{-1}) in $\mathrm{Hom}_R(fR, eR/(fJ)^{(n_1)})$ (or in $\mathrm{Hom}_R(eR/(fJ)^{(n_1)}, fR)$). However there are no extensions of θ by Theorem 0, a contradiction.

If R is left QF-2 in the above, then any element in $\mathrm{Hom}_R(S, S)$ is extensible to an element in $\mathrm{Hom}_R(eR, eR)$ by [10], Lemma 13, however this fact is not true in lemma 1 (see Proposition 3 and Example 2 below). Under the assumption (#) we can state the content of Lemma 1 as follows:

let S' be a simple submodule in gR, then any element in $\mathrm{Hom}_R(S, S')$ *is extensible to an element in* $\mathrm{Hom}_R(fR, gR)$, *where g is any primitive idempotent.*

Finally we study (#) for any R-modules M and N. We start with studying a structure of N. Let $eR \supset eJ \approx \sum \oplus (f_iR)^{(n_1)} \oplus \cdots$ be as in (0). We consider the condition:

(*) the properties in Lemma 1 and Theorem 0 hold.

We fix primitive idempotents e and $f = f_i$ above. Take a projective module T and put $T = (eR)^{(p)} \oplus (fR)^{(q)} \oplus \sum g_jR$, where $g_jR \not\approx eR$ and $g_jR \not\approx fR$ for all j.

Lemma 2. *Assume* (*). *Let T, e and f be as above. If T/C is indecomsable, then $T = fR$ or $q = 0$.*

Proof. Assume $q \geqq 1$. Consider the decomposition $T = (fR) \oplus T'$, where $T' = (eR)^{(p)} \oplus (fR)^{(q-1)} \oplus \sum \oplus g_jR$, and use the same notations as in [10], Lemma 10 for this decomposition. We may assume $C \subset J(T)$ (note that T is a lifting module). Suppose $C_1 = 0$ and $C^1 = fJ$. Put $C^1 = xR$; $xk = x$ for a primitive idempotent k and let $\theta: C^1 = C^1/C_1 \approx C^2/C_2 \subset J(T')/C_2$ be the ismorphism. Then $\theta(x) = x_1 + x_2 + x_3 + C_2$, where $x_1 \in (eJ)^{(p)}$, $x_2 \in (fJ)^{(q-1)}$, $x_3 \in \sum \oplus g_jJ$ and $x_ik = x_1$ for all i. Since $x_1J = 0$ for all i form iii) in Theorem 0, $\theta: xR \approx (x_1 + x_2 + x_3)R \subset T'$, i.e., $(x_1 + x_2 + x_3)R \oplus C_2 = C^2$. Therefore θ is extensible to an element θ' in $\mathrm{Hom}_R(fR, T')$ by the properties in Lemma 1 (note $(x_1 + x_2 + x_3)R \subset x_1R \oplus x_2Rx_3R$). As a consequence $T = T' \oplus fR(\theta') \supset C = C \cap T' \oplus C \cap fR(\theta')$, provided $C_1 = 0$. If $C_1 \neq 0$, $C_1 = C^1$, and hence $T = fR$, since T/C is indecomposable. Accordingly we know that if T/C is indecomposable, $T = fR$ or $q = 0$.

We consider following modules: $Z = (fR)^{(m)} \oplus fJ$ and $U = V \oplus fJ$, where V is a submodule of $(fR)^{(m)}$. Similarly to Lemma 2 we have

Lemma 3. *Assume* iii) *in* Theorem 0. *Let U be as above and X a* *submodule of* U. *Then* $X \subset V$ *or* $U = V \oplus fJ(\theta)$ *and* $X \supset fJ(\theta)$, *where* $\theta \in \mathrm{Hom}_R(fJ, V)$.

More generally we consider $Z^* = (fR)^{(p)} \oplus (fJ)^{(q)}$ and $U^* = V^* \oplus (fJ)^{(q)}$, where V^* is a submodule of $(fR)^{(p)}$.

Corollary. *Assume* iii) *in* Theorem 0. *Let* U^* *be as above and* X^* *a submodule of* U^*. *Then we obtain the following decomposition of* U^*: $U^* = V^* \oplus (fJ)^{(q')} \oplus Y_1 \oplus \cdots \oplus Y_{q-q'}$, *and* $X^* \supset Y_1 \oplus \cdots \oplus Y_{q-q'}$, *where* $Y_i \approx fJ$ *for all* i, $X^* \subset V^* \oplus Y_1 \oplus \cdots \oplus Y_{q-q'}$, $Z^* = (fR)^{(p)} \oplus (fJ)^{(q')} \oplus Y_1 \oplus \cdots \oplus Y_{q-q'}$ *and* $U^* = V^* \oplus (fJ)^{(q')} \oplus Y_1 \oplus \cdots \oplus Y_{q-q'}$.

This corollary means that there exists an automorphism σ of Z^* such that

$$Z^* = (fR)^{(p)} \oplus (fJ)^{(q')} \oplus (fJ)^{(q-q')} \supset \sigma(U^*) = V^* \oplus (fJ)^{(q')} \oplus (fJ)^{(q-q')} \supset \sigma(X^*) \oplus (fJ)^{(q-q')}.$$

We shall denote the above situation by the diagram:

$$\begin{array}{lllll} & & p & q-q' & q' \\ & Z^* & fR & fJ & fJ \\ & & \cup & & \\ (1) & U^* & V^* & fJ & fJ \\ & & \cup & & \\ & X^* & X' & fJ & 0 \end{array}$$

Next we study a structure of a submodule M' of an R-module M under $(*)$. Let P be a projective cover of M, i.e.,

(2) $\qquad M \approx P/Q$ and $M' \approx P'/Q$ for some submodule P' of P.

Then we have a decomposition of P such that $P = P_1 \oplus P_2$, $P' = P_1 \oplus P_2 \cap P'$ and $P_2' = P_2 \cap P' \subset J(P_2)$.
Let

(2') $\qquad P_2 = (e_1R)^{(a_1)} \oplus (e_2R)^{(a_2)} \oplus \cdots \oplus (f_{11}R)^{(b_{11})} \oplus \cdots \oplus (f_{1s_1}R) \oplus (f_{21}R)^{(b_{21})} \oplus \cdots \oplus (f_{2s_2}R)^{(b2s_2)} \oplus \cdots \oplus \Sigma \oplus gR$, where the e_i, the f_1 are given in (0) and $gJ^2 = 0$ ($e_iR \not\approx e_jR$, $f_{ik}R \not\approx f_{js}R$ if $i \neq j$ and $gR \not\approx f_{ik}R$ for all i).

Consider $J(P_2)$ and rearrange it as follows:

$$J(P_2) = (D_{11}{}^{(a_1)} \oplus (f_{11}J)^{(b_{11})}) \oplus (D_{12}{}^{(a_1)} \oplus (f_{12}J)^{(b_{12})}) \oplus \cdots \oplus (D_{1s_1}{}^{(a_1)} \oplus (f_{1s_1}J)^{(b_{1s_1})}) \oplus (D_{21}{}^{(a_2)} \oplus (f_{21}J)^{(b_{21})}) \oplus \cdots \oplus \Sigma \oplus gJ,$$ where $D_{ij} = (f_{ij}R)^{(n_{ij})}$.

Put $E_{ij'} = D_{ij}{}^{(a_i)} \oplus (f_{ij})^{(b^{ij})}$ and $F = \Sigma \oplus gJ$. Then from $(*)$ we know that any simple sub-factor modules of $E_{ij'}$ are not isomorphic to any ones of $E_{st'}$ and F for

$\{i, j\} \neq \{s, t\}$. Hence we obtain

$$
(3) \qquad
\begin{aligned}
&P_2' = \sum \oplus P_{2ij}' \oplus P_0\\
&Q^2 = \sum \oplus Q_{ij'}^{2'} \oplus Q_0^2 \text{ and}\\
&Q_2 = \sum \oplus Q_{2ij}' \oplus Q_{20} \text{ (see [10], Lemma 10 for } Q_1 \text{ and } Q^2),
\end{aligned}
$$

where $E_{ij}' \supset P_{2ij}' \supset Q_{ij}^{2'} \supset Q_{2ij}'$ and $F \supset P_0 \supset Q_0^2 \supset Q_{20}$. We may observe $E_{11}' \supset P_{211}' \supset Q_{11}^{2'} \supset Q_{211}'$ for the fixed (1,1) without loss of generality. From Corollary to Lemma 3 we have

$$
(4) \qquad
\begin{array}{llccccc}
 & & a_1 & d_1 & c_1 & b_1 & a_0\\
E_{11}' & D_{11} & & f_{11}J & f_{11}J & f_{11}J & f_{11}J\\
 & \cup & & & & & \\
E_{211}' & P_{211} & & f_{11}J & f_{11}J & f_{11}J & 0\\
 & \cup & & & & & \\
Q^2{}_{11}' & Q^2{}_{11} & & f_{11}J & f_{11}J & 0 & 0\\
 & \cup & & & & & \\
Q_{211}' & Q_{211} & & f_{11}J & 0 & 0 & 0 \quad ,
\end{array}
$$

where $b_{11} = a_0 + b_1 + c_1 + d_1$.
Next we observe $D_{11} \supset P_{211} \supset Q_{11}^2 \supset Q_{211}$. We put $f_{11} = f$. Then from [4], Lemma 5

$$
\begin{array}{lcccc}
 & e_1 & & e_2 & e_3\\
P_{211} & fR & & fJ & 0
\end{array}
$$

where $e_1 + e_2 + e_3 = a_1 n_{11}$.
Further from Corollary to Lemma 3 we have

$$
(5) \qquad
\begin{array}{llcccc}
 & & e_1 & h_1 & g_1 & e_2'\\
P_{211} & fR & & fJ & fJ & fJ\\
 & \cup & & & & \\
Q^2{}_{11} & H & & fJ & fJ & 0\\
 & \cup & & & & \\
Q_{211} & I & & fJ & 0 & 0 \quad ,
\end{array}
$$

where $e_2 = h_1 + g_1 + e_2'$.
We observe the left side of the above diagram : $(fR)^{(e_1)} \supset H \supset I$. From ($*$) and [4], Lemma 5 we have

$$
\begin{array}{lccc}
 & i_1 & i_2 & e_1'\\
(fR)^{(e_1)} \ (=E) & fR & fR & fR\\
 & & & \\
H & fR & fJ & 0 \quad ,
\end{array}
$$

where $e_1 = i_1 + i_2 + e_1'$.
We apply Lemma 3 to I and $H = H_1 \oplus fJ$, where $H_1 = (fR)^{(i_1)} \oplus (fJ)^{(i_2-1)}$. Then $I \subset H_1$ of $I \supset fJ(h)$ for some $h \in \mathrm{Hom}_R(fJ, H_1) \subset \mathrm{Hom}_R(fJ, E_1)$, where $E_1 =$

$((fR)^{(i_1)}\oplus(fR)^{(i_2-1)})\oplus fR\subset E$. From $(*)$ h is extensible to $\tilde{h}\in\mathrm{Hom}_R(fR, E)$. Hence $E=E_1\oplus fR(\tilde{h})\supset H_1\oplus fJ(h)$ and $I\supset fJ(h)$. Repeating this argument we may assume

$$(6)\qquad \begin{array}{c|c|c|c|c|}
 & i_1 & j_2' & i_2' & e_1' \\
E & fR & fR & fR & fR \\
H & fR & fJ & fJ & 0 \\
 & \cup & & & \\
I & I' & fJ & 0 & 0
\end{array},$$

where $i_2=i_1+i_2'$.
Applying again [4], Lemma 5 to the left corner of the above diagram we have finally

$$(7)\qquad \begin{array}{c|c|c|c|c|c|c|c|c|}
 & k_1 & k_2 & i_1' & j_1 & i_2' & e_1' & h_1 & g_1 \\
P_2' & fR & fR & fR & fR & fR & fR & fJ & fJ \\
Q^2 & fR & fR & fR & fJ & fJ & 0 & fJ & fJ \\
Q_2 & fR & fJ & 0 & fJ & 0 & 0 & fJ & 0
\end{array}$$

$$\begin{array}{|c|c|c|c|c}
e_2' & d_1 & c_1 & b_1 & \\
fJ & fJ & fJ & fJ & P_0 \\
 & & & & \cup \\
0 & fJ & fJ & 0 & Q^2{}_0 \\
 & & & & \cup \\
0 & fJ & 0 & 0 & Q_{20}
\end{array},$$

where $i_1=k_1+k_2+i_1'$ and f rums over all the idempotents in (0). From the above we have

$$(8)\qquad \begin{array}{c|c|c|c|c|c|c|c|c|}
 & k_1 & k_2 & i_1' & j_1 & i_2' & e_1' & h_1 & g_1 \\
P_2'/Q_2 & 0 & fR/fJ & fR & fR/fJ & fR & fR & 0 & fJ \\
Q^2/Q_2 & 0 & fR/J & fR & 0 & fJ & 0 & 0 & fJ
\end{array}$$

$$\begin{array}{|c|c|c|c|c}
e_2' & d_1 & c_1 & b_1 & \\
fJ & 0 & fJ & fJ & P_0/Q_{20} \\
 & & & & \cup \\
0 & 0 & fJ & 0 & Q^2{}_0/Q_{20}
\end{array}$$

Now we come back to (2). $M'=P'/Q$ and $P'Q=(P_1/Q_1\oplus P_2'/Q_2)/(Q/Q_1\oplus Q_2))$ and $Q/(Q_1\oplus Q_2)=Q^2/Q_2(\theta)$ for some $\theta\in\mathrm{Hom}_R(Q^2/Q_2,Q^1/Q_1)$. Since P_0 is semisimple, Q_0^2/Q_{20} is a direct summand of P_0/Q_{20}. Now $P_2'/Q_2=\Sigma_f((fR/fJ)^{(k_{2f})}\oplus\cdots\oplus(fR)^{(i_{2'f})}\oplus\cdots)\oplus P_0/Q_{20}\supset Q^2/Q_2=\Sigma_f((fR/fJ)^{(k_2)}\oplus\cdots\oplus(fJ)^{(i_{2'f})}\oplus\cdots)\oplus Q_0^2/Q_{20}$. We compare direct summands of P_2'/Q_2 and Q^2/Q_2. Then we know that only one summand $(fJ)^{(i_{2'f})}$ of Q^2/Q_2 is a proper submodule of $(fR)^{(i_{2'f})}$, which is a direct summand of P_2'/Q_2 for each f. Consider $\theta|(fJ)^{(i_{2'f})}$. Since $Q\subset J(P)$, we know from a similar argument in the proof of Lemma 2 that $\theta|(fJ)^{(i_{2'f})}$ is induced from an $\theta'\in\mathrm{Hom}_R(fJ^{(i_{2'f})}, P_1)$, and hence $\theta|(fJ)^{(i_{2'f})}$ is extensible to $\Theta\in\mathrm{Hom}_R((fR)^{(i_{2'f})}, P_1/Q_1)$. Therefore

Lemma 4. *Let M' be as above and assume $(*)$. Then*
$M'=P'/Q\approx P_1/Q_1\oplus\Sigma_f\oplus((fR)^{(e'_{f1})}\oplus(fR/fJ)^{((j'_1+i'_2)_f)}\oplus(fJ)^{((b_1+e'_2)_f)})\oplus\widetilde{S}$, *where $\widetilde{S}$ is a direct sum of simple components of* $\mathrm{Soc}(R)$.

Theorem. *Let R be artinian. Then the following are equivalent*:
1) (#) *holds whenever M is local.*
2) (#) *holds for any finitely generated R-modules.*
3) *R is a right almost hereditary ring with $(*)$.*

Proof. 1)→3) This is given by Lemma 1 and Theorem 0.

3)→2). Assume that M is N-projective. Put $M=P/Q$, where P is a projective cover of M. For any submodule M' of M we can suppose $M'=P'/Q$ for some $P'\subset P$. From Lemma 4, M' is a direct sum of the following modules:

1) P_1/Q_1, 2) projective module, 3) simple component of $\mathrm{Soc}(R)$ and 4) fR/fJ, where fR is given in (0).

From the proof of Theorem 1 in [6], p.813 we know that P_1/Q_1 is N-projective in cases 2) and 3) from $(*)$. We assume 4), i.e. $M'=fR/fJ$. First we suppose that N is indecomposable. For the fixed f above (and hence e) we apply Lemma 2. Let $N=T/C$; T is a projective cover of N. We use the same notations as in Lemma 2. If $T=fR$, then M' is trivially almost N-projective (cf. Theorem 0). Hence we assume $q=0$ from Lemma 2. Take any element θ in $\mathrm{Hom}_R(fR, T)$. Then $\theta=\theta_1+\theta_2$ where $\theta_1\in\mathrm{Hom}_R(fR, (eR)^{(p)})$, and $\theta_2\in\mathrm{Hom}_R(fR,\Sigma\oplus g_jR)$. Here we recall the proof of Lemma 4. First we consider the decomposition: $e_iR\supset e_iJ\approx\Sigma_k\oplus(f_{ik}R)^{(n_{ik})}\oplus\Sigma\oplus S_{ij}$ as in (0). Let

$$\mu_k^i:\ e_iJ\rightarrow(f_{ik}R)^{(n_{ik})}$$

be the projection of e_iJ onto the kth component $(f_{ik}R)^{(n_{ik})}$. Next we take the decompositon of P_2 in (2'). Let

$$\xi_q^p:\ J(P_2)\rightarrow e_pJ$$

be the projection of $J(P_2)$ onto the radical e_pJ of the qth component of $(e_pR)^{(a_p)}$

in (2'). we recall the situation where the case 4) occurs. If we carefully observe it, then we know that it comes from P_{2jk} and (6), i.e., $f_{jk}=f$, $e=e_j$ and $e_jR\supset e_jJ\approx (f_{jk}R)^{(n_{jk})}\oplus\cdots$, and $0\neq\mu_k^j\xi_x^j(Q^2)\subset(f_{jk}R)^{(n_{jk})}$ for some x(note $Q^2\subset J(P_2)$). Since $Q^2\subset J(P_2)$, there exists a simple submodule S in Q^2 such that $\mu_k^j\xi_k^j(S)\neq 0$ from Theorem 0 and [10], Corollary to Lemma 2. Further since S is simple, $\xi_x^j(S)\subset \sum_q\oplus(f_{jq}R)^{(n_{jq})}\oplus\sum_t\oplus S_{jt})$ from $(*)$. To the above e and f we consider a homorphism

$$(9)\qquad \Theta:\ P\xrightarrow{\pi}P_2\xrightarrow{\lambda}eR\xrightarrow{\theta}eR\subset T,$$

where π is the projection, λ is the projection onto eR such that $\lambda|J(P_2)=\xi_k^j$, θ is any homorphism and the last eR is the any direct component of $(eR)^{(p)}$ in T. Since P/Q is $N=T/C$-projective, $\Theta(Q)\subset C$. Further since $\xi_k^j(S)$ in non-zero and simple and $\mu_k^j\xi_k^j(S)\neq 0$, $eRe\xi_x^j(S)\supset(f_{jk}J)^{(n_{jk})}$ by Theorem 0. Moreover $\xi_x^j(S)=\lambda\pi(S)$, and hence $(f_{jk}J)^{(pn_{jk})}\subset\sum_{\theta\in(eRe)^{(p)}}\theta\xi_x^j(S)=\sum_\theta\theta\lambda\pi(S)=\sum_\theta\Theta(S)\subset\sum_\theta\Theta(Q)\subset C$. As a consequence $\theta_1(fJ)\subset(f_{jk}J)^{(pn_{jk})}\subset C$, and clearly $\theta_2(fJ)=0$. Accordingly M' is T/C-projective. Finally let $N=\sum\oplus N_i$; the N_i are indecomposable. Then M' is almost N_i-projective as above. If $M'=fR/fJ$ is not N_i-projective, $N_i=fR/A$ from [3], Theorem 1. Hence M' is almost N-projective by [5], Theorem. Thus we have shown the implicartion.

2)$\rightarrow$1). This is trivial.

Here we apply Theorem to special hereditary algebras. Let R be a hereditary algebra over a field K. Assume

$$(10)\qquad eRe=eK \text{ for any primitive idempotent } e.$$

Corollary. *Let R be a basic hereditary algebra as above. Then the following are equivalent*:

1) (#) *holds when M and N are local.*
2) (#) *holds when M is local and N is a direct sum of local modules.*
3) i) $J^3=0$, ii) *J is a direct sum of uniserial modules, and* iii) $R/\mathrm{Soc}(R)$ *is left serial.*

Furthermore the following are equivalent:

4) (#) *holds for any R-modules.*
5) i) 3) *holds,* ii) *J^2 is square-free and* iii) *any simple component ($\approx fJ$) of J^2 is never isoomorphic to any simple ones which are not contained in J^2, except fJ in fR, where f is a primitive idempotent given in* (0).

In this case (H) *in* [6] *holds.*

Proof. 1)$\longleftrightarrow$2) Since $\mathrm{Soc}(R)$ is projective, this is clear from [10], Theorem 2.

1)$\longleftrightarrow$3) Since R is hereditary, iii) in Theorem 0 always holds and i), ii) in the proposition are equivalent to i) in Theorem 0. Further iii) in the proposition is equivalent to ii), iv) in Theorem 0.

4)$\longleftrightarrow$5) This is clear from the assumption (10), Lemma 1 and Theorem 0.

The last statement is clear from [6], Theorem 2.

3. QF-2 rings

In this section we study a left QF-2 ring with (#) as right R-modules (cf. [10], Proposition 3).

Lemma 5. *Let R be left* QF-2. *Further assume that* (#) *holds as right R-modules when M is local and N is a direct sum of local modules. Then* $\mathrm{Soc}(R)$ *is almost projective, and hence R is right almost hereditary,* (cf. Example 4 below).

Proof. Let $eR\supset eJ$ be as (0). Then for any submodule X of eJ we have $X=\sum_i \oplus X_i \oplus X'$ by Theorem 0 and [10], Lemma 13, where $X_i=X\cap(f_iR)^{(n_i)}$ and $X'=X\cap(\sum\oplus S_j)$. Further $X_i\approx(f_iR)^{(m_i)}\oplus(f_iJ)^{(m'_i)}$ by [4], Lemma 5, where $n_i\geqq m_i+m'_i$. Let Y be a submodule of X_i. Then after changing direct decomposition of $(f_iR)^{(m_i)}\oplus(f_iJ)^{(m'_i)}$, we can assume $Y=\sum_i\oplus f_iR\cap Y\oplus\sum_j\oplus f_jJ\cap Y$ again by [4], Lemma 5. Now we prove the lemms. Let gR/gJ be monomorphic to $\mathrm{Soc}(R)$ for a primitive idempotent g. Then gR is uniserial by [10], Lemma 9. First we shall show that gR is injective if $gJ\neq 0$. Let k be any primitive idempotent and take any diagram

$$\begin{array}{ccc} 0\to & K & \to kR \\ & \downarrow{\scriptstyle\rho} & \\ & gR & \end{array}$$

In order to show that gR is injective, we may assume by [8], Lemma $1^{\#}$ that $\rho(K)$ is simple and $K\subset kJ$.

a) $kJ^2\neq 0$.

Then $kR\supset kJ$ have the structure (0). Then from the initial observation and ([4], Lemma 5), $K/\rho^{-1}(0)$ is isomorphic to one of S_j, f_iJ and f_iR/f_iJ for some i and j in (0). However the last case does not occur by assumption. Hence $gR\subsetneqq kR$ or $gR\approx kR$ by [10], Corollary to Lemma 13, provided $\rho\neq 0$. In the former case $gR\approx f_jR$ in (0) for some j. On the other hand $\overline{f}_iR\approx gR/gJ$ is not isomorphic to any simple component of $\mathrm{Soc}(R)$, a contradiction. Therefore $\rho=0$ in this case. Assume $gR\approx kR$. Then kR is uniserial, and hence ρ is a monomorphism by assumption and K is simple. Accordingly ρ is extensible to an element in $\mathrm{Hom}_R(kR, gR)$ by [10], Lemma 13.

b) $kJ^2=0$.

Then $kR \subsetneqq gR$ or $kR \approx gR$ by [10], Corollary to Lemma 13, provided $\rho \neq 0$, and hence kR is uniserial. Then ρ is extensible to an element in $\mathrm{Hom}_R(kR, gR)$ by [10], Lemma 13.
Thus we have shown that gR is injective. Finally we shall show that gR/gJ is injective if $gJ^2 \neq 0$. In the above diagram we replace gR with gR/gJ^2.

a') $kJ^2 \neq 0$.

Then since $K/\rho^{-1}(0) \approx \{S_j, f_kJ, f_kR/f_kJ\}$ as the initial observation and gJ is projective, $f_iR/f_iJ \approx gJ/gJ^2$ for some i by Theorem 0. Hence $gR \approx kR$ by Theorem 0. As a consequence we may assume $gR = kR$. Since gJ is projective, ρ is given by an element θ' in $\mathrm{Hom}_R(gJ, gJ)$ (which induces $\mathrm{Hom}_{RR}(gJ^2, gJ^2)$). Then $\theta' \in \mathrm{Hom}_R(gJ^2, gJ^2)$ is extensible to θ in $\mathrm{Hom}_R(gR, gR)$ by [10], Lemma 13. Now consider $(\theta - \theta')|gJ$. Since $(\theta - \theta')(gJ^2) = 0$, $(\theta - \theta')|gJ = 0$ by Theorem 0. Hence ρ is extensible to $\nu\theta: gR \to gR/gJ^2$, where $\nu: gR \to gR/gJ^2$ is the natural epimorphism.

b') $kJ^2 = 0$.

Then $\rho = 0$ by assumption. Therefore gR/gJ is almost projective by [9], Theorem 1.
Thus J is almost projective from (∗), and hence R is right almost hereditary.

Proposition 2. *Let R be a left* QF-2 *ring. Then the following are equivalent*:

1) *R is a right almost herditary ring such that $J^3 = 0$ and if $eJ^2 \neq 0$ for a primitive idempotent e, then eJ has the decomposition* (0).

2) *R is right almost hereditary and* (#) *holds when M and N are local.*

3) (#) *holds when M is local and N is a direct sum of local modules.*

4) (#) *holds for any R-modules M and N,* (cf. Example 4 below).

Proof. 1)$\longleftrightarrow$2). This is given in [10], Proposition 3.

2)$\to$3). This is clear from Proposition 1.

3)$\to$4). Since R is right almost hereditary by Lemma 5, we obtain 4) by Theorem and [10], Lemma 13.

4)$\to$2). This is clear from Theorem.

We shall add one more property when R is left QF-2. Let $eR \supset eJ \approx (fR)^{(n)} \oplus \cdots$ as in (0), and put $eJ \supset \sum_{i \leq n} \oplus u_i fR \approx fR^{(n)}$, where $u_i fR \approx fR$. We identify $(fR)^{(n)}$ with $\sum \oplus u_i fR$.

Lemma 6. *Assume* 1) *and* 2) *in* [10], Lemma 13 *and* (0). *Let N_1 and N_2 be submodules in $(fR)^{(n)}$, which are isomorphic to fR and hence $\theta: N_1 \to N_2$ be an isomorphism. Then θ is given by an element z in eRe.*

Proof. Let $eR\supset fR\supset S=\mathrm{Soc}(fR)$. Then from 1), 2) and [10], Lemma 6 we obtain

a) every automorphism of S is extensible uniquely to an automorphism of fR (cf. a') in the proof of Lemma 5).

Put $S_i=\mathrm{Soc}(N_i)$ for $i=1, 2$.

b) Assume $N_1=u_1fR$ and $S_1=S_2$. Let $S_1=xR$ and $N_2=yR$; $y=eyf \in(fR)^{(n)}$. Then $y=u_1w_1+\cdots+u_nw_n$; the w_i are units or zero in fRf by 2). Then $x=yr=u_1w_1+\cdots u_nw_nr$ for some $r\in R$. On the other hand $x=u_1r'$ for some $r' \in fR$. Hence $w_2=\cdots=w_n=0$ (cf. the proof of [10], Lemma 13), and $N_1=N_2$.

c) Assume $N_1=u_1fR$ and θ': $N_2\rightarrow N_1$. Then $\theta'|S_2$ is extensible to $z_l\in \mathrm{Hom}_R(eR, eR)$, the left-sided multiplication of z, by 1).

Further $zN_2=N_1$ from b), and z is a unit by [10], Lemma 6. Consider $z_l\theta'^{-1}|S_1 =1_{S_1}(z_l\theta'^{-1}: N_1\rightarrow N_1)$. Then from a) $z_l\theta'^{-1}=1_{N_1}$, and $z_l|N_2=\theta'$.

Since $u_1fR\approx N_1\approx N_2$, we obtain a unit z_i in eRe such that z_{1l}: $N_1\rightarrow u_1fR$ and z_{2l}: $N_2\rightarrow u_1fR$ from c). Hence again by c) $z_{2l}\theta z_{1l}^{-1}=z_l$ for some z, and $\theta=(z_2^{-1}zz_1)_l$.

Concerningly Proposition 2, we have

Proposition 3. *Let R be artinian. Assume that $J^3=0$ and $(eR\supset)eJ$ has the demomposition* (0). *Then the following are equivalent*:

1) i) *Let S_i be a simple submodule of h_iR for $i=1, 2$. If θ: $S_1\rightarrow S_2$ is isomorphic, then θ is extensible to an element in* $\mathrm{Hom}_R(h_1R, h_2R)$ *or in* $\mathrm{Hom}_R(h_2R, h_1R)$, *where h_1, h_2 are primitive idempotents,* ii) *fR/fJ is never monomorphic to* $\mathrm{Soc}(R)$, *where f appears in* (0).

2) R is left QF-2. (cf. Example 3.)

Proof. 2)$\rightarrow$1). This is clear from [10], Lemma 13.

1)$\rightarrow$2). Let S_1 and S_2 be simple left R-modules of $Rh(Jh\neq0)$ for a primitive idempotent h. Suppose $S_i\approx R\bar{k}_i$ for $i=1. 2$, where the k_i are primitive idempotents. Put $S_i=Rx_i$ with $k_ix_ih=x_i\in J$. Then $k_iR\supset x_iR$ and there exists a homomorphism ϕ_i:$hR\rightarrow x_iR$.

a) $k_iJ^2=0$ for $i=1, 2$. Then $x_1R\approx \bar{h}R$ since $x_i\in k_iJ$ and the x_iR are local. Hence there exists z in k_2Rk_1 (or in k_1Rk_2) such that $zx_1=x_2$ (or $zx_2=x_1$) by assumption. As a consequence $S_2=S_1$.

b) $k_1J^2\neq0$ and $k_2J^2=0$. Then $x_2R\approx \bar{h}R$ as above. If $x_1R\approx \bar{h}R$, then $S_1=S_2$ as in a). Suppose that x_1R is not simple. Since $x_1R\subset k_1J$ and x_1R is local, x_1R is projective by (0), and hence $x_1R\approx hR$, which is a contradiction to iii) in Theorem 0 for $x_2R\approx \bar{h}R$.

c) $k_iJ^2\neq0$. Since x_iR is local, x_iR is simple or projective by iii) in Theorem 0. Hence again from iii) in Theorem 0 we obtain two cases $\alpha)x_1R\approx x_2R\bar{h}R$ and $\beta)x_1R\approx x_2R\approx hR$ (and $k_1R\approx k_2R$). Then from Lemma 6 and the arguemt in a) we obtain $S_1=S_2$ in both cases. Hence R is left QF-2.

We note the following fact:

the class of rings with (#) for local modules M and $N \supsetneqq$ the class of rings with (#) for local module M and any direct sum of local modules N $\supsetneqq$ the class of rings with (#) for any finitely generated R-modules. See the following examples.

4. Examples

Let $L \supset K$ be fields.

1.

$$R_1=\begin{pmatrix} K & K & K & K \\ 0 & K & K & 0 \\ 0 & 0 & K & P \\ 0 & 0 & 0 & R \end{pmatrix}, \text{ where } P=L,\ K \text{ or } O \text{ and } e_{13}e_{34}P=0=e_{23}e_{34}P.$$

If $P=L$, then R_1 satisfies the conditions in Theorem 0, but the conditions in [10], Theorem 2. If $P=K$, R_1 satisfies the conditions in [10], Theorem 2, but R_1 is not almost hereditary. If $P=0$, then R_1 satisfies the conditions in Theorem.

2.

$$R_2=\begin{pmatrix} K & L & L \\ 0 & L & L \\ 0 & 0 & L \end{pmatrix}.$$

R_2 satisfies the condition in Theorem, but not left QF-2.

3. $R_3=eK \oplus fK \oplus aK \oplus bK \oplus cK \oplus caK$, where $\{e, f\}$ is the set of mutually orthogonal primitive idempotents with $1=e+f$, $a=eaf$, $b=ebf$, $c=fce$, and $ca=cb$. Then $R(=R_3)$ is a left QF-2 ring with $J^3=0$, but 1) in Proposition 3 does not hold as right R-modules. However R satisfies 1) in Proposition 3 as left R-modules, but not right QF-2.

4. As above $R_4=eK \oplus fK \oplus gK \oplus aK \oplus abK \oplus bK \oplus cK$, where $a=eaf$, $b=fbe$ and $c=ecg$. Then $R(=R_4)$ is left serial and (#) holds for local modules M and N, however R is not right almost hereditary.

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