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ON LAMBEK TORSION THEORIES, III

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In this note, developing our previous work [8] with S. Takashima, we will characterize rings R for which every finitely generated submodule of the injective envelope $E({}_RR)$ is torsionless. Those characterizations would yield recent results of Gómez Pardo and Guil Asensio [6, Theorems 1.5 and 2.2]. Also, we will provide a necessary and sufficient condition for an extension ring Q of a ring R to be a quasi-Frobenius maximal two-sided quotient ring of R.

Throughout this note, R stands for an associative ring with identity, modules are unitary modules, and torsion theories are Lambek torsion theories. Sometimes, we consider right R-modules as left R^{op}-modules, where R^{op} denotes the opposite ring of R, and we use the notation ${}_RX$ (resp. X_R) to stress that the module X considered is a left (resp. right) R-module. We denote by Mod R the category of left R-modules and by $(\ )^*$ both the R-dual functors. For a module X, we denote by $E(X)$ its injective envelope and by $\varepsilon_X: X \to X^{**}$ the usual evaluation map. A module X is called torsionless (resp. reflexive) if ε_X is a monomorphism (resp. an isomorphism). For an $X \in \text{Mod } R$, we denote by $\tau(X)$ its Lambek torsion submodule. Namely, $\tau(X)$ is a submodule of X such that $\text{Hom}_R(\tau(X), E({}_RR)) = 0$ and $X/\tau(X)$ is cogenerated by $E({}_RR)$. A module X is called torsion (resp. torsionfree) if $\tau(X) = X$ (resp. $\tau(X) = 0$). A submodule Y of a module X is called a dense (resp. closed) submodule if X/Y is torsion (resp. torsionfree).

Here we recall some definitions. Let Y be a submodule of a module X. Then X is called a rational extension of Y if $\text{Hom}_R(X/Y, E(X)) = 0$. Let Q be an extension ring of R, i.e., Q is a ring containing R as a subring with common identity. Then Q is called a left (resp. right) quotient ring of R if ${}_RQ$ (resp. Q_R) is a rational extension of ${}_RR$ (resp. R_R). A left quotient ring Q of R is called a maximal left quotient ring of R if $E({}_RQ)/Q$ is torsionfree. As an extension ring of R, a maximal left quotient ring of R is isomorphic to the biendomorphism ring of $E({}_RR)$ (see, e.g., Lambek [10] for details). An extension ring Q of R is called a maximal two-sided quotient ring of R if it is both a maximal left quotient ring and a maximal right quotient ring of R. A ring homomorphism $R \to Q$ is called a left (resp. right) flat epimorphism if the induced functor ${}_RQ \otimes_R$- (resp. -$\otimes_R Q_R$) is a localization functor of Mod R (resp. Mod R^{op}), i.e., Q_R (resp. $_RQ$) is flat and $Q\otimes_R Q \simeq Q$ canonically (see, e.g., Silver [17], Lazard [11] and Popescu and Spircu [15] for details). A module X is called τ-finitely generated if it contains a finitely generated dense submodule. A finitely generated module X is called τ-finitely presented (resp. τ-coherent) if for every epimorphism (resp. homomorphism) $\pi: Y \to X$ with Y finitely generated, Ker π is τ-finitely generated. A module X is called τ-noetherian (resp. τ-artinian) if it satisfies the ascending (resp. descending) chain condition on closed submodules. Finally, a ring R is called left (resp. right) τ-noetherian if $_RR$ (resp. R_R) is τ-noetherian, left (resp. right) τ-artinian if $_RR$ (resp. R_R) is τ-artinian, and left (resp. right) τ-coherent if $_RR$ (resp. R_R) is τ-coherent.

1. τ-absolutely pure and τ-semicompact rings. In this section, we characterize rings R for which every finitely generated submodule of $E(_RR)$ is torsionless.

Lemma 1.1 (Hoshino [7, Theorem A]). *For a ring R the following are equivalent.*

(a) $\tau(X) = \mathrm{Ker}\ \varepsilon_X$ *for every finitely presented* $X \in \mathrm{Mod}\ R$.

(a)$^{\mathrm{op}}$ $\tau(M) = \mathrm{Ker}\ \varepsilon_M$ *for every finitely presented* $M \in \mathrm{Mod}\ R^{\mathrm{op}}$.

Following [8], we call a ring R τ-absolutely pure if it satisfies the equivalent conditions in Lemma 1.1. We call a homomorphism $\pi: X \to Y$ a τ-epimorphism if Cok π is torsion. Then we call a module X τ-semicompact if for every inverse system of τ-epimorphisms $\{\pi_\lambda: X \to Y_\lambda\}_{\lambda\in\Lambda}$ with each Y_λ torsionless, the induced homomorphism $\varprojlim \pi_\lambda: X \to \varprojlim Y_\lambda$ is a τ-epimorphism. Finally, we call a ring R left (resp. right) τ-semicompact if $_RR$ (resp. R_R) is τ-semicompact.

REMARKS. (1) The τ-semicompactness is just the R-linear compactness, in the sense of Gómez Pardo [5], relative to Lambek torsion theory.

(2) Let Mod R/τ denote the quotient category of Mod R over the full subcategory Ker(Hom$_R$(-, $E(_RR)$)). Assume that the image of $_RR$ in Mod R/τ is linearly compact in the sense of Gómez Pardo [5]. Then R is left τ-semicompact.

Theorem 1.2. *For a ring R the following are equivalent.*

(a) *Every finitely generated submodule of $E(_RR)$ is torsionless.*

(b) $\tau(X) = \mathrm{Ker}\ \varepsilon_X$ *for every finitely generated* $X \in \mathrm{Mod}\ R$.

(c) $\mathrm{Ext}^1_R(X,R)$ *is torsion for every finitely generated* $X \in \mathrm{Mod}\ R$.

(d) *R is τ-absolutely pure and right τ-semicompact.*

Proof. (a)$\Leftrightarrow$(b). See Hoshino [7, Lemma 5].

(b)$\Rightarrow$(c). This is due essentially to Ohtake [14, Lemma 2.3]. Let $0 \to Y \to F \to X \to 0$ be an exact sequence in Mod R with F finitely generated free and let $\pi: Y^* \to \mathrm{Ext}^1_R(X,R)$ denote the canonical epimorphism. Let $h \in Y^*$ and form a push-out diagram:

$$\begin{array}{ccccccccc} 0 & \to & Y & \to & F & \to & X & \to & 0 \\ & & {}^{h}\downarrow & & \downarrow & & \| & & \\ 0 & \to & R & \xrightarrow{\phi} & Z & \to & X & \to & 0. \end{array}$$

Since Z is finitely generated, Ker ε_Z is torsion. Thus $\phi^{**} \circ \varepsilon_R = \varepsilon_Z \circ \phi$ is monic, so is ϕ^{**}. Hence $(\mathrm{Cok}\phi^*)^* \simeq \mathrm{Ker}\phi^{**} = 0$. Since $\pi(h)R_R$ is an epimorphic image of $\mathrm{Cok}\phi^*$, $(\pi(h)R_R)^* = 0$ and thus $\mathrm{Ext}^1_R(X,R)$ is torsion.

(c) ⇒ (b). Let $X \in \mathrm{Mod}\ R$ be finitely generated. Let Y be a submodule of Ker ε_X and let $j: Y \to X$ denote the inclusion. Then $j^* = 0$ and Y^* embeds in $\mathrm{Ext}^1_R(X/Y, R)$. Thus Y^* is torsion, so that $Y^* = 0$. Hence Ker ε_X is torsion and $\tau(X) = \mathrm{Ker}\ \varepsilon_X$.

(c) ⇔ (d). This is easily deduced from [8, Lemma 2.7].

REMARK. The equivalence (a) ⇔ (d) of Theorem 1.2 would yield a result of Gómez Pardo and Guil Asensio [6, Theorem 2.2].

Corollary 1.3 (cf. Sumioka [20, Theorem 1]). *Let R be left perfect. Then the following are equivalent.*

(a) *Every finitely generated submodule of $E({}_RR)$ is torsionless.*
(b) *R contains a faithful and injective left ideal.*

Proof. (a) ⇒ (b). By Storrer [18] R contains an idempotent e with ReR a minimal dense right ideal. It is obvious that ${}_RRe$ is faithful. Since by Theorem 1.2 $\mathrm{Ext}^1_R(X,Re) \simeq \mathrm{Ext}^1_R(X,R) \otimes_R Re = 0$ for every finitely generated $X \in \mathrm{Mod}\ R$, ${}_RRe$ is injective.

(b) ⇒ (a). Obvious.

Corollary 1.4. *Let R be τ-absolutely pure, left and right τ-semicompact. Then both* Ker ε_X *and* Cok ε_X *are torsion for every finitely generated* $X \in \mathrm{Mod}\ R$.

Proof. Let $X \in \mathrm{Mod}\ R$ be finitely generated. By Theorem 1.2 Ker ε_X is torsion. We know from the argument of Jans [9, Theorem 1.1] that Cok $\varepsilon_X \simeq \mathrm{Ext}^1_R(M,R)$ with $M \in \mathrm{Mod}\ R^{\mathrm{op}}$ finitely generated. Thus again by Theorem 1.2 Cok ε_X is torsion.

REMARK. Assume that R is a maximal left quotient ring of itself, i. e., $E({}_RR)/R$ is torsionfree. Then $\mathrm{Ext}^1_R(X,Y) = 0$ for all torsion $X \in \mathrm{Mod}\ R$ and reflexive

$Y \in \text{Mod } R$. Thus Corollary 1.4 would yield a result of Gómez Pardo and Guil Asensio [6, Theorem 1.5].

Corollary 1.5. *Let R be τ-absolutely pure and left τ-semicompact. Then every finitely generated $X \in \text{Mod } R$ is τ-semicompact.*

Proof. Let $X \in \text{Mod } R$ be finitely generated. Since every factor module of a τ-semicompact module is τ-semicompact, we may assume that X is free. Then the argument of [8, Lemma 2.7] applies.

2. Flat epimorphic extension rings. Throughout this section, Q stands for an extension ring of R.

The following lemmas seem to be known (cf. Silver [7], Lazard [11], Popescu and Spircu [15], Morita [13] and so on). However, for the benefit of the reader, we include proofs.

Lemma 2.1. *The following are equivalent.*

(1) *The inclusion $R \to Q$ is a left flat epimorphism.*
(2) *$Q \otimes_R X = 0$ for every submodule X of ${}_RQ/R$.*

Proof. (1)$\Rightarrow$(2). Obvious.

(2)$\Rightarrow$(1). Let $\pi: Q \otimes_R Q \to Q$ denote the multiplication map. Then ${}_Q\text{Ker } \pi \simeq {}_Q Q \otimes_R (Q/R) = 0$. Next, let $F_1 \to F_0 \to X \to 0$ be an exact sequence in Mod R with each F_i finitely generated free and put $Y = \text{Im}(F_1 \to F_0)$. We have a sequence of embeddings $\text{Tor}_1^R(Q,X) \hookrightarrow \text{Tor}_1^R(Q/R,X) \hookrightarrow (Q/R) \otimes_R Y$. Let us form a pull-back diagram:

$$\begin{array}{ccc} (Q/R)\otimes_R F_1 & \twoheadrightarrow & (Q/R)\otimes_R Y \\ \cup & & \cup \\ Z & \twoheadrightarrow & \text{Tor}_1^R(Q,X). \end{array}$$

Since $(Q/R) \otimes_R F_1$ is isomorphic to a finite direct sum of copies of ${}_RQ/R$, it follows by induction that $Q \otimes_R Z = 0$. Thus, since $Q \otimes_R Q \simeq Q$ canonically, $\text{Tor}_1^R(Q,X) \simeq Q \otimes_R \text{Tor}_1^R(Q,X) = 0$.

Lemma 2.2. *The following are equivalent.*

(1) *Q is a left quotient ring of R.*
(2) (a) *${}_Q Q \otimes_R (Q/R)$ is torsion.*
(b) *${}_Q\text{Tor}_1^R(Q,X)$ is torsion for every $X \in \text{Mod } R$.*

Proof. Note that $\text{Hom}_Q(Q \otimes_R (Q/R), E({}_QQ)) \simeq \text{Hom}_R(Q/R, \text{Hom}_Q({}_QQ_R, E({}_QQ)))$, and that $\mathrm{Hom}_Q(\mathrm{Tor}_1^R(Q,X), E({}_QQ)) \simeq \mathrm{Ext}_R^1(X, \mathrm{Hom}_Q({}_QQ_R, E({}_QQ)))$ for every $X \in \mathrm{Mod}\ R$.

(1) ⇒ (2). Obvious.

(2) ⇒ (1). It follows that ${}_R\mathrm{Hom}_Q({}_QQ_R, E({}_QQ))$ is injective. Thus $E({}_RQ)$ embeds in $\mathrm{Hom}_Q({}_QQ_R, E({}_QQ))$. It then follows that $\mathrm{Hom}_R(Q/R, E({}_RQ)) = 0$.

The next lemma generalizes results of Cateforis [2, Proposition 2.2] and Masaike [12, Proposition 3] (cf. also Morita [13, Theorem 7.2]).

Lemma 2.3. *The following are equivalent.*

(1) *The inclusion $R \to Q$ is a left flat epimorphism.*

(2) (a) *Q is a left quotient ring of R.*
(b) *${}_QQ \otimes_R X$ is torsionfree for every submodule X of ${}_RQ$.*

Proof. (1) ⇒ (2). By Lemma 2.2 (a) follows. It is obvious that (b) holds.

(2) ⇒ (1). Let Y be a submodule of ${}_RQ/R$. Since ${}_RY$ is torsion, so is ${}_QQ \otimes_R Y$. Next, let us form a pull-back diagram:

$$\begin{array}{ccccccccc} 0 & \to & R & \xrightarrow{j} & Q & \to & Q/R & \to & 0 \\ & & \| & & \cup & & \cup & & \\ 0 & \to & R & \xrightarrow{\phi} & X & \to & Y & \to & 0, \end{array}$$

where $j: R \to Q$ is an inclusion. Since ${}_QQ \otimes_R j$ is a split monomorphism, so is ${}_QQ \otimes_R \phi$. Thus ${}_QQ \otimes_R Y$ is torsionfree, so that $Q \otimes_R Y = 0$. By Lemma 2.1 the assertion follows.

Lemma 2.4. *The following are equivalent.*

(1) (a) *Q is a maximal left quotient ring of R.*
(b) *$E({}_QQ)$ is an injective cogenerator in* Mod *Q.*

(2) (a) *${}_RQ/R$ is torsion.*
(b) *$Q \otimes_R X = 0$ for every torsion $X \in$* Mod *R.*

Proof. (1) ⇒ (2). Obvious.

(2) ⇒ (1). By Lemma 2.1 the inclusion $R \to Q$ is a left flat epimorphism. Thus by Lemma 2.2 Q is a left quotient ring of R. Next, let $X \in \mathrm{Mod}\ Q$ be torsion. Then ${}_RX$ is torsion and thus ${}_QX \simeq {}_QQ \otimes_R X = 0$. Hence $E({}_QQ)$ is an injective cogenerator in Mod Q, so that Q is a maximal left quotient ring of R.

3. Flatness of the injective envelope. Throughout this section, Q stands for a left quotient ring of R.

Lemma 3.1. *Let R be left τ-noetherian and let $X \in \operatorname{Mod} R$ be flat. Then ${}_Q Q \otimes_R X$ is torsionfree.*

Proof. Let I be a dense left ideal of R. By Faith [4, Proposition 3.1] I contains a finitely generated subideal J with I/J torsion. Then R/J is finitely presented torsion, so that $\operatorname{Hom}_R(R/J, Q \otimes_R X) \simeq \operatorname{Hom}_R(R/J, Q) \otimes_R X = 0$. Thus $\operatorname{Hom}_R(R/I, Q \otimes_R X) = 0$. Hence ${}_R Q \otimes_R X$ is torsionfree, so is ${}_Q Q \otimes_R X$.

Corollary 3.2. *Let R be left τ-noetherian. Let $n \geq 1$ and let $X \in \operatorname{Mod} R$ with* weak $\dim_R X \leq n$. *Then* $\operatorname{Tor}_n^R(Q, X) = 0$.

Proof. Let $\cdots \to F_1 \to F_0 \to X \to 0$ be an exact sequence in Mod R with each F_i free and put $Y = \operatorname{Cok}(F_{n+1} \to F_n)$. Then Y is flat and thus by Lemma 3.1 ${}_Q Q \otimes_R Y$ is torsionfree. On the other hand, by Lemma 2.2 ${}_Q \operatorname{Tor}_n^R(Q, X)$ is torsion. It follows that $\operatorname{Tor}_n^R(Q, X) = 0$.

Lemma 3.3. *Let $X \in \operatorname{Mod} Q$ with ${}_Q Q \otimes_R X$ torsionfree. Then ${}_Q Q \otimes_R X \simeq {}_Q X$ canonically.*

Proof. Let $\pi : Q \otimes_R X \to X$ denote the canonical epimorphism. Then ${}_R \operatorname{Ker} \pi \simeq {}_R(Q/R) \otimes_R X$ is torsion, so is ${}_Q \operatorname{Ker} \pi$. It follows that $\operatorname{Ker} \pi = 0$.

Proposition 3.4. *Let R be left τ-noetherian. Then every $X \in \operatorname{Mod} Q$ with ${}_R X$ flat is flat. In particular, $E({}_Q Q)$ is flat whenever $E({}_R R)$ is.*

Proof. Let $X \in \operatorname{Mod} Q$ with ${}_R X$ flat. Then by Lemmas 3.1 and 3.3 ${}_Q Q \otimes_R X \simeq {}_Q X$ canonically. Since both $-\otimes_Q Q_R$ and $-\otimes_R X$ are exact, so is $-\otimes_Q X$.

Proposition 3.5. *For a ring R the following are equivalent.*

(1) *Arbitrary direct products of copies of $E({}_R R)$ are flat.*

(2) *R is τ-absolutely pure and right τ-coherent.*

Proof. (1) $\Rightarrow$ (2). By Hoshino and Takashima [8, Lemma 1.4] R is τ-absolutely pure. Next, let $0 \to M \to F \to R$ be an exact sequence in Mod R^{op} with F finitely generated free. By Colby and Rutter [3, Theorem 1.3] M contains a finitely generated submodule N with $(M/N) \otimes_R E({}_R R) = 0$. It suffices to show that M/N is torision. For an $L \in \operatorname{Mod} R^{\mathrm{op}}$, there exists a natural homomorphism

$$\theta_L : L \otimes_R E({}_R R) \to \operatorname{Hom}_R(L^*, E({}_R R))$$

such that $\theta_L(x \otimes y)(\alpha) = \alpha(x) y$ for $x \in L$, $y \in E({}_R R)$ and $\alpha \in L^*$. Now, let L be a cyclic submodule of M/N and let $\pi : R \to L$ be epic in Mod R^{op}. Since $\theta_L \circ (\pi \otimes_R E({}_R R))$

$=\mathrm{Hom}_R(\pi^*,E({}_RR))\circ\theta_R$ is epic, so is θ_L. Note that $L\otimes_R E({}_RR)=0$. Thus $\mathrm{Hom}_R(L^*,E({}_RR))=0$ and hence $L^*=0$. It follows that M/N is torsion.

(2)$\Rightarrow$(1). See Hoshino and Takashima [8, Proposition 1.6].

4. Quasi-Frobenius qoutient rings. In this section, we provide a necessary and sufficient condition for an extension ring Q of R to be a quasi-Frobenius maximal two-sided qoutient ring of R.

Lemma 4.1. *Let R be left τ-noetherian and let Q be a maximal left quotient ring of R. Assume that* weak dim ${}_RQ\leqq 1$. *Then the inclusion $R\rightarrow Q$ is a ring epimorphism.*

Proof. We claim that $(Q/R)\otimes_R Q=0$. Let I be a dense left ideal of R. By Faith [4, Proposition 3.1] I contains a finitely generated subideal J with I/J torsion. Note that J is also a dense left ideal of R. It follows that $(Q/R)_R$ is an epimorphic image of the direct sum $\oplus\mathrm{Hom}_R(R/J,Q/R)_R$, where J runs over all finitely generated dense left ideals of R. Let J be a finitely generated dense left ideal of R. Since $\mathrm{Hom}_R(R/J,Q/R)_R\simeq\mathrm{Ext}^1_R(R/J,R)$, we have only to show that $\mathrm{Ext}^1_R(R/J,R)\otimes_R Q=0$. For an $X\in\mathrm{Mod}\ R$, there exists a natural homomorphism

$$\delta_X: X^*\otimes_R Q\rightarrow\mathrm{Hom}_R(X,Q)$$

such that $\delta_X(\alpha\otimes q)(x)=\alpha(x)q$ for $\alpha\in X^*$, $q\in Q$ and $x\in X$. As we remarked in [8], there exists an epimorphism $\pi: X\rightarrow J$ with X finitely presented and $\mathrm{Ker}\,\pi$ torsion. Note that by Auslander [1, Proposition 7.1] δ_X is monic. Since π^* is an isomorphism, $\mathrm{Hom}_R(\pi,Q)\circ\delta_J=\delta_X\circ(\pi^*\otimes_R Q)$ is monic, so is δ_J. Next, let $j: J\rightarrow R$ denote the inclusion. Since $\mathrm{Hom}_R(j,Q)$ is an isomorphism, so is $\mathrm{Hom}_R(j,Q)\circ\delta_R$ $=\delta_J\circ(j^*\otimes_R Q)$. Thus δ_J is epic. Hence δ_J is an isomorphism, so is $j^*\otimes_R Q$. It follows that $\mathrm{Ext}^1_R(R/J,R)\otimes_R Q\simeq\mathrm{Cok}(j^*\otimes_R Q)=0$.

In case $Q=R$, the next theorem is due to Faith [4, Corollary 5.4].

Theorem 4.2. *For an extension ring Q of R the following are equivalent.*

(1) *Q is a quasi-Frobenius maximal two-sided quotient ring of R.*

(2) (a) *R is left τ-noetherian.*

  (b) *${}_RQ/R$ is torsion.*

  (c) *Q_R is injective.*

Proof. (1)$\Rightarrow$(2). Obvious.

(2)$\Rightarrow$(1). For an $X\in\mathrm{Mod}\ R$, there exists a natural homomorphism

$$\theta_X : Q\otimes_R X \rightarrow \mathrm{Hom}_R(X^*, Q)$$

such that $\theta_X(q\otimes x)(\alpha)=q\alpha(x)$ for $q\in Q$, $x\in X$ and $\alpha\in X^*$. Since Q_R is injective, θ_X is an isomorphism for every finitely presented $X\in \mathrm{Mod}\ R$. Let I be a dense left ideal of R. By Faith [4, Proposition 3.1] I contains a finitely generated subideal J with I/J torsion. Then R/J is finitely presented torsion, so that $Q\otimes_R(R/J)\simeq \mathrm{Hom}_R((R/J)^*, Q)=0$. Thus $Q\otimes_R(R/I)=0$. It follows that $Q\otimes_R X=0$ for every torsion $X\in \mathrm{Mod}\ R$. Hence by Lemma 2.4, Q is a maximal left quotient ring of R, and $E({}_QQ)$ is an injective cogenerator in Mod Q. Thus by Lemma 2.1 Q_R is flat as well as injective, so that $E(R_R)$ is flat. Hence by Hoshino and Takashima [8, Proposition 1.7] and Masaike [12, Proposition 2] Q is a right quotient ring of R. It follows that Q is a right selfinjective maximal right quotient ring of R. On the other hand, since R is left τ-noetherian, so is Q. Thus Q is left noetherian. Hence by Faith [4, Theorem 2.1] Q is quasi-Frobenius.

Corollary 4.3. *Let R be left and right noetherian and let Q be a maximal left quotient ring of R. Then the following are equivalent.*

(1) *Q is a quasi-Frobenius maximal two-sided quotient ring of R.*

(2) *${}_RQ$ is flat and* inj dim ${}_RQ\leqq 1$.

Proof. (1)$\Rightarrow$(2). By Lemma 2.3 ${}_RQ$ is flat. Also, ${}_RQ$ is injective by Lambek [10, §5].

(2)$\Rightarrow$(1). By Lemmas 4.1 and 2.2 Q is a right quotient ring of R. Next, we claim that ${}_RQ$ is injective. Since

$$\begin{aligned}\mathrm{Tor}_2^R(E(R_R),X)&\simeq \mathrm{Hom}_R(\mathrm{Ext}_R^2(X,R),E(R_R))\\&\simeq \mathrm{Hom}_R(\mathrm{Ext}_R^2(X,R),\mathrm{Hom}_Q({}_RQ_Q,E(Q_Q)))\\&\simeq \mathrm{Hom}_Q(\mathrm{Ext}_R^2(X,R)\otimes_R Q,E(Q_Q))\\&\simeq \mathrm{Hom}_Q(\mathrm{Ext}_R^2(X,Q),E(Q_Q))\\&=0\end{aligned}$$

for every finitely generated $X\in \mathrm{Mod}\ R$, we have weak dim $E(R_R)\leqq 1$. Thus by Hoshino [7, Propositions F and C] every finitely generated submodule of $E({}_RR)$ is torsionless. Let $X\in \mathrm{Mod}\ R$ be finitely generated. Since by Theorem 1.2 $X/\tau(X)$ is torsionless, there exists an exact sequence $0\rightarrow X/\tau(X)\rightarrow F\rightarrow Y\rightarrow 0$ in Mod R with F free. Thus $\mathrm{Ext}_R^1(X,Q)\simeq \mathrm{Ext}_R^1(X/\tau(X),Q)\simeq \mathrm{Ext}_R^2(Y,Q)=0$. Hence ${}_RQ$ is injective and by Theorem 4.2 the assertion follows.

REMARK. Let R be left noetherian and let $X\in \mathrm{Mod}\ R$ be flat. Then $\mathrm{Ext}_R^i(Y,R)\otimes_R X\simeq \mathrm{Ext}_R^i(Y,X)$ for all $i\geqq 0$ and finitely generated $Y\in \mathrm{Mod}\ R$, so that inj dim ${}_RX\leqq$ inj dim ${}_RR$. Thus, together with Lemma 2.3, Corollary 4.3 would yield a result of Sato [16, Theorem].

5. Appendix. Throughout this section, Q stands for an extension ring of R. We make some remarks on submodules of Q_R.

The argument of Sumioka [19, Proposition 6] suggests the following lemma.

Lemma 5.1. *The following are equivalent.*

(1) *Q is a left quotient ring of R.*

(2) (a) *${}_RQ/R$ is torsion.*

(b) *$R\cap I\neq 0$ for every nonzero two-sided ideal I of Q.*

Proof. (1)$\Rightarrow$(2). Obvious.

(2)$\Rightarrow$(1). Put ${}_QE=\mathrm{Hom}_R({}_RQ_Q,E({}_RR))$. Then ${}_RE\simeq E({}_RR)$ canonically, so that the composite of ring homomorphisms $\mathrm{End}\,(E({}_RR))\to\mathrm{End}\,({}_QE)\to\mathrm{End}\,({}_RE)$ is an isomorphism. Thus $\mathrm{End}\,({}_QE)=\mathrm{End}\,({}_RE)$ and hence $\mathrm{Biend}\,({}_QE)=\mathrm{Biend}\,({}_RE)$. Let $\phi: Q\to\mathrm{Biend}\,({}_QE)$ denote the canonical ring homomorphism. Since ${}_RE$ is faithful, $R\cap\mathrm{Ker}\,\phi=0$ and thus $\mathrm{Ker}\,\phi=0$. Since $\mathrm{Biend}\,({}_RE)$ is a maximal left quotient ring of R, the assertion follows.

Lemma 5.2 (cf. Masaike [12, Proposition 2]). *Assume that Q is a right quotient ring of R. Let M be a submodule of Q_R containing R and put $I=\{a\in R|aM\subset R\}$. Then M is torsionless if and only if $({}_RR/I)^*=0$.*

Proof. Let $j:R_R\to M_R$ denote the inclusion. Then j is an essential monomorphism, so that $\mathrm{Ker}\,\varepsilon_M=0$ if and only if $\mathrm{Ker}\,j^{**}=0$. It suffices to show that $\mathrm{Ker}\,j^{**}\simeq({}_RR/I)^*$. Identify $(R_R)^*$ with ${}_RR$. We claim that $\mathrm{Im}\,j^*=I$. It is obvious that $I\subset\mathrm{Im}\,j^*$. Conversely, let $h\in M^*$. Since $E(Q_Q)_R\simeq E(R_R)$ is injective, h extends to some $\phi: Q_R\to E(Q_Q)_R$. It is easy to see that ϕ is Q-linear. Thus $h(1)x=\phi(1)x=\phi(x)=h(x)\in R$ for all $x\in M$ and hence $j^*(h)=h(1)\in I$.

For an $M\in\mathrm{Mod}\,R^{\mathrm{op}}$, there exists a natural homomorphism

$$\eta_M: M\to\mathrm{Hom}_Q(\mathrm{Hom}_R(M,Q),Q)$$

such that $\eta_M(x)(\alpha)=\alpha(x)$ for $x\in M$ and $\alpha\in\mathrm{Hom}_R(M,Q)$, and for an $X\in\mathrm{Mod}\,R$ there exists a natural homomorphism

$$\zeta_X: X^*\to\mathrm{Hom}_Q(Q\otimes_RX,Q)$$

such that $\zeta_X(\alpha)(q\otimes x)=q\alpha(x)$ for $\alpha\in X^*$, $q\in Q$ and $x\in X$. Also, for L, $M\in\mathrm{Mod}\,R^{\mathrm{op}}$ there exists a natural homomorphism

$$\delta_{L,M}: L\otimes_RM^*\to\mathrm{Hom}_R(M,L)$$

such that $\delta_{L,M}(x\otimes\alpha)(y)=x\alpha(y)$ for $x\in L$, $\alpha\in M^*$ and $y\in M$.

For each $M\in\operatorname{Mod} R^{\mathrm{op}}$, we have a commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{\eta_M} & \operatorname{Hom}_Q(\operatorname{Hom}_R(M,Q),Q) \\ {\scriptstyle \varepsilon_M}\downarrow & & \downarrow{\scriptstyle \operatorname{Hom}_Q(\delta_{Q,M},Q)} \\ M^{**} & \xrightarrow{\zeta_{M^*}} & \operatorname{Hom}_Q(Q\otimes_R M^*,Q) \end{array}$$

which yields the following lemma.

Lemma 5.3. *Let $M\in\operatorname{Mod} R^{\mathrm{op}}$. Assume that both η_M and $\operatorname{Hom}_Q(\delta_{Q,M},Q)$ are monic. Then M is torsionless.*

Also, for each $M\in\operatorname{Mod} R^{\mathrm{op}}$, we have a commutative diagram with exact rows:

$$\begin{array}{ccccccc} & & R\otimes_R M^* & \to & Q\otimes_R M^* & \to & (Q/R)\otimes_R M^* & \to 0 \\ & & {\scriptstyle \delta_{R,M}}\downarrow\wr & & \downarrow{\scriptstyle \delta_{Q,M}} & & \downarrow{\scriptstyle \delta_{Q/R,M}} & \\ 0 & \to & \operatorname{Hom}_R(M,R) & \to & \operatorname{Hom}_R(M,Q) & \to & \operatorname{Hom}_R(M,Q/R). & \end{array}$$

Note that, in case M is finitely generated, $\operatorname{Hom}_R(M,Q/R)$ embeds in a direct sum of copies of ${}_RQ/R$. Thus Snake lemma yields the following two lemmas.

Lemma 5.4. *Assume that ${}_RQ/R$ is torsion. Then both ${}_R\operatorname{Ker}\delta_{Q,M}$ and ${}_R\operatorname{Cok}\delta_{Q,M}$ are torsion for every finitely generated $M\in\operatorname{Mod} R^{\mathrm{op}}$.*

Lemma 5.5. *Assume that the inclusion $R\to Q$ is a left flat epimorphism. Then $\delta_{Q,M}\simeq {}_QQ\otimes_R\delta_{Q,M}$ is an isomorphism for every finitely generated $M\in\operatorname{Mod} R^{\mathrm{op}}$.*

We are now in a position to formulate results of Masaike [12] as follows.

Proposition 5.6 (Masaike [12]). *For an extension ring Q of R the following hold.*

(1) *If Q is a left quotient ring of R, every finitely generated submodule of Q_R is torsionless.*

(2) *If the inclusion $R\to Q$ is a left flat epimorphism, every finitely generated submodule of Q_R embeds in a free module.*

(3) *Assume that Q is a right quotient ring of R. Then Q is a left quotient ring of R if and only if every finitely generated submodule of Q_R is torsionless.*

Proof. (1) By Lemmas 5.3 and 5.4.

(2) By Lemma 5.5.

(3) By Lemmas 5.1 and 5.2.

References

[1] M. Auslander: *Coherent functors*, Proc. Conf. Cat. Algebra, 189–231, Springer, Berlin 1966.

[2] V.C. Cateforis: *Two-sided semisimple maximal quotient rings*, Trans. Amer. Math. Soc. **149** (1970), 339–349.

[3] R.R. Colby and E.A. Rutter, Jr.: *Π-flat and Π-projective modules*, Arch. Math. **22** (1971), 246–251.

[4] C. Faith: *Rings with ascending condition on annihilators*, Nagoya Math. J. **27** (1966), 179–191.

[5] J.L. Gómez Pardo: *Counterinjective modules and duality*, J. Pure and Appl. Algebra **61** (1989), 165–179.

[6] J.L. Gómez Pardo and P.A. Guil Asensio: *Morita dualities associated with R-dual functors*, J. Pure and Appl. Algebra **93** (1994), 179–194.

[7] M. Hoshino: *On Lambek torsion theories*, Osaka J. Math. **29** (1992), 447–453.

[8] M. Hoshino and S. Takashima: *On Lambek torsion theories II*, Osaka J. Math. **31** (1994), 729–746.

[9] J.P. Jans: *Duality in noetherian rings*, Proc. Amer. Math. Soc. **12** (1961), 829–835.

[10] J. Lambek: *On Utumi's ring of quotients*, Canad. J. Math. **15** (1963), 363–370.

[11] D. Lazard: *Epimorphismes plats d'anneaux*, C. R. Acad. Sci. Paris **266** (1968), 314–316.

[12] K. Masaike: *On quotient rings and torsionless modules*, Sci. Rep. Tokyo Kyoiku Daigaku **A11** (1971), 26–30.

[13] K. Morita: *Localization in categories of modules I*, Math. Z. **114** (1970), 121–144.

[14] K. Ohtake: *A generalization of Morita duality by Localizations*, preprint

[15] N. Popescu and T. Spircu: *Sur les epimorphismes plats d'anneaux*, C. R. Acad. Sci. Paris **268** (1969), 376–379.

[16] H. Sato: *On localizations of a* 1-*Gorenstein ring*, Sci. Rep. Tokyo Kyoiku Daigaku **A13** (1977), 188–193.

[17] L. Silver: *Noncommutative localizations and applications*, J. Algebra **7** (1976), 44–76.

[18] H.H. Storrer: *Rings of quotients of perfect rings*, Math. Z. **122** (1971), 151–165.

[19] T. Sumioka: *On finite dimensional QF-3′ rings*, Proceedings of 10th Symposium on Ring Theory, 79–105, Okayama Univ., Okayama, Japan 1978.

[20] T. Sumioka: *On QF-3 and 1-Gorenstein rings*, Osaka J. Math. **16** (1979), 395–403.

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