Albu, T. and Wisbauer, R.
Osaka J. Math.
28 (1991), 295-304

GENERATORS IN GROTHENDIECK CATEGORIES WITH RIGHT PERFECT ENDOMORPHISM RINGS

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(Received June 7, 1990)

It is well-known that Grothendieck categories need not have projective objects (e.g. [12], 18.12). However, projective objects can be obtained from certain finiteness conditions. By a theorem of Năstăsescu, a Grothendieck category with an artinian generator has a finitely generated projective generator (see [10], [1]). His proof refers to the Gabriel-Popescu theorem as well as to the theory of Δ-injective modules. In this paper we give direct proofs of more general statements.

After preliminary results in section 1 we prove in section 2: Assume U is a generator in a Grothendieck category $\mathfrak{C}$ and the endomorphism ring of U is right perfect. Then there exists a projective generator in $\mathfrak{C}$. If the generator U is a coproduct of small objects U_λ in $\mathfrak{C}$ and $\hat{S}$, the ring of all endomorphisms of U with $(U_\lambda)f=0$ almost everywhere, is right perfect, then there exists a projective generator in $\mathfrak{C}$ which is a coproduct of small objects, and $\mathfrak{C}$ is equivalent to a full module category over a ring with enough idempotents.

In section 3 we obtain a new characterization of QF categories (in the sense of Harada [5]) by observing: In a locally finitely generated Grothendieck category $\mathfrak{C}$ an object U is a noetherian injective generator if and only if U is an artinian projective cogenerator.

A result of Auslander on categories of finite representation type ([3], Theorem 4.4) is generalized in section 4: A Grothendieck category $\mathfrak{C}$ of bounded representation type has a projective generator and is equivalent to a full module category over a ring with enough idempotents of bounded representation type.

Finally, in section 5, we interpret our results in a special case: For a left module M over an associative ring R denote by $\sigma[M]$ the full Grothendieck subcategory of R-Mod subgenerated by M. Assume $M=\oplus_\Lambda M_\lambda$, with finitely generated M_λ's, is a generator in $\sigma[M]$ and the ring $\hat{S}$ of all $f\in\mathrm{End}_R(M)$ with $(M_\lambda)f=0$ almost everywhere, is right perfect. Then there exists a projective left $\hat{S}$-module P which is a direct sum of local modules, such that $M\otimes_{\hat{S}}P$ is a projective generator in $\sigma[M]$, and $\sigma[M]$ is equivalent to $\widehat{\mathrm{End}}_{\hat{S}}(P)$-Mod. The observation on QF categories in section 3 yields new descriptions of quasi-projective noetherian QF modules in the sense of Hauger-Zimmermann [7], which are exactly quasi-projective noetherian *strongly quasi-injective* modules in the sense of Menini-Orsatti [9].

1. Preliminary results

$\mathfrak{C}$ will always denote a Grothendieck category and we write morphisms in $\mathfrak{C}$ on the right side of the objects. Recall that an object V of $\mathfrak{C}$ is said to be *small* in $\mathfrak{C}$ if the functor $\mathrm{Hom}_{\mathfrak{C}}(V, -)$ commutes with coproducts.

For $U = \oplus_{\Lambda} U_{\lambda}$ with small U_{λ}'s in $\mathfrak{C}$ and $N \in \mathfrak{D}$ define

$$\widehat{\mathrm{Hom}}_{\mathfrak{C}}(U,N) = \{f \in \mathrm{Hom}_{\mathfrak{C}}(U,N) \mid (U_{\lambda})f = 0 \text{ for almost all } \lambda \in \Lambda\}.$$

For $N = U$ we write $\widehat{\mathrm{Hom}}_{\mathfrak{C}}(U, U) = \widehat{\mathrm{End}}_{\mathfrak{C}}(U) =: \hat{S}$. This is a subring and right ideal in $S = \mathrm{End}_{\mathfrak{C}}(U)$.

$\hat{S}$ is a ring with enough idempotents and we denote by $\hat{S}$-Mod the category of all left $\hat{S}$-modules X with $X = \hat{S}X$. This is exactly $\sigma[_{\hat{S}}\hat{S}]$, the full subcategory of $\hat{S}$-modules subgenerated by $\hat{S}$ (e.g. [12], 49.1). $\hat{S}$ is right perfect if and only if it has *dcc* on cyclic (finitely generated) left ideals (e.g. [12], 49.9). In this case, since $\hat{S}$ is (left and right) semi-perfect, every simple (finitely generated) module in $\hat{S}$-Mod has a projective cover.

Let us collect some information about the functor $\widehat{\mathrm{Hom}}_{\mathfrak{C}}(U, -)$:

Lemma 1.1. *Assume $U = \oplus_{\Lambda} U_{\lambda}$, with small U_{λ}'s, is a generator in $\mathfrak{C}$. Then*

(1) *The functor*

$$F := \widehat{\mathrm{Hom}}_{\mathfrak{C}}(U, -) : \mathfrak{C} \to \hat{S}\text{-Mod}$$

is full and faithful and commutes with products and coproducts.

(2) *F has a left adjoint G: $\hat{S}$-Mod $\to \mathfrak{C}$, i.e. for $K \in \hat{S}$-Mod and $L \in \mathfrak{C}$ there exist (functorial) isomorphisms*

$$\psi : \mathrm{Hom}_{\hat{S}}(K, F(L)) \simeq \mathrm{Hom}_{\mathfrak{C}}(G(K), L),$$

and G is an exact functor which commutes with coproducts.

(3) *There are functorial morphisms (for $N \in \mathfrak{C}$, $L \in \hat{S}$-Mod)*

$$\mu_N : GF(N) \to N, \quad \nu_L : L \to FG(L).$$

μ_N is an isomorphism for every $N \in \mathfrak{C}$ and ν_L is an isomorphism for every projective $L \in \hat{S}$-Mod.

(4) *If U is projective, then $\widehat{\mathrm{Hom}}_{\mathfrak{C}}(U, -)$: $\mathfrak{C} \to \hat{S}$-Mod is an equivalence of categories.*

Proof. (1), (2) This is shown in [8], Theorem 2.11. It can also be derived from the proofs of [12], 51.7.

(3) The first ismorphism is given in [8], Theorem 2.11. Since $\nu_{\hat{S}}$ is obviously an isomorphism and F and G commute with coproducts, ν_L is also an isomorphism for any projective L (compare [12], 51.6, (2)).

(4) This is shown in [8], Corollary 2.12 and also by the proof of [12], 51.11.

For our main result we will need the following

Lemma 1.2. *Assume* $U=\oplus_\Lambda U_\lambda$, *with small* U_λ*'s, is a generator in* $\mathfrak{C}$. *If* $\hat{S}=\widehat{\mathrm{End}}_{\mathfrak{C}}(U)$ *is a right perfect ring, then the class*

$$\mathfrak{T}=\{X\in\hat{S}\text{-Mod}\mid G(X)=0\}$$

is a hereditary torsion class (i.e. is closed under direct sums, submodules, factor modules and extensions) and there exist an injective module Q *and a projective module* P *in* $\hat{S}$-Mod *with the properties*

$$(i)\qquad \begin{aligned}\mathfrak{T}&=\{X\in\hat{S}\text{-Mod}\mid \mathrm{Hom}_{\hat{S}}(X,Q)=0\}\\ &=\{X\in\hat{S}\text{-Mod}\mid \mathrm{Hom}_{\hat{S}}(P,X)=0\}.\end{aligned}$$

(*ii*) *P is a direct sum of cyclic local $\hat{S}$-modules, $G(P)$ is a direct sum of small objects in $\mathfrak{C}$, and* $\widehat{\mathrm{End}}_{\mathfrak{C}}(G(P))\simeq\widehat{\mathrm{End}}_{\hat{S}}(P)$.

Proof. Since G is an exact functor, the assertions about $\mathfrak{T}$ are easily verified.

(*i*) Let $\mathfrak{E}$ denote a representative set of all simple modules in $\hat{S}$-Mod not contained in $\mathfrak{T}$ and denote by Q the injective hull of the direct sum of all objects in $\mathfrak{E}$.

Note that in general $\mathfrak{E}$ is an infinite set. If $\mathfrak{E}=\emptyset$ then $Q=0$, all simple $\hat{S}$-modules are in $\mathfrak{T}$ and we conclude $\mathfrak{T}=\hat{S}$-Mod: Indeed, for $X\in\hat{S}$-Mod denote by X' the greatest submodule of X belonging to $\mathfrak{T}$. If $X'\neq X$, then X/X' contains a simple submodule Y/X'. Applying the exact functor G to the sequence

$$0\to X'\to Y\to Y/X'\to 0$$

we conclude $Y\in\mathfrak{T}$, contradicting the choice of X'. This implies $X=X'\in\mathfrak{T}$.

Hence assume $\mathfrak{E}\neq\emptyset$. Suppose $0\neq f\in\mathrm{Hom}_{\hat{S}}(X,Q)$ for some $X\in\mathfrak{T}$. Since Q has an essential socle we find a submodule $X'\subset X$ with $(X')f=E$ for some $E\in\mathfrak{E}$. Since $(X')f\in\mathfrak{T}$ this is a contradiction and hence

$$\mathfrak{T}\subset\{X\in\hat{S}\text{-Mod}\mid \mathrm{Hom}_{\hat{S}}(X,Q)=0\}.$$

Now assume $\mathrm{Hom}_{\hat{S}}(X,Q)=0$ for a non-zero $\hat{S}$-module X. X has non-zero socle and, by definition of Q, $\mathrm{Soc}(X)$ belongs to $\mathfrak{T}$.

Consider the ascending Loewy series $\{s_\alpha(X)\}_{\alpha\geq 0}$ of X with $s_0(X)=0$, $s_1(X)=$

$\mathrm{Soc}(X)$ and so on (see [11], p. 182).

If $s_1(X)\neq X$, then $\mathrm{Hom}(X/s_1(X), Q)=0$ and, as above, we see that $\mathrm{Soc}(X/s_1(X))=s_2(X)/s_1(X)$ belongs to $\mathfrak{T}$. Since $\mathfrak{T}$ is closed under extensions we conclude $s_2(X)\in\mathfrak{T}$. By transfinite induction, we obtain that $s_\alpha(X)\in\mathfrak{T}$ for all ordinals α. Since $\hat{S}$ is right perfect, we know that $X=s_\gamma(X)$ for some ordinal γ, hence $X\in\mathfrak{T}$ and the first equality is established (compare [11], Chap. VIII, 3.3 and 6.3).

Every module in $\mathfrak{E}$ has a projective cover in $\hat{S}$-Mod and we denote the direct sum of all these projective covers by P.

Assume $0\neq f\in\mathrm{Hom}_{\hat{S}}(P,X)$ for some $X\in\hat{S}$-Mod. Then $(P')f\neq 0$ for at least one of the local summands P' of P. For a maximal submodule $K\subset(P')f$ we have an epimorphism

$$P'\to(P')f\to(P')f/K\,.$$

Since a local module has only one simple factor module (see [12], 19.7), $(P')f/K$ cannot be in $\mathfrak{T}$ (by construction of P) and hence is isomorphic to a submodule of Q. This yields a non-zero morphism $P\to(P')f\to Q$.

Finally, consider $0\neq g\in\mathrm{Hom}_{\hat{S}}(X,Q)$ for $X\in\hat{S}$-Mod. Then for some simple submodule $E\subset Q$, we find a submodule $X'\subset X$ with $(X')g=E$ and we have the diagram with exact line

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow & & \\ X' & \to & E & \to & 0\,. \end{array}$$

By projectivity of P, this can be extended commutatively by a (non-zero) morphism $P\to X'\subset X$, establishing the second equality.

(*ii*) If ${}_{\hat{S}}P'$ is small, we see from

$$\mathrm{Hom}_{\hat{S}}(P',F(L))\simeq\mathrm{Hom}_{\mathfrak{C}}(G(P'),L)$$

and Lemma 1.1 that $G(P')$ is small in $\mathfrak{C}$.

Since G commutes with coproducts, $G(P)$ is a coproduct of small objects. ${}_{\hat{S}}P$ being a direct sum of finitely generated projectives we have, by Lemma 1.1,

$$\widehat{\mathrm{Hom}}_{\mathfrak{C}}(G(P),G(P))\simeq\widehat{\mathrm{Hom}}_{\hat{S}}(P,\widehat{\mathrm{Hom}}_{\mathfrak{C}}(U,G(P)))\simeq\widehat{\mathrm{End}}_{\hat{S}}(P)\,.$$

2. Projective generators in $\mathfrak{C}$

With the foregoing preparation we are now able to prove:

Theorem 2.1. *Assume $U=\oplus_\Lambda U_\lambda$, with small U_λ's, is a generator in $\mathfrak{C}$ and $\hat{S}=\widehat{\mathrm{End}}_{\mathfrak{C}}(U)$ is a right perfect ring.*

Then there exists a projective left $\hat{S}$-module P which is a direct sum of local modules such that $G(P)$ is a projective generator in $\mathfrak{C}$ and

$$\widehat{\mathrm{Hom}}_{\mathfrak{C}}(G(P), -) : \mathfrak{C} \to \widehat{\mathrm{End}}_{\hat{S}}(P)\text{-Mod}$$

is an equivalence of categories.

Proof. For the projective $\hat{S}$-module P as defined in Lemma 1.2, we first show that $G(P)$ is a generator in $\mathfrak{C}$.

For $L \in \hat{S}$-Mod consider the canonical morphism $\alpha: P^{(\Omega)} \to L$ with $\Omega = \mathrm{Hom}_{\hat{S}}(P, L)$. Then $\mathrm{Hom}_{\hat{S}}(P, \mathrm{Coker}\, \alpha) = 0$ and hence $\mathrm{Coker}\, \alpha \in \mathfrak{T}$ (Lemma 1.2). By definition of $\mathfrak{T}$ and exactness of G, we get an epimorphism

$$G(P^{(\Omega)}) \simeq G(P)^{(\Omega)} \to G(L).$$

Since every object in $\mathfrak{C}$ is isomorphic to some $G(L)$, this implies that $G(P)$ is a generator in $\mathfrak{C}$.

To prove that $G(P)$ is projective we have to show that any diagram in $\mathfrak{C}$ with exact line

$$(*) \qquad \begin{array}{ccccc} & & G(P) & & \\ & & \downarrow f & & \\ K & \xrightarrow{p} & L & \to & 0 \end{array}$$

can be extended commutatively by some morphism $G(P) \to K$ (compare [1], Proposition 8.6):

With the morphism (isomorphism) ν_P (see Lemma 1.1) the functor F yields the diagram

$$(**) \qquad \begin{array}{ccccccc} & & P & \xrightarrow{\nu_P} & FG(P) & & \\ & & & & \downarrow F(f) & & \\ F(K) & \to & \mathrm{Im}\, F(p) & \subset & F(L) & \to & \mathrm{Coker}\, F(p). \end{array}$$

From the lower line of (*) we get, by Lemma 1.1, the exact commutative diagram

$$\begin{array}{ccccccc} GF(K) & \to & GF(L) & \to & G(\mathrm{Coker}\, F(p)) & \to & 0 \\ \downarrow\simeq & & \downarrow\simeq & & & & \\ K & \to & L & \to & 0 & & \end{array}$$

This yields $G(\mathrm{Coker}\, F(p)) = 0$, so $\mathrm{Hom}_{\hat{S}}(P, \mathrm{Coker}\, F(p)) = 0$ by Lemma 1.2, and hence the image of ν_P composed with $F(f)$ lies in $\mathrm{Im}\, F(p)$. Since P is projective, we find a morphism $P \to F(K)$ yielding a commutative diagram. Applying the functor G to the completed diagram (**) we regain the diagram (*) (up to isomorphism) extended in the desired way.

Now by Lemma 1.1, (4), $\widehat{\mathrm{Hom}}_{\mathfrak{C}}(G(P), -)$ is an equivalence.

For an arbitrary generator U (not necessarily a coproduct of small objects), we replace $\hat{S} = \widehat{\mathrm{End}}_{\mathfrak{C}}(U)$ by $S = \mathrm{End}_{\mathfrak{C}}(U)$ and obtain the following version with literally the same proof. Observe that in this case $\mathrm{Hom}_{\mathfrak{C}}(G(P), -)$ need not be an equivalence:

Theorem 2.2. *Let $\mathfrak{C}$ be a Grothendieck category and U a (finitely generated) generator in $\mathfrak{C}$ with right perfect endomorphism ring S. Then $\mathfrak{C}$ has a (finitely generated) projective generator.*

Since an artinian generator in a Grothendieck category has finite length, its endomorphism ring is semi-primary and we obtain Năstăsescu's result in [10] (also [1], 12.12):

Corollary 2.3. *Let $\mathfrak{C}$ be a Grothendieck category with an artinian generator. Then $\mathfrak{C}$ has a finitely generated projective generator and hence is equivalent to a full module category over a unital ring.*

3. QF categories

In Harada [5], a Grothendieck category with a generating set of small objects is called *QF category* if every projective object is injective in $\mathfrak{C}$.

As noted in [5], in general this property is not equivalent to all injectives being projective in $\mathfrak{C}$. For example, the category of torsion modules over $\boldsymbol{Z}$ has no non-zero projectives (e.g. [12], 18.12) and hence trivially is *QF* without injectives being projective.

However, in special cases we have many characterizations of *QF* categories known for *QF* rings. For this we need a Lemma which is shown with the proof of [12], 48.10, combined with results on semiperfect objects in Grothendieck categories (see [4]):

Lemma 3.1. *Let $\mathfrak{C}$ be a Grothendieck category with a set of finitely generated generators. If $\mathfrak{C}$ has a projective cogenerator with finitely generated socle, then*

(1) *there exists an injective and projective generator in $\mathfrak{C}$;*
(2) *every cogenerator is a generator in $\mathfrak{C}$;*
(3) *every generator is a cogenerator in $\mathfrak{C}$.*

We can now essentially follow the proof of [12, 48.14] to show:

Theorem 3.2. *Let $\mathfrak{C}$ be a Grothendieck category with a set of finitely generated generators. For any object $U \in \mathfrak{C}$ with $S = \mathrm{End}_{\mathfrak{C}}(U)$, the following properties are equivalent:*

(*a*) *U is a noetherian, injective generator in $\mathfrak{C}$;*
(*b*) *U is an artinian, projective cogenerator in $\mathfrak{C}$;*
(*c*) *U is a noetherian, projective cogenerator in $\mathfrak{C}$;*
(*d*) *U is a projective cogenerator in $\mathfrak{C}$ and S_S is artinian;*

(*e*) *U is an injective generator in $\mathfrak{C}$ and ${}_SS$ is artinian;*

(*f*) $\mathrm{Hom}_{\mathfrak{C}}(U, -)\colon \mathfrak{C}\to S$-Mod *is an equivalence and*

(*i*) *S is a noetherian QF ring, or*

(*ii*) *every projective object is injective in $\mathfrak{C}$, or*

(*iii*) *every injective object is projective in $\mathfrak{C}$.*

Proof. (a)$\Rightarrow$(b) Since $S^{(\Lambda)}\simeq\mathrm{Hom}_{\mathfrak{C}}(U, U^{(\Lambda)})$ is injective in S-Mod, S is a semi-primary ring (see also [5], Corollary 1), and by Theorem 2.2, there exists a finitely generated projective generator U' in $\mathfrak{C}$. Since U' is U-generated, it is a direct summand of a finite direct sum of copies of U and hence U' is also injective in $\mathfrak{C}$. $S'=\mathrm{End}_{\mathfrak{C}}(U')$ is left noetherian and semi-primary and hence, by Hopkins theorem, left artinian. Since $\mathrm{Hom}_{\mathfrak{C}}(U', -)\colon \mathfrak{C}\to S'$-Mod is an equivalence, U' and U are artinian. Therefore $U'=\hat{E}_1\oplus\cdots\oplus\hat{E}_k$ with simple objects E_i and $\hat{E}_i$ the injective hull of E_i in $\mathfrak{C}$. Without restriction we may assume $E_i\not\simeq E_j$ for $i\neq j$. Denoting the Jacobson radical of an object V by $J(V)$ (e.g. [4]) we have

$$U'/J(U')\simeq\oplus_{i\geq k}\hat{E}_i/J(\hat{E}_i)\,.$$

This implies that every simple object in $\mathfrak{C}$ is of the form $\hat{E}_i/J(\hat{E}_i)$. Hence either of the sets $\{\hat{E}_1/J(\hat{E}_1), \cdots, \hat{E}_k/J(\hat{E}_k)\}$ and $\{E_1, \cdots, E_k\}$ is a representative set of all simple objects in $\mathfrak{C}$. Therefore U' is a projective cogenerator in $\mathfrak{C}$. Since U' is a subobject of a finite coproduct of copies of U, U is a cogenerator in $\mathfrak{C}$. Also, U is a direct summand of a coproduct of copies of U' and hence it is projective.

Observe that under the given conditions S is left noetherian and left self-injective, i.e. S is a (noetherian) QF ring (e.g. [12], 48.15).

(b)$\Rightarrow$(c) By Lemma 3.1, U is a generator in $\mathfrak{C}$ and hence it is noetherian by the general Hopkins-Levitzki Theorem (e.g. [1], 7.6).

(c)$\Rightarrow$(d) By Lemma 3.1, there exist injective generators in $\mathfrak{C}$ and hence U is an injective and projective generator and $\mathrm{Hom}_{\mathfrak{C}}(U, -)\colon \mathfrak{C}\to S$-Mod defines an equivalence. Therefore S is left noetherian and self-injective, i.e. a QF ring.

(d)$\Rightarrow$(b) Since S is semiperfect, we may assume U to be a finite coproduct of projective objects with local endomorphism rings. By [4], Proposition 1 with Corollary 1, these objects are finitely generated and hence U is finitely generated.

Since U is a cogenerator, every subobject of U is an annihilator subobject (see [5]). By the order reversing bijection between the annihilator subobjects of U and the annihilator submodules of S_S (compare [12], 28.1), we obtain that U is a noetherian object.

(b)$\Rightarrow$(a) follows from the proof of (b)$\Rightarrow$(c).

(a)$\Rightarrow$(e) is clear from the above implications (S is a QF ring).

(e)$\Rightarrow$(a) As a left artinian ring, S is left noetherian. It is obvious that, for a generator U, *acc* on left ideals in S implies *acc* on subobjects of U.

(a)$\Leftrightarrow$(f) (i) follows from the proof (a)$\Rightarrow$(b).

The equivalence of (i), (ii) and (iii) in (f) is clear from the characterization of noetherian QF rings (e.g. [12], 48.15).

4. Categories of bounded type

A Grothendieck category $\mathfrak{C}$ with a family of generators $\{U_\lambda\}_\Lambda$ of finite length is called of *bounded representation type*, if there exists a finite upper bound for the length of finitely generated indecomposable modules. As a consequence of the Harada-Sai Lemma ([6], Lemma 12), in this case $\widehat{\mathrm{End}}_{\mathfrak{C}}(\oplus_\Lambda U_\lambda)$ is a semi-primary ring (e.g. [12], 54.1) and we obtain from Theorem 2.1:

Corollary 4.1. *A Grothendieck category $\mathfrak{C}$ of bounded representation type has a projective generator and is equivalent to a full module category over a ring with enough idempotents and of left bounded representation type.*

A Grothendieck category $\mathfrak{C}$ with a family of generators of finite length is called of *finite representation type*, if there is only a finite number of non-isomorphic finitely generated indecomposable objects in $\mathfrak{C}$. For these categories the above Corollary is well-known (e.g. [3], Theorem 4.4, [12], 54.2). In general, bounded representation type does not imply finite representation type for Grothendieck categories (e.g. [12], Section 54).

5. Categories of type $\sigma[M]$

In module categories the adjoint functor considered in Lemma 1.1 can be described by an ordinary tensor product and hence our previous results can be somewhat refined in this case. We also describe some conditions which ensure right perfect endomorphism rings.

Let R be an associative ring with unit and R-Mod the category of unital left R-modules. For a left R-module M, denote by $\sigma[M]$ the full subcategory of R-Mod whose objects are submodules of M-generated modules. Morphisms of left modules are written on the right (and vice versa). For basic definitions see [12]. Since $\sigma[M]$ is a Grothendieck category we get from Theorem 2.1:

Theorem 5.1. *Assume $M=\oplus_\Lambda M_\lambda$, with M_λ's finitely generated R-modules, is a generator in $\sigma[M]$. If $\hat{S}=\widehat{\mathrm{End}}_R(M)$ is a right perfect ring, then there exists a projective left $\hat{S}$-module P, which is a direct sum of local modules, such that $M\otimes_{\hat{S}}P$ is a projective generator in $\sigma[M]$ and*

$$\widehat{\mathrm{Hom}}_R(M\otimes_{\hat{S}}P, -)\colon \sigma[M]\to\widehat{\mathrm{End}}_{\hat{S}}(P)\text{-Mod}$$

is an equivalence of categories.

The module M is called *semi-projective* if for every $f\in S=\mathrm{End}_R(M)$, $Sf=$

$\mathrm{Hom}_R(M,(M)f)$. A finitely generated, semi-projective module with *dcc* on cyclic or M-cyclic submodules has a right perfect endomorphism ring (e.g. [12], 31.10, 43.10) and hence

Corollary 5.2. *Assume the R-module M is a generator in* $\sigma[M]$. *If M is finitely generated, semi-projective with dcc on* (M-)*cyclic submodules, then* $\sigma[M]$ *has a finitely generated projective generator.*

M is called *semi-injective* if for every $f \in S$, $fS = \mathrm{Hom}_R(M/\mathrm{Ker}\, f, M)$. Submodules of M which are intersections of kernels of endomorphisms are called *annihilator submodules*. A semi-injective module M with *acc* on annihilator submodules has a semi-primary endomorphism ring (e.g. [12], 31.12) and we get

Corollary 5.3. *Assume M is a generator in* $\sigma[M]$. *If M is semi-injective and has acc on annihilator submodules, then* $\sigma[M]$ *has a projective generator.*

QF modules (*Quasi-Frobenius modules*) were defined in Hauger-Zimmermann [7] (also [12], 48.2). In particular, a noetherian module M is a QF module if it is an injective cogenerator in $\sigma[M]$. Modules M which are injective cogenerators in $\sigma[M]$ are also called *strongly quasi-injective* and characterized in Menini-Orsatti [9], Theorem 6.7.

It is known that a finitely generated M-projective module M is a noetherian QF module if and only if M is noetherian, injective and generator in $\sigma[M]$ (e.g. [12], 48.14). By our Theorem 3.2, we obtain new characterizations of noetherian QF modules.

Recall that, for finitely generated M, *M-projective* is equivalent to *projective in* $\sigma[M]$.

Corollary 5.4. *For any R-module M and* $S = \mathrm{End}_R(M)$, *the following are equivalent:*

(*a*) *M is a noetherian, injective generator in* $\sigma[M]$;
(*b*) *M is an artinian, projective cogenerator in* $\sigma[M]$;
(*c*) *M is a noetherian, projective cogenerator in* $\sigma[M]$;
(*d*) *M is a projective cogenerator in* $\sigma[M]$ *and* S_S *is artinian;*
(*e*) *M is an injective generator in* $\sigma[M]$ *and* ${}_S S$ *is artinian.*

Let $\{V_\lambda\}_\Lambda$ be a representative set of the finitely generated modules in $\sigma[M]$. Then $V = \bigoplus_\Lambda V_\lambda$ is a generator in $\sigma[M]$ and if $\sigma[M]$ is of bounded representation type, then $\hat{\mathrm{End}}_R(V)$ is a semiprimary ring. The following version of Corollary 4.1 extends (a)$\Leftrightarrow$(g) in [12, 54.2] from modules of finite representation type to modules of bounded representation type:

Corollary 5.5. *For an R-module M the following are equivalent:*

(*a*) $\sigma[M]$ *is of bounded representation type;*
(*b*) $\sigma[M]$ *is equivalent to a full module category over a ring with enough*

idempotents and of left bounded representation type.

Acknowledgement. This paper was written during a visit of the first author at the Institute of Mathematics of the University of Düsseldorf with financial support by the Alexander von Humboldt-Stiftung. He is very indebted to both institutions.

References

[1] T. Albu and C. Năstăsescu: *Relative Finiteness in Module Theory,* Marcel Dekker, New York, 1984.

[2] F. Anderson and K. Fuller: *Rings and Categories of Modules,* Springer, GTM, 1973.

[3] M. Auslander: *Representation Theory of Artin Algebras II,* Comm. Algebra **1** 1974, 269–310.

[4] M. Harada: *Perfect Categories I,* Osaka, J. Math. **10** 1973, 329–341.

[5] M. Harada: *Perfect Categories IV,* Osaka J. Math. **10** 1973, 585–596.

[6] M. Harada and Y. Sai: *On Categories of Indecomposable Modules I,* Osaka J. Math. **7** 1970, 323–344.

[7] G. Hauger and W. Zimmermann: *Quasi-Frobenius-Moduln,* Arch. Math. **24** 1975, 379–386.

[8] C. Menini: *Gabriel-Popescu Type Theorems and Graded Modules,* in *Perspectives in Ring Theory,* ed. by F. van Oystaeyen and L. Le Bruyn, Kluver Acad. Publ., 1988, 239–251.

[9] C. Menini and A. Orsatti: *Good Dualities and Strongly Quasi-Injective Modules,* Ann. Mat. Pura Appl. **127** 1981, 187–230.

[10] C. Năstăsescu: *Δ-anneaux et modules Δ-injectifs. Applications aux categories localement artiniennes,* Comm. Algebra **9** 1981, 1981–1996.

[11] B. Stenström: *Rings of Quotients,* Springer, Grundlehren **217** 1975.

[12] R. Wisbauer: *Grundlagen der Modul-und Ringtheorie,* Verlag R. Fischer, München, 1988.

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