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ON ALMOST RELATIVE INJECTIVES ON ARTINIAN MODULES

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We have introduced a concept of almost relative projectives (resp. injectives) in [7] (resp. [2]) which is deeply related with lifting modules [9] (resp. extending modules [10]). When we study further those modules, we have understood that it is necessary to generalize [2], Theorem to a case of artinian modules. Namely, we shall give the following theorem (Theorem 2): let U and $\{U_i, I_j\}_{i=1}^{m}{}_{j=1}^{k}$ be LE and artinian modules such that U is I_j-injective for all j and U is almost U_i-injective but not U_i-injective for all i. Then U is almost $\sum_i \oplus U_i \oplus \sum_j \oplus I_j$-injective if and only if $\sum_i \oplus U_i$ is an extending module.

1. Preliminaries

Let R be a ring with identity. Every module M is a unitary right R-module. In this paper we mainly study modules with non-zero socles. We shall denote an injective hull and the socle of M by $\mathrm{E}(M)$ and $\mathrm{Soc}(M)$, respectively. Let N be a submodule of M. If $N \cap N' \neq 0$ for any non-zero submodule N' of M, then N is called an essential submodule of M. If every proper submodule is essential in M, then we call M a uniform module.

We start with definition of almost injective modules following [2]. We take two R-modules U and U_0. Let V be a submodule of U and i the inclusion. Consider the following diagram and two conditions 1) and 2):

(0)
$$\begin{tikzcd} U \arrow[d, dashed, "\pi"'] \arrow[dr, dashed, "\tilde{h}"] & V \arrow[l, "i"'] \arrow[d, "h"] & 0 \arrow[l] \\ U' & U_0 \arrow[l, dashed, "\tilde{h}"] & \end{tikzcd}$$

1) There exists $\tilde{h}\colon U \to U_0$ such that $\tilde{h}i = h$ or
2) There exist a non-zero direct summand U' of U and $\tilde{h}\colon U_0 \to U'$ such that $\tilde{h}h = \pi i$, where $\pi\colon U \to U'$ is the projection of U onto U'.

U_0 is called *almost U-injective* if the above 1) or 2) holds true for the diagram (0) with any V and any h [2] (U_0 is called *U-injective* if we have only 1) [1]).

We frequently use the following property:

(§) Assume that U is indecomposable and U_0 is almost U-injective. If the h given in (0) is not a monomorphism, the case 2) does not occur, and hence there exists $\tilde{h}$: $U\to U_0$ with $\tilde{h}i=h$.

We use sometimes this property without any references.

We shall exhibit some properties dual to ones on almost relative projectives, whose proofs are categorical. Hence we shall skip their proofs,

The following is useful in this paper.

Theorem 1 (dual to [6], Theorem 1). *Let U be an indecomposable and non-uniform module and U_0 an R-module. If U_0 is almost U-injective, then U_0 is U-injective.*

We always assume every module contains non-zero socle unless otherwise stated. We shall study almost relative injectivity among uniform modules with non-zero socle.

Let U_1 and U_2 be uniform modules with isomorphic socles S_1 and S_2, respectively. If for any isomorphism f: $S_1\to S_2$, f or f^{-1} is extensilbe to an element in $\mathrm{Hom}_R(U_1, U_2)$ or in $\mathrm{Hom}_R(U_2, U_1)$, then we say that $U_1\oplus U_2$ has *the extending property of simple modules* (briefly EPSM). If $\mathrm{End}_R(U_i)$ is a local ring for $i=1$, 2 i.e., the U_i are LE modules, then this concept coincides with usual one in [5], §9.6.

Proposition 1 (dual to [6], Proposition 2). *Let E be an indecomposable injective module and U_1, U_2 submodules of E. Assume that either U_1 or U_2 is artinian. Then U_1 is almost U_2-injective if and only if* i): $J(T)U_2\subset U_1$ *and* ii): $U_1\oplus U_2$ *has* EPSM, *where* $T=\mathrm{End}_R(E)$. *In this case if U_1 is not U_2-injective, then U_2 is U_1-injective.*

REMARK 1. In Proposition 1, if we assume $U_1\subset U_2$, we know that the assumption of "artinian" is superfluous for the first half. If U_i does not contain a simple socle for $i=1, 2$, $U_1\oplus U_2$ trivially have EPSM. Let Z be the ring of integers. Then $U_1=U_2=Z$ trivially satisfy i) and ii) in Proposition 1. However Z is not almost Z-injective as Z-modules.

Proposition 2 (dual to [8], Proposition 1). *Let U_0, U_1 and U_2 be R-modules and U_1, U_2 indecomposable. Assume that U_0 is almost U_1-injective, but not U_1-injective. Then* 1): *if U_0 is U_2-injective, U_1 is U_2-injective.* 2): *If U_0 is almost U_2-injective, but not U_2-injective, then we obtain the following fact*: i); *if* $\mathrm{Soc}(U_1)\not\approx\mathrm{Soc}(U_2)$, *$U_1$ is U_2-injective and* ii); *if* $0\neq\mathrm{Soc}(U_1)\approx\mathrm{Soc}(U_2)$, *then U_1 is almost U_2-injective (and U_2 is almost U_1-injective) if and only if $U_1\oplus U_2$ has* EPSM.

2. Main theorem

In this section we shall give the main theorem which is a generalization of [2], Theorem.

Lemma 1 (dual to [6], Proposition 5). *Let U_0, U_1 and U_2 be R-modules with non-zero socle. Assume that i): U_1 and U_2 are* LE *modules, ii): U_0 is almost U_1-injective but not U_1-injective and iii): U_0 is almost $U_1 \oplus U_2$-injective. Further assume that there exists an isomorphism f of a simple sub-factor module V_2/V_2' of U_2 onto a simple submodule of U_1. Then f is extensible to an element $\tilde{f}$: $U_2 \to U_1$ (or f^{-1} is extensible to an element $\tilde{f}'$: $U_1 \to U_2$, in this case $V_2' = 0$).*

Corollary. *Let U_0, U_1 and U_2 be as in Lemma* 1 *and satisfy i), ii) and iii) in Lemma* 1. *Then U_1 is almost U_2-injective.*

Proof. If U_0 is U_2-injective, U_1 is U_2-injective by Proposition 2. Hence we may assume that U_0 is almost U_2-injective, but not U_2-injective. Further we may assume from Proposition 2 and Theorem 1 that $\mathrm{Soc}(U_1) \approx \mathrm{Soc}(U_2)$ is simple. It is clear from Lemma 1 that $U_1 \oplus U_2$ has EPSM. Hence U_1 is almost U_2-injective by Proposition 2.

Lemma 2. *Let U_1 and U_2 be artinian and uniform modules with isomorphic socles. Assume that U_2 is almost U_1-injective. If an isomorphism f of $S_1 = \mathrm{Soc}(U_1)$ onto $S_2 = \mathrm{Soc}(U_2)$ is extensible to an element F: $U_1 \to U_2$, then U_2 is U_1-injective.*

Proof. Let g: $S_1 \to S_2$ be any isomorphism. Since U_2 is almost U_1-injective, g is extensible to G: $U_1 \to U_2$ or g^{-1} is extensible to G': $U_2 \to U_1$. We assume the latter case. Then $G'F$ is an endomorphism of U_1 and $G'F \mid S_1 = g^{-1}f \mid S_1$ is an isomorphism. Hence $G'F$ is a monomorphism, and so an isomorphism, since U_1 is artinian. Therefore G' is an isomorphism, and hence G'^{-1} is an extension of g. We shall show that U_2 is U_1-injective. Take any diagram with V_1 a submodule of U_1:

$$\begin{array}{ccccc} U_1 & \xleftarrow{i} & V_1 & \leftarrow & 0 \\ & & \downarrow h & & \\ & & U_2 & & \end{array}$$

Assume that h is a monomorphism. Then $h \mid S_1$ is extensible to H: $U_1 \to U_2$ from the initial part. Since $\ker(h\text{-}Hi) \supset S_1$, there exists $\tilde{h}$: $U_1 \to U_2$ with $\tilde{h}i = h - Hi$, and hence $h = (\tilde{h} + H)i$. If h is not a monomorphism, then we obtain $\tilde{h}'$: $U_1 \to U_2$ with $\tilde{h}'i = h$ by definition.

Let $\{U_i\}_{i=1}^{t}$ be a set of artinian and uniform modules with $\mathrm{Soc}(U_i) = S_i$ as in Lemma 2. Assume that U_i is almost U_j-injective for any pair (i, j). If an isomorphism of S_1 onto S_2 is extensible to F: $U_1 \to U_2$, we denote it by $U_1 \leqslant U_2$. Then if $U_1 \leqslant U_2$ and $U_2 \leqslant U_1$, $U_1 \approx U_2$ from the above proof. Hence the relation

$\leqslant$ defines a total order on the isomorphism classes of $\{U_i\}$, and $U_i \geqslant U_j$ is equivalent to U_i being U_j-injective. We give one more remark. Let $\{A_i\}$ be a set of uniform modules with isomorphic socles. Then we may assume that the A_i are submodules of $E=\mathrm{E}(A_1)$. If A_i is A_j-injective, $A_j \subseteq A_i$ (cf. [3], Lemma 9). Hence if we assume that for every pair (i, j) either A_i is A_j-injective or A_j is A_i-injective, then $\{A_i\}$ is linearly ordered with respect to inclusion. Further if $A_i \supseteq A_j$, A_i is A_j-injective by assumption.

We remember here the definition of extending modules [10]. Let X be an R-module. If for any submodule Y of X, there exists a direct decomposition of X such that $X=X_1 \oplus X_2$ and X_1 is an essential extension of Y, then X is called an extending module [10].

Let $\{D_i\}_{i=1}^{p}$ be a set of indecomposable R-modules and U_0 an R-module. Assume that U_0 is almost $\Sigma_i \oplus D_i$-injective. Then U_0 is almost D_i-injective for all i (cf. [2]). We shall divide $\{D_i\}$ into two disjoint parts $\{D_i\}=\{U_j\} \cup \{I_k\}$ as follows:

(*) 1) U_0 is I_k-injective for all k.

2) U_0 is almost U_j-injective but not U_j-injective for all j.

We note that all U_j are uniform Theorem 1. The following theorem is a generalization of [2], Theorem.

Theorem 2. *Let U_0 be an R-module, $\{U_j; I_k\}_{j=1, k=1}^{n, m}$ a set of R-modules satisfying (*). Assume that the U_j are* LE *R-modules for all j. If U_0 is almost $(\sum_{j=1}^{n} \oplus U_j) \oplus (\sum_{k=1}^{m} \oplus I_k)$-injective, then $\sum_{j=1}^{n} \oplus U_j$ is an extending module. We assume further that the U_j are artinian. Then the converse is true.*

Proof. The first part is clear from Corollary to Lemma 1 and [3], Theorem 4. Conversely, we assume that $\sum_{j=1}^{n} \oplus U_j$ is an extending module. If Soc $(U_0)=0$, $n=0$ for the U_j are artinian, and Theorem 2 is clear by [1]. Hence we may assume $\mathrm{Soc}(U_0) \neq 0$. Now U_i is almost U_j-injective by [3], Theorem 4 $(i \neq j)$. Put $I=\sum_{k=1}^{m} \oplus I_k$, $U=\sum_{j=1}^{n} \oplus U_j$ and $W=U \oplus I$. Take any diagram with row exact:

$$\begin{array}{ccc} W = U \oplus I & \xleftarrow{i} & V \leftarrow 0 \\ & & \downarrow h \\ & & U_0 \end{array}$$

We shall show

(1) either a): there exists $\tilde{h}$: $W \to U_0$ with $\tilde{h}i=h$ or b): there exist an non-zero indecomposable direct summand U_1' of W and $\tilde{h}'$: $U_0 \to U_1'$ with $\pi_1' i = \tilde{h}'h$, where π_1': $W \to U_1'$ is the projection.

Since our proof is very long, we shall divide it into several steps.

Step 1 *Reduction.* Taking a complement of V in W, we may assume from the proof of Theorem in [2] that

(♯) V is essential in W.

Step 2 *Refinement of diagrams.* Put $U-U_j=U_1\oplus\cdots\oplus U_{j-1}\oplus U_{j+1}\oplus\cdots\oplus U_n$ and $V_i=U_i\cap V$. Consider three diagrams:

(2–j)
$$\begin{array}{ccc} U_j & \overset{i}{\leftarrow} & V_j \leftarrow 0 \\ & & \downarrow h|V_j \\ & & U_0 \end{array}$$

(2–j*)
$$\begin{array}{ccc} U-U_j & \overset{i}{\leftarrow} & (U-U_j)\cap V \leftarrow 0 \\ & & \downarrow h|(U-U_j)\cap V \\ & & U_0 \end{array}$$

and

(2–j**)
$$\begin{array}{ccc} I & \overset{i}{\leftarrow} & I\cap V \leftarrow 0 \\ & & \downarrow h|I\cap V \\ & & U_0 \end{array}$$
(cf. [2], $(2-k)$ in p. 689)

Since U_0 is I_k-injective, there exists always $\tilde{h}_I\colon I\to U_0$ with $\tilde{h}_I i|(I\cap V)=h|(I\cap V)$ by [1].

Step 3 *Existence of $\tilde{h}_j\colon U_j\to U_0$ for all j.* First we shall show under (♯) that

(3) if there exists $\tilde{h}_j\colon U_j\to U_0$ in (2–j) such that $\tilde{h}_j(i|V_j)=h|V_j$ for each j, then there exists $\tilde{h}_U\colon U\to U_0$ such that $\tilde{h}_U(i|(U\cap V))=h|(U\cap V)$. Hence there exists $\tilde{h}\colon W\to U_0$ with $\tilde{h}i=h$, i.e. (1)–a)).

Using [2], Lemma C, we can prove (3) in a similar manner to step 3 in [8] by induction on n, the number of direct summands U_i.

Step 4 *Existence of $\tilde{h}_j\colon U_0\to U_j$ for some j.* From Step 3 the following case remains: for some j there exist no homomorphisms $\tilde{h}'_j\colon U_j\to U_0$ with $\tilde{h}'_j(i|V_j)=h|V_j$, and hence

there exists $\tilde{h}_j\colon U_0\to U_j$ with $\tilde{h}_j(h|V_j)=i|V_j$, i.e.

(4)
$$\begin{tikzcd} U_1 & V_1 \arrow[l, "i"'] & 0 \arrow[l] \\ & U_0 \arrow[u, "h|V_1"'] \arrow[ul, "\tilde{h}_1"] & \end{tikzcd}$$

is commutative.

Under the assumption (4) we shall show that we obtain the second half b) in (1). We pick one $U_{i'}$ in the set $\boldsymbol{U}$ consisting of all the U_j satisfying (4), and take the subset $\boldsymbol{T}=\{U_{k'}\mid \mathrm{Soc}(U_{k'})\approx\mathrm{Soc}(U_{i'})\}$ of $\boldsymbol{U}$. Now we finally choose a largest one in $\boldsymbol{T}$ with respect to the relation $\leqslant$ given after Lemma 2, say U_1. Then U_1 is U_j-injective for any U_j $(\neq U_1)$ in $\boldsymbol{T}$ by Lemma 2.

Here we fix U_1 and $\tilde{h}_1$, i.e.

$$(4') \qquad \begin{tikzcd} U_j & V_j \arrow[l, "i"'] \arrow[d, "h|V_j"] & 0 \arrow[l] \\ & U_0 \arrow[ul, "\tilde{h}_j"] & \end{tikzcd}$$

is commutative.

Step 5–1) $m=0$. We assume $W=U$. Then we shall show the following (5) by induction on p under the assumption (♯) and (4′).

There exists a new direct decomposition of $W_p=U_1\oplus U_2\oplus\cdots\oplus U_p=U_1\oplus(\tilde{U}_2\oplus\cdots\oplus\tilde{U}_p)$ such that $\tilde{h}_1\, h|(W_p\cap V)=\pi_1(p)\, i|(W_p\cap V)$, i.e.

$$(5) \qquad \begin{tikzcd} W_p \arrow[ddr, "\pi_1(p)"'] & V\cap W_p \arrow[l] \arrow[d, "h|V\cap W_p"] & 0 \arrow[l] \\ & U_0 \arrow[d, "\tilde{h}_1"] & \\ & U_1 & \end{tikzcd}$$

is commutative,

where $\pi_i(p)$: $W_p\to U_1$ is the projection with respect to the second decomposition of W_p.

Here we assume temporatily that Step 5–1) is completed.

Step 5–2) $m\neq 0$. We shall show the following (5′) again by induction on p under the assumption (♯), (4′) and Step 5–1).

(5′) There exists a new direct decomposition of $W_p=U_1\oplus\cdots\oplus U_n\oplus I_1\oplus\cdots\oplus I_p=U_1\oplus(\tilde{U}_2\oplus\cdots\oplus\tilde{U}_n\oplus\tilde{I}_1\oplus\cdots\oplus\tilde{I}_p))$ such that $\tilde{h}_1\, h|(W_p\cap V)=\pi_1(p)\, i|(W_p\cap V)$.

If we take $W_p=W$ in (5) or (5′), then we obtain b) of (1). Now if $p=1$ in Step 5–1), then (4′) is nothing but (5). Further Step 5–1) is similar to Step 5–2), and hence we may show them in the cases $p=n$ and $p=m$. By X we denote U_n if $m=0$ (resp. I_m if $m\neq 0$). Put

$$(6) \qquad g=\tilde{h}_1\, h\colon V\to U_1,\ W_*=W-X\,.$$

Step 6 *New decompositions by induction.* We may show (5) and (5′) on $W_p=W$ under the assumption that W_* satisfies (5) and (5′). Namely we obtain a new direct decomposition of W_* by induction hypothesis

$$(7) \qquad \begin{aligned} W_* &= U_1\oplus(U_2'\oplus\cdots\oplus U_{n-1}') \quad \text{if} \quad m=0 \quad \text{or} \\ &= U_1\oplus(U_2'\oplus\cdots\oplus U_n'\oplus I_1'\oplus\cdots\oplus I_{m-1}') \quad \text{if} \quad m\neq 0\,, \end{aligned}$$

and a commutative diagram:

(7′)
$$\begin{array}{ccc} W_* & \overset{i}{\leftarrow} & V\cap W_* \leftarrow 0 \\ & \searrow^{\pi_1'} & \downarrow h|V\cap W_* \\ & & U_0 \\ & & \downarrow \tilde{h} \\ & & U_1 \end{array}$$

where π_1': $W_*\to U_1$ is the projection with respect to the direct decompotition above, i.e.,

(7″)
$$g|(V\cap W_*)=\pi_1' i|(V\cap W_*)$$

We fix the direct decomposition of W_* (7). Using (7) we have a decomposition $W=W_*\oplus X$. We consider the diagram:

(8)
$$\begin{array}{ccc} X & \overset{i}{\leftarrow} & X\cap V \leftarrow 0 \\ & & \downarrow h \\ & & U_0 \quad g|(X\cap V) \\ & & \downarrow \tilde{h}_1 \\ & & U_1 \end{array}$$

Step 7 *Existence* $\tilde{h}_x$: $X\to U_1$. We divide the argument into two cases in (8).
Step 7–1) $X=U_n$ $(m=0)$.
i) $X\notin \boldsymbol{T}$. Since U_0 is almost X-injective, we have the following two cases.
i–1) There exists $\tilde{h}_x'$: $X\to U_0$ which makes the following diagram commutative:

$$\begin{array}{ccc} X & \overset{i}{\leftarrow} & X\cap V \leftarrow 0 \\ & \searrow^{\tilde{h}_x'} & \downarrow h|(X\cap V) \\ & & U_0 \end{array}$$

Putting $\tilde{h}_x=\tilde{h}_1\tilde{h}_x'$, we obtain

$$\tilde{h}_x\colon X\to U_1 \text{ with } \tilde{h}_x i|(X\cap V)=g|(X\cap V).$$

i–2) There exists $\tilde{h}_x'$: $U_0\to X$ satisfying (4). Since $X\notin \boldsymbol{T}$, $\mathrm{Soc}(X)\not\approx\mathrm{Soc}(U_1)$ from the definition of $\boldsymbol{T}$. Hence U_1 is X-injective by Proposition 2, and so there exists

$$\tilde{h}_x\colon X\to U_1 \text{ in (8) with } \tilde{h}_x i|(X\cap V)=g|(X\cap V).$$

ii) $X\in \boldsymbol{T}$. From the choice of U_1 and Lemma 2 U_1 is X-injective. Hence we are in the same situation as in *i*–2).
Step 7–2) $X=I_m$. Then U_1 is X-injective by Proposition 2. Thus in any cases we obtain

(9) $$\tilde{h}_x\colon X \to U_1 \text{ with } \tilde{h}_x i\,|(X\cap V) = g\,|(X\cap V)\,.$$

Step 8 *Final decomposition* We shall apply [2], the proof of Lemma C to (7′) and (9) (use the assumption (#) and note the definition of f'_k in [2], p.689). Taking $\pi'_1\oplus\tilde{h}_x$: $W=W_*\oplus X\to U_1$, we can get a homomorphism $\tilde{h}'_x$: $X\to U_1$ such that

(10) $$g = \pi'_1\oplus(\tilde{h}_x+\tilde{h}'_x)\,|\,V$$

from [2], the proof of Lemma C (cf. $\hat{f}$ in p. 690). Put $X(-\tilde{h}_x-\tilde{h}'_x)=\{-\tilde{h}_x(y)-\tilde{h}'_x(y)+y \mid y\in X\}$ and $W=U_1\oplus(\tilde{U}_2\oplus\cdots)\oplus X(-\tilde{h}_x-\tilde{h}'_x)$. Let π_1 be the projection of W onto U_1 with respect to the above decomposition. We shall show

(5) and (5′) $$\pi_1 i\,|\,V = g = \hat{f}_1 h\,.$$

Let θ^* and θ_x be the projections of W onto W_* and X with respect to $W=W_*\oplus X$, respectively. Put $V^*=\theta_*(V)$ and $V^x=\theta_x(V)$. Then for any element v in V we have $v=v^*+v^x$, where $v^*\in V^*$ and $v^x\in V^x$. Further $v=v^*+(\tilde{h}_x(v^x)+\tilde{h}'_x(v^x))+(-\tilde{h}_x(v^x)-\tilde{h}'_x(v^x)+v^x)$, where $v^*+\tilde{h}_x(v^x)+\tilde{h}(v^x)\in W_*$ and $-\tilde{h}_x(v^x)-\tilde{h}'_x(v^x)+v^x\in X(-\tilde{h}_x-\tilde{h}'_x)$. Hence $\pi_1(v)=\pi_1(v^*)+\tilde{h}_x(v^x)+\tilde{h}'_x(v^x)$. Since $\pi_1|W_*=\pi'_1|W_*$, $\pi_1(v)=\pi'_1(v^*)+\tilde{h}_x(v^x)+\tilde{h}'_x(v)=g(v)$ from (10).

Corollary 1. *Let U_0, $\{U_j, I_k\}$ be modules satisfying (*) as in Theorem 2. We assume further that* $\mathrm{Soc}(U_j)\neq 0$ *for all j and that for every pair (i,j) either U_i is U_j-injective or U_j is U_i-injective (e.g.* $\mathrm{Soc}(U_i)\not\approx\mathrm{Soc}(U_j)$*). Then U_0 is almost $\sum_j\oplus U_j\oplus\sum_k\oplus I_k$-injective if and only if $\sum_j\oplus U_j$ is an extending module*

Proof. We note that in the above proof we used only once the assumption, artinian, in Step 4. However it is available to use the same argument in Step 4 from the remark before Theorem 2. Hence the proof is clear from the proof of Theorem 2.

Corollary 2 ([2], Theorem). *Let $\{U_0, U_1\}_{i=1}^n$ be* LE, *artinian and uniform modules. Then U_0 is almost $\sum_i\oplus U_i$-injective if and only if*

1) *U_0 is almost U_i-injective for all i.*

2) *For any pair U_i, U_j $(i\neq j)$ either U_0 is simultaneously U_i and U_j-injective or $U_i\oplus U_j$ has* EPSM.

Proof. From Proposition 2 and the assumptions 1) and 2) we know that if U_0 is not U_k-injective for $k=i,j$, then U_i and U_j are almost relative injective each other, i.e., $U_i\oplus U_j$ is an extending module. Hence U_0 is almost $\sum_i\oplus U_i$-injective by Theorem 2 and [3], Theorem 4. Conversely if U_0 is almost $\sum_i\oplus U_i$-injective, we have trivially 1). If U_0 is not U_i-injective, then U_i is almost U_j-injective by Corollary to Lemma 1, Hence $U_i\oplus U_j$ has EPSM.

REMARK. The second part of the proof of Theorem 2 is categrical. Hence it is available to get a dual version for almost relative projectives.

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