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RIGHT PERFECT RINGS WITH THE EXTENDING PROPERTY ON FINITELY GENERATED FREE MODULES

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In [3], [4] Harada has studied the following conditions:

(∗) Every non-small right R-module contains a non-zero injective submodule.

(∗)* Every non-cosmall right R-module contains a non-zero projective direct summand.

And he has found two new classes of rings which are characterized by ideal theoretic conditions: one is perfect rings with (∗) and the other one is semi-perfect rings with (∗)*. In [9], Oshiro has studied these rings by using the lifting and extending property of modules, and defined H-rings and co-H-rings related to (∗) and (∗)*, respectively.

A ring R is called a right H-ring if R is right artinian and R satisfies (∗). Dually, R is called a right co-H-ring if R satisfies (∗)* and the ACC on right annihilator ideals.

A right R-module M is said to be an extending module if for any submodule A of M there is a direct summand A^* of M containing A such that A_R is essential in A^*_R. If this "extending property" holds only for uniform submodules of M, so M is called a module with the extending property for uniform modules.

The following theorem is proved by Oshiro in [9, Theorem 3.18].

Theorem. *For a ring R the following conditions are equivalent:*

1) *R is a right co-H-rings.*

2) *Every projective right R-module is an extending module.*

3) *Every right R-module is expressed as a direct sum of a projective module and a singular module.*

4) *The family of all projective right R-modules is closed under taking essential extensions, i.e. for any exact sequence $O\to P\xrightarrow{\varphi} M$, where P is projective and im φ is essential in M, M is projective.*

In this paper we shall consider the case that R is a right perfect ring with

$(*)^*$ and give a new characterization of right co-H-rings. More precisely we shall prove the following theorems.

Theorem I. *Let R be a right perfect ring. Then the following conditions are equivalent:*

1) *R satisfies $(*)^*$.*
2) *$R_R^k := \underbrace{R_R \oplus \cdots \oplus R_R}_{k \text{ summands}}$ is an extending module for each $k \in \mathfrak{N}$.*
3) *R_R^2: is an extending module.*

Theorem II. *A ring R is a right co-H-ring if and only if*

1) *R is right perfect,*
2) *R satisfies the ACC on right annihilator ideals and*
3) *$R_R^2 := R_R \oplus R_R$ is an extending module.*

In the case that R is right non-singular, the condition 2) of Theorem II can be omitted as the following theorem shows.

Theorem III. *Let R be a right non-singular, right perfect ring. Then the following conditions are equivalent:*

1) *R_R^2 is an extending module.*
2) *R has finite right Goldie dimension and R_R^2 has the extending property for uniform modules.*
3) *R is a right co-H-ring.*
4) *R is Morita-equivalent to a finite direct sum of upper triangular matrix rings over division rings.*

We note that the equivalence between 3) and 4) is proved by Oshiro in [9, Theorem 4.6].

1. Preliminaries. Throughout this paper we assume that R is an associative ring with identity and all R-modules are unitary right R-modules. For a module M over R we write M_R (${}_RM$) to indicate that M is a right (left) R-module. We use $\mathrm{E}(M)$, $\mathrm{J}(M)$, $\mathrm{Z}(M)$ to denote the injective hull, the Jacobson radical and the singular submodule of M, respectively. For a submodule N of a non-singular module M (i.e. $\mathrm{Z}(M)=O$), $\mathrm{E}_M(N)$ denotes the unique maximal essential extension of N in M.

For two R-modules M and N, the symbol $M \underset{\sim}{\subsetneq} N$ means that M is R-isomorphic to a submodule of N. The symbols $M \subsetneq_e N$, $M \subsetneq^{\oplus} N$ mean that M is an essential submodule, respectively a direct summand of N. The descending Loewy chain $\{\mathrm{J}_i(M)\}$ of a module M is defined as follows:

$$\mathrm{J}_0(M) = M, \quad \mathrm{J}_1(M) = \mathrm{J}(M), \quad \mathrm{J}_2(M) = \mathrm{J}(\mathrm{J}_1(M)), \cdots$$

An R-module M is said to be small if M is small in $E(M)$, and if M is not small, M is called non-small. Dually, M is called a co-small module if for any projective module P and any epimorphism $f: P \rightarrow M$, $\ker f$ is an essential submodule of P. M is called a non-cosmall module if M is not cosmall. For basic properties of these modules we refer to [3], [4], [10].

Let R be a ring and e be a primitive idempotent of R. We say that e is a right non-small idempotent if eR is a non-small R-module (cf. [4]), and e is called a right-t-idempotent if for any primitive idempotent f of R, every R-monomorphism of eR in fR is an R-isomorphism. For a ring R we shall use the following symbols:

$$N_r(R) = \{e \in R \mid e \text{ is a right non-small idempotent of } R\}$$
$$T_r(R) = \{e \in R \mid e \text{ is a right-t-idempotent of } R\} \ .$$

Following [11], a ring R is called a right QF-3 ring if $E(R_R)$ is projective.

The following results are useful for our investigation in this paper.

Theorem A ([3, Theorem 3.6]). *A semiperfect ring R satisfies $(*)^*$ if and only if for a complete set $\{e_i\} \cup \{f_j\}$ of orthogonal primitive idempotents of R with each e_iR is non-small and each f_jR is small.*

a) Each e_iR is an injective R-module.

b) For each e_iR, there exists $t_i \geq 0$ such that $J_t(e_iR)$ is projective for all $0 \leq t \leq t_i$ and $J_{t_i+1}(e_iR)$ is a singular module.

c) For each f_jR, there exists an e_iR such that $f_jR \underset{\sim}{\subsetneq} e_iR$.

Lemma B ([3], [10]). *The following statements hold about non-cosmall modules:*

1) *An R-module M is non-cosmall if $M \neq Z(M)$.*

2) *If an R-module M contains a non-zero projective submodule, then M is non-cosmall.*

A ring R is called right perfect if each of its right modules has a projective cover (see [2, Theorem P] or [1, Theorem 28.4]).

Lemma C ([7]). *If R is a right perfect ring, then R has ACC on principal right ideals.*

From the definition of non-cosmall modules and Lemma B we have:

Lemma D. *For a ring R and a cardinal α the following conditions are equivalent:*

1) *Every α-generated right R-module is a direct sum of a projective module and a singular module.*

2) *$R_R^{(\alpha)} := \bigoplus_I R_R$ is an extending module where card $I = \alpha$.*

2. The main results. We start our investigation by proving the following lemma.

Lemma 1. *Let R be a semiperfect ring. Then $\boldsymbol{N}_r(R) \subset \boldsymbol{T}_r(R)$.*

Proof. Let $e \in \boldsymbol{N}_r(R)$. In order to show that $e \in \boldsymbol{T}_r(R)$ it suffices to prove that for any primitive idempotent f of R, every R-monomorphism of eR to fR is isomorphic. Since R is semiperfect, $J(fR)$ is a unique maximal submodule of fR. Let α be an R-monomorphism of eR to fR and suppose that $\alpha(eR) \neq fR$. Then $\alpha(eR)$ is small in fR. It follows that $\alpha(eR)$ is a small module by [3, Proposition 1.1]. Hencd eR is also a small module, a contradiction. Therefore $\alpha(eR) = fR$.

We note that M is an extending module if and only if every closed submodule of M is a direct summand. Hence we have

Lemma 2. *Let R be a ring and P be a projective right R-module. If P is an extending module, so is every direct summand of P.*

Proposition 3. *Let R be a right perfect ring and $e \in \boldsymbol{I}_r(R)$. If $R_R \oplus eR_R$ is an extending module, then eR_R is injective.*

Proof. Since R is right perfect.

$$R_R = e_1R \oplus \cdots \oplus e_nR\,,$$

where $\{e_i\}_{i=1}^n$ is a set of mutually orthogonal primitive idempotents of R. Since $R_R \oplus eR_R$ is an extending module by assumption, Lemma 2 shows that R_R is an extending module, furthermore each e_iR is an extending module. It follows that fR_R is uniform for each primitive idempotent f of R.

Now we prove the injectivity of eR_R with e as in Proposition 3. Let U be a right ideal of R and α be any R-homomorphism of U in eR. We show that α is extended to one in $\mathrm{Hom}_R(R_R, eR)$. We can assume that U_R is essential in R_R. Since $M := R_R \oplus eR$ is an extending module, there is a direct summand U^* of M such that $\{u - \alpha(u) \mid u \in U\} \subset_e U^*$. Then $U^* \cap eR = 0$ and $U^* \oplus eR \subset_e M$. Write $M = U^* \oplus M'$. Sinc eR is uniform, M' is indecomposable. On the other hand, since R has the Azumaya's diagram, so does M $(= e_1R \oplus \cdots \oplus e_nR \oplus eR)$. Therefore there is some primitive idempotent e_i such that $M' \cong e_iR$. Let ε be an R-isomorphism of M' onto e_iR. Consider the projection $U^* \oplus M' \xrightarrow{p} M'$ and put $p_1 = p|_{eR}$. Then p_1 is a monomorphism. Hence we have a sequence

$$0 \rightarrow eR \xrightarrow{p_1} M' \xrightarrow{\varepsilon} e_iR\,,$$

therefore $\varphi:=\varepsilon p_1$ is an R-monomorphism of eR in e_iR. By the definition of e, φ is an isomorphism,. From this, $e_iR=\varphi(eR)=(\varepsilon p_1)(eR)=\varepsilon(p_1eR)$. Hence $p_1(eR)=M'$, since ε is an isomorphism. It follows that $p(eR)=M'$, and then $U^*\oplus eR=U^*\oplus M'=M$. Let π be the projection of $U^*\oplus eR$ on eR. Then $\pi|_R$ is an extension of α. The proof of Proposition 3 is complete.

Corollary 3′. *Let R be a right perfect ring. If $R_R\oplus R_R$ is an extending module, then R is a right QF-3 ring. (This is equivalent to the fact that eR_R is injective for each $e\in N_r(R)$.)*

Proof. Let R be right perfect. In view of [3, remark after Proposition 1.2], $N_r(R)\neq\phi$. Let $e\in T_r(R)$. By Lemma 1, $e\in T_r(R)$. By Lemma 2, $R_R\oplus eR_R$ is an extending module. Hence eR_R is injective by Proposition 3. Therefore R is right QF-3 by [3, Theorem 1.3].

Theorem 4. *Let R be a right perfect ring. Then the following conditions are equivalent:*

1) *R satisfies* $(*)^*$.

2) *R_R^k is an extending n odule for each $k\in\mathfrak{N}$.*

2) *bis Every k-generated right R-module is expressed as a direct sum of a projective module and a singular module.*

3) *R_R^2 is an extending module.*

3) *bis Every 2-generated right R-module is expressed as a direct sum of a projective module and a singular module.*

Proof. It is casy to see that the equivalence of 2) and 2) bis, as well as of 3) and 3) bis follows Lemma D.

1)⇒2). Remark: Let R be a right perfect ring. Then $R_R=e_1R\oplus\cdots\oplus e_nR$ where $\{e_i\}_{i=1}^n$ is a set of mutually orthogonal primitive idempotents of R, furthermore $\mathrm{End}_R(e_iR)$ is local. Consider the module $F:=R_R^k$ and let B be a direct summand of F. Then B is projective, and hence there is a decomposition $B=B_1\oplus\cdots\oplus B_t$ with B_i indecomposable. Since F has the Azumaya's diagram, it follows that $t\leq kn$.

Now suppose that R is right perfect and 1) holds. Let M be a submodule of F. Consider the set

$$\boldsymbol{M}=\{M'\mid M\subset M'\subset^{\oplus}F\}\ .$$

By the remark above, it follows that $\boldsymbol{M}$ has a minimal element, M^* say. We shall show that $M\subset_e M^*$. Indeed, if not, then $C:=M^*/M$ is a non-cosmall module, since M^* is projective. By 1), C contains a non-zero direct summand which is projective, say $C=M_1/M\oplus M_2/M$ with M_1/M is non-zero projective. Form this $M^*/M_2\cong(M^*/M)/(M_2/M)\cong M_1/M$. Therefore $M^*=M_2\oplus D$, where $D\cong M_1/M$ is a non-zero projective module. Thus $M\subset M_2\subset^{\oplus}M^*$ with

$M_2 \neq M^*$, a contradiction to the minimality of M^* in $\boldsymbol{M}$. Hence we must have $M \subset_e M^*$. This shows that F is an extending module, i.e. it holds 2).

3) follows from 2) immediately.

3)⇒1). Assume 3). In order to show that 1) holds, we shall show that R satisfies a), b) and c) of Theorem A. We check it in three steps.

Step 1. Since R is right perfect, in view of the remark of Harada in [3, after proposition 1.2], there exists a complete set $\{g_i\}$ of mutually orthogonal primitive idempotents such that $1=\sum g_i$. Furthermore we can devide $\{g_i\}$ into two parts $\{g_i\}=\{e_i\}_{i=1}^{n} \cup \{f_j\}_{j=1}^{m}$, where each e_iR is non-small and each f_jR is small and we always have $n \geqslant 1$. Hence by 3) and Corollary 3′, it follows that each e_iR is injective. Thus R satisfies the condition a) of Theorem A.

Step 2. Put $e:=e_i$ where e_iR is injective. The following remark will be used in the step. Let $U \subset eR$ with $U=uR+vR$ for some u, v in U. Then Lemma D shows that U is either projective or U is singular.

Now let $J_1=\mathrm{J}(eR)$. We shall show that if J_1 is not a singular module, then J_1 is projective. Assume that J_1 is not singular. Then there is an $0 \neq x \in J_1$ such that xR is not a singular module. As noted above, xR is projective. Then the set $\boldsymbol{P}$ of non-zero projective submodules of J_1 is non-empty. For any P in $\boldsymbol{P}$, P is uniform. Since R is right perfect, $P \cong fR$ for some primitive idempotent f in R. In particular we see from this that every element of $\boldsymbol{P}$ is a principal right ideal of R. By Lemma C, the ACC holds in $\boldsymbol{P}$. Let P^* be a maximal element in $\boldsymbol{P}$. We show that $P^*=J_1$. Assume that $P^* \neq J_1$. Then there exists an $x \in J_1$ but $x \notin P^*$. Put $P^*=pR$ for some $p \in P^*$, and consider the module $B:=xR+pR$. Then by the remark in the step, B is either projective or singular. But clearly B can not be singular, so B is projective. Therefore B is contained in $\boldsymbol{P}$, a contradiction to the maximality of P^*. Hence we must have $P^*=J_1$, i.e. J_1 is projective. Let $J_2=\mathrm{J}(J_1)$. Since J_1 is projective and cyclic, J_2 is the unique maximal submodule of J_1. Using the same argument as above, we see that J_2 is either projective or singular. If J_2 is projective, J_2 is a cyclic module and $\mathrm{J}(J_2)$ is the unique maximal submodule of J_2. Continuing this way we have a descending chain of cyclic projective right R-modules J_i

$$eR=J_0 \supset J_1 \supset \cdots \tag{1}$$

where J_i/J_{i+1} is simple for each $i=0, 1, \cdots$ Now, since J_i is projective and cyclic, for each J_i there is a primitive idempotent f_i of R such that $J_i \cong f_iR$. This and the fact that R is right perfect show that in (1) there are J_i and J_j such that $J_i \cong J_j$. Let φ be this isomorphism. Then φ can be extended to an R-monomorphism of eR to eR. We know also from (1) that the composition length of eR/J_i is i and that of eR/J_j is j. From these and the fact that eR is indecomposable injective, we must have $J_i=J_j$. Thus J_{i+1} is singular. We have shown that R satisfies b) of Theorem A.

Step 3. In this step we shall show that R satisfies the condition c) of Theorem A. Let f be any primitive idempotent of R. Then fR is uniform by Lemma 2. By Corollary 3, R is right QF-3, hence the injective hull $E(fR)$ of fR is projective. Therefore $E(fR)\cong\oplus_I e_iR$. It follows $E(fR)\cong e_iR$ for some e_i. This shows that $fR\underset{\sim}{\subsetneq} e_iR$.

Now using Theorem A we get that R satisfies $(*)^*$. Thus 1) holds. The proof of Theorem 4 is complete.

Let R be a right co-H-ring. Then it is easy to see that R is right perfect and $R_R\oplus R_R$ is an extending module. Moreover, by Theorem 4 we have the following result.

Theorem 5. *A ring R is a right co-H-ring if and only if R satisfies the following conditions:*

1) *R is right perfect.*
2) *R has the ACC on right annihilator ideals.*
3) *$R_R\oplus R_R$ is an extending module.*

REMARK. In [8], Kato has given an example for a semiprimary ring R which is an injective cogenerator in the category of all right R-modules but R is not a QF-ring. It follows that R is not a right co-H-ring. However by Theorem 4, R satisfies $(*)^*$. This shows that the class of rings considered in Theorem 4 properly contains the class of all right co-H-rings.

We now consider the case where R is right non-singular, i.e. $Z(R_R)=0$. Let M be a module and U be a submodule of M. By Zorn's Lemma, there is a maximal essential extension $E_M(U)$ of U in M. As is well-known, if M is non-singular, $E_M(U)$ is determined uniquely.

Lemma 6. *Let M be a non-singular module with finite Goldie dimension. If M has the extending property for uniform modules, then M is an extending module.*

Proof. Let M be a module having the properties as in Lemma 6. By [5] we know that every direct summand of M has also the extending property for uniform modules. Let A be a non-zero submodule of M. Then A has also finite Goldie dimension, k say. Clearly, the Goldie dimension of $E_M(A)$ is also k. Let V_1 be a uniform submodule of A. Then $E_M(V_1)\subsetneq E_M(A)$, since $Z(M)=0$. Hence by assumption we have

$$M = E_M(V_1)\oplus M_1\,, \tag{1}$$

From this, $E_M(A)=E_M(V_1)\oplus A_1$ where $A_1=E_M(A)\cap M_1$. If $A_1\neq 0$, A_1 contains a uniform submodule V_2 for which we have $E_M(V_2)\subsetneq M_1$ and $M_1=E_M(V_2)\oplus M_2$, $E_M(V_2)\subsetneq E_M(A)$. By (1), $M=E_M(V_1)\oplus E_M(V_2)\oplus M_2$. Then $E_M(A)=E_M(V_1)\oplus E_M(V_2)\oplus A_2$ with $A_2=E_M(A)\cap M_2$. Since the Goldie dimension of A is k, we get after k steps

$$M = \mathrm{E}_M(V_1) \oplus \cdots \oplus \mathrm{E}_M(V_k) \oplus M_k$$

with $\mathrm{E}_M(A) = \mathrm{E}_M(V_1) \oplus \cdots \oplus \mathrm{E}_M(V_k)$, proving the extending property of M.

Theorem 7. *For a right non-singular ring R the following conditions are equivalent*:

1) *R is right perfect and $R_R \oplus R_R$ is an extending module.*

2) *R is right perefect with finite right Goldie dimension and $R_R \oplus R_R$ has the extending property for uniform modules.*

3) *R is (right and left) perfect and $R_R \oplus R_R$ has the extending property for uniform modules.*

4) *R is a right co-H-ring.*

5) *R is Morita-equivalent to a finite direct sum of upper triangular matrix rings over division rings.*

Proof. 4)⇔5) is proved by Oshiro [9, Theorem 4.6]. 4)⇒3) is clear.

3)⇒2). Assume 3). R has a decomposition $R_R = e_1R \oplus \cdots \oplus e_nR$ where $\{e_i\}_{i=1}^n$ is a set of mutually orthogonal primitive idempotents of R. Since R is left perfect, every e_iR contains a minimal submodule. Moreover by [5, Proposition 1], each e_iR has also the extending property for uniform modules. Then it is easy to see that the Goldie dimension of R_R is finite. Hence we have 2).

2)⇒1) holds by Lemma 6.

1)⇒4). Assume 1). R has the above decomposition related to e_iR's. By Theorem 4, R satisfies $(*)^*$. Hence Theorem A shows that e_iR has finite composition length if e_iR is injective, since $\mathrm{Z}(R_R)=0$. Now if e_iR is not injective, we consider $\mathrm{E}(e_iR)$. Clearly $\mathrm{E}(e_iR)$ is non-co-small. Since e_iR is uniform, $\mathrm{E}(e_iR)$ must be projective by $(*)^*$. Hence there is a primitive idempotent f with $\mathrm{E}(e_iR) \cong fR$, therefore $\mathrm{E}(e_iR)$ has finite length. These facts show that R is right artinian. Therefore R is a right co-H-ring. The proof of Theorem is complete.

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