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# TORSION FREE EXTENDING MODULES

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## 1. Introduction

We recall that a module is extending if every complement submodule is a direct summand. In [6] we showed that, over a commutative domain $R$, a non-torsion module is extending if and only if it is of the form "injective $\oplus$ extending torsion free reduced", and that a torsion free reduced module is extending if and only if it is a finite direct sum of uniform submodules each pair of which is extending.

In Theorem 1, we provide now a characterization of the extending property for such pairs, and thereby complete the description of non-torsion extending $R$-modules. (For the torsion case, cf. [7]). The drawback of this characterization, viz. that it is formulated in terms of local data, is removed in Theorems 5 and 6, under the assumption that a certain natural overring $S$ of $R$ is noetherian. The subsequent corollaries state what can be said if $R$ itself is noetherian.

The last section presents a number of examples, which demonstrate that our various technical conditions cannot be relaxed.

Throughout this paper $R$ will be a commutative domain with quotient field $K$.

A submodule $N$ of a module $M$ is a complement submodule, if there is another submodule $N'$ such that $N$ is maximal with respect to $N \cap N' = 0$.

Let $T$ be an overring of $R$. The conductor of $R$ in $T$ is the largest ideal of $R$ which is also an ideal of $T$.

## 2. Direct sums of two uniform modules over commutative domains

In this section we characterize all torsion free (reduced) extending modules which are direct sums of two uniform submodules.

Let $M_1$ and $M_2$ be torsion free reduced uniform $R$-modules. Since the $M_i$ are embeddable into the quotient field $K$ of $R$, we may assume $M_i \subsetneq K$. Let $A := \{q \in K: qM_1 \subset M_2\}$ and $B := \{q \in K: qM_2 \subset M_1\}$. For any $R$-submodule $X$ of $K$, let $O(X) := \{q \in K: qX \subset X\}$. Denote $O(M_1) \cap O(M_2)$ by $S$. If $M_i \neq 0$, then $A \cong \hom_R(M_1, M_2)$, and $B \cong \hom_R(M_2, M_1)$, and $O(M_i) \cong \mathrm{end}_R(M_i)$.

**Theorem 1.** *Let $M=M_1\oplus M_2$ be a torsion free reduced R-module, where the $M_i$ are uniform. Then the following statements are equivalent:*
1) *$M$ is extending;*
2) *for every maximal ideal $P$ of $S$, $O(A_P)$ coincides with $O(B_P)$, and is a valuation ring with maximal ideal $W\subset A_PB_P$. If $A_P\cong W\cong B_P$, then $O(A_P)$ is discrete.*

Proof. 1) $\Rightarrow$ 2): Let $M$ be extending. By [6], Corollary 8, we have that $A$ and $B$ are non-zero. By [6], Theorem 7, we obtain $q^{-1}A\cap S+qB\cap S=S$ for each $0\neq q\in K$. It follows that $q^{-1}A_P\cap S_P+qB_P\cap S_P=S_P$ for every maximal ideal $P$ of $S$, and hence $q\in A_P$ or $q^{-1}\in B_P$.

By the same argument as in [7], Theorem 20, we can show that $O(A_P)$ coincides with $O(B_P)$ and is a valuation ring with maximal ideal $W\subset A_PB_P$, and that if $A_P\cong W\cong B_P$, then $O(A_P)$ $(=O(B_P))$ is discrete.

2) $\Rightarrow$ 1): The same argument as in [7], Theorem 20, shows that $q\in A_P$ or $q^{-1}\in B_P$ for all $0\neq q\in K$ and every maximal ideal $P$ of $S$. It follows that $q^{-1}A_P\cap S_P+qB_P\cap S_P=S_P$, and hence $q^{-1}A\cap S+qB\cap S=S$. Therefore $M$ is extending, by [6], Theorem 7.

**Corollary 2.** *Let $M=M_1\oplus M_2$ be an extending R-module as in Theorem 1. Then $O(A)$ coincides with $O(B)$, and is integrally closed.*

Proof. By Theorem 1, $O(A_P)=O(B_P)$ is a valuation ring for all maximal ideals $P$ of $S$. Since any valuation ring is integrally closed, we have that $\bigcap_P O(A_P)$ $(=\bigcap_P O(B_P))$ is integrally closed. It is clear that $AO(A_P)\subset A_P$ for all $P$; hence $A[\bigcap_P O(A_P)]\subset \bigcap_P A_P=A$, i.e. $\bigcap_P O(A_P)\subset O(A)$. Thus $\bigcap_P O(A_P)=O(A)$. Similarly $\bigcap_P O(B_P)=O(B)$.

**Corollary 3.** *Let $N$ be a uniform torsion free reduced R-module. Then $N^n$ is extending if and only if $O(N)$ is a Prüfer domain.*

Proof. Theorem 1, and [6], Theorem 11.

**Corollary 4.** *Let $P$ be a maximal ideal of a commutative domain $R$. Then the following statements are equivalent:*
1) *$P\oplus R$ is extending;*
2) *$O(P)$ is a Prüfer domain and $P$ is a maximal ideal of $O(P)$. $R_Q$ is a valuation ring for all maximal ideals $Q$ different from $P$.*

Proof. Theorem 1, Corollary 3, and observing that $(R:P)P=R$ or $(R:P)P=P$.

## 3. Direct sums of uniform modules over noetherian domains

**Theorem 5.** *Let $M=M_1\oplus M_2$ be a torsion free reduced R-module, where*

*the $M_i$ are uniform. Let $S$ be noetherian. Then the following statements are equivalent:*
1) *$M$ is extending;*
2) *$O(A)$ coincides with $O(B)$, and is a Dedekind domain. $AB$ is a product of distinct maximal ideals of $O(A)$. There is a one-to-one correspondence between the maximal ideals of $O(A)$ and the maximal ideals of $S$, via contraction;*
3) *the integral closure $S'$ of $S$ is Dedekind and is a (maximal) equivalent order. There is a one-to-one correspondence, via contraction (and extension), between the maximal ideals of $S$ and of $S'$. The conductor $D$ of $S$ in $S'$ is a product of distinct maximal ideals of $S'$ (or $S$). $A$ and $B$ are $S'$-modules with $AB=D$.*

Proof. 1) $\Rightarrow$ 2). Let $M$ be extending. By Corollary 2, $O(A)=O(B)$ is integrally closed. By [6], Corollary 8, $AB$ is a non-zero ideal of $S$ which is also an ideal of $O(A)$. Then $AB$ is contained in the conductor $D$ of $S$ in $O(A)$. Since $S$ is noetherian and $D\neq 0$, we have that $O(A)$ is a finitely generated $S$-module. It follows that $O(A)$ is noetherian and integral over $S$, and hence $O(A)$ is the integral closure of $S$.

Since $A$ is a fractional ideal of $S$ and hence finitely generated, $O(A)_P=O(A_P)$ is a rank one discrete valuation ring, for every maximal ideal $P$ of $S$, by Theorem 1. Since $O(A)_P$ is integral over $S_P$, it follows that $S_P$ is one dimensional for all $P$. Hence $S$ is one dimensional, and thus $O(A)$ is Dedekind.

We show that for each maximal ideal $P$ of $S$ there exists a unique maximal ideal $\mathfrak{P}$ of $O(A)$ such that $P=\mathfrak{P}\cap S$. The existence of such $\mathfrak{P}$ is due to $O(A)$ being integral over $S$. The uniqueness follows from the fact that $O(A)_P=O(A)_{\mathfrak{P}}$ for any maximal ideal $\mathfrak{P}$ of $O(A)$ lying over $P$. This establishes the one-to-one correspondence, via contraction.

Now we show that $AB$ is a product of distinct maximal ideals of $O(A)$. Since $O(A)_P=O(A)_{\mathfrak{P}}$, for every maximal ideal $P$ of $S$, where $\mathfrak{P}\cap S=P$, we have $(AB)_{\mathfrak{P}}=(AB)_P$. Hence $(AB)_{\mathfrak{P}}=\mathfrak{P}_{\mathfrak{P}}$ for any maximal ideal $\mathfrak{P}$ of $O(A)$ containing $AB$, by Theorem 1. On the other hand, $AB$ is an ideal of $O(A)$, $AB=\Pi\mathfrak{P}^{n(\mathfrak{P})}$. It follows that $(AB)_{\mathfrak{P}}=\mathfrak{P}_{\mathfrak{P}}^{n(\mathfrak{P})}$, and by comparison we conclude that $n(\mathfrak{P})=1$.
2) $\Rightarrow$ 3). By 2), $O(A)=O(B)=:S'$ is Dedekind and is the integral closure of $S$, hence a maximal equivalent order.

Now let $D$ be the conductor of $S$ in $S'$; it is clear that $AB\subset D$. Since $A$ and $B$ are non zero, $M_1S'$ and $M_2S'$, as $S'$-modules, can be embedded in each other. By [5], Lemma 12, $M_1S'=M_2I$ where $I$ is a fractional ideal of $S'$. Since $M_iD\subset M_i$ $(i=1, 2)$, it follows that $M_2ID\subset M_1$ and $M_1I^{-1}D\subset M_2$. Hence $ID\subset B$ and $I^{-1}D\subset A$, and thus $D^2\subset AB\subset D$. Since $AB$, by 2), is a product of distinct maximal ideals of $S'$, we deduce $AB=D$.

Now we show that for any maximal ideal $P$ of $S$, the unique maximal ideal of $S'$ lying over $P$ is $\mathfrak{P}=PS'$. This means that the inverse of the one-to-one correspondence via contraction (in Condition 2)) is given by extension. Since

$P=\mathfrak{P}\cap S\subset\mathfrak{P}$, we have $PS'\subset\mathfrak{P}$. Hence $PS'=\mathfrak{P}^n$, $n\geq 1$. If $n>1$, then $\mathfrak{P}D\subset\mathfrak{P}\cap S=P\subset PS'=\mathfrak{P}^n$, and thus $D\subset\mathfrak{P}^{n-1}\subset\mathfrak{P}$. Therefore $D\subset\mathfrak{P}\cap S\subset\mathfrak{P}^n$, which contradicts that $D$ is a product of distinct maximal ideals of $S'$.
3) $\Rightarrow$ 1): Condition 2) of Theorem 1 can be easily verified, by using that $A$ and $B$ are $S'$-modules, and $AB=D$.

For more than two uniform modules $M_i$, we use the notations $A_{ij}:=\{q\in K: qM_i\subset M_j\}$, $A_{ii}=O(M_i)$, and $S_{ij}=O(M_i)\cap O(M_j)$.

We combine Theorem 6 with Theorem 11 from [6] to obtain the following:

**Theorem 6.** *Let $M=\bigoplus_{i=1}^{n} M_i$ be a torsion free reduced R-module, where the $M_i$ are uniform. Let $S:=\bigcap_{i=1}^{n} O(M_i)$ be noetherian. Then the following statements are equivalent:*
1) *$M$ is extending;*
2) *the integral closure $S'$ of $S$ is Dedekind, and is a (maximal) equivalent order. There is a one-to-one correspondence, via contraction (and extension) between the maximal ideals of $S'$ and of $S$. The conductor $D$ of $S$ in $S'$ is a product of distinct maximal ideals of $S'$ (or $S$). For all $i\neq j$, the $A_{ij}$ are $S'$-modules, and $D\subset A_{ij}A_{ji}\subset S'$;*
3) *there is a Dedekind domain $R\subset L\subset K$, maximal ideals $\mathfrak{P}_k (k=1, 2, \cdots, n)$ of $L$, and subfields $F_k$ of $L/\mathfrak{P}_k$ such that the $A_{ij}$ are $L$-modules and $\bigcap_{k=1}^{n}\mathfrak{P}_k\subset A_{ij}A_{ji}\subset L$ $(i\neq j)$, and $S$ is the full inverse image of $\bigoplus_{k=1}^{n} F_k$ under the natural homomorphism $L\to L/\bigcap_{k=1}^{n}\mathfrak{P}_k\cong\bigoplus_{k=1}^{n} L/\mathfrak{P}_k$.*

Proof. 1) $\Rightarrow$ 2): Let $M$ be extending. Then $M_i\oplus M_j$ is extending for all $i\neq j$. It is clear that $I:=A_{12}A_{23}\cdots A_{n-1n}A_{n1}$ is a non zero ideal of $S$ which is also an ideal of $S_{ij}$. Since $S$ is noetherian, $S_{ij}$ is noetherian for all $i\neq j$. By Theorem 5, the $O(A_{ij})$ are Dedekind. Since $o\neq I\subset D$ (the conductor of $S$ in $O(A_{ij})$), it follows that $O(A_{ij})$ is the integral closure of $S$; and hence all $O(A_{ij})$ coincide. We denote this ring by $S'$.

We show that $(S_{ij}:D)D=D_{ij}$, where $D_{ij}$ is the conductor of $S_{ij}$ in $S'$. It is clear that $(S_{ij}:D)D\subset D_{ij}$. Now let $x\in D_{ij}$ be arbitrary. For any $y\in(S':D)$, we have $yD\subset S'$ and hence $xyD\subset xS'\subset S_{ij}$. It follows that $xy\in(S_{ij}:D)$, hence $x(S':D)\subset(S_{ij}:D)$. Thus $x\in xS'=x(S':D)D\subset(S_{ij}:D)D$. Therefore $(S_{ij}:D)\,D=D_{ij}$, and hence $O(D)=O(S_{ij}:D)=S'$.

By Theorem 5 and since $O(D)=O(S_{ij}:D)=S'$, we have that $D\oplus S_{ij}$ is extending for all $i\neq j$. 2) follows, by Theorem 5, once we show that $D\oplus S$ is extending. Since $S$ is noetherian, it is enough to show that $D_P\oplus S_P$ is extending, for every maximal ideal $P$ of $S$. To this end we consider two cases:
*Case* 1. $P\supset D$. Since $D\oplus S_{ij}$ is extending for all $i\neq j$, we have that $D_P\oplus(S_{ij})_P$

is extending. Since $O(D_P)=O(D)_P=S'_P$ is local, we have, by [6], Theorem 7, that $D_P$ and $q(S_{ij})_P$ are comparable for every $q\in K$. If $q(S_{ij})_P\subset D_P$ for some $i\neq j$, then $qS_P\subset q(S_{ij})_P\subset D_P$. On the other hand, if $D_P\subset q(S_{ij})_P$ for all $i\neq j$, then $D_P\subset \bigcap_{i\neq j} q(S_{ij})_P=qS_P$. It follows that $D_P$ and $qS_P$ are comparable for every $q\in K$, and hence, by [6], Theorem 7, $D_P\oplus S_P$ is extending whenever $P\supset D$.

*Case* 2. $P\not\supset D$. Then $D_P=S_P=S'_P$ is a rank one discrete valuation ring, and thus $D_P\oplus S_P$ is extending.

2) $\Rightarrow$ 3): From 2), the conductor of $S$ in $S'$ is a product of distinct maximal ideals of $S'$, i.e. $D=\prod_{i=1}^{n}\mathfrak{P}_i=\bigcap_{i=1}^{n}\mathfrak{P}_i$. Let $\mathfrak{P}_i\cap S=:P_i$, it follows that $D=\bigcap_{i=1}^{n}P_i$; and hence $\bigoplus_{i=1}^{n}k_i:=\bigoplus_{i=1}^{n}S/P_i\cong S/\bigcap_{i=1}^{n}P_i=S/D\subset S'/\bigcap_{i=1}^{n}\mathfrak{P}_i\cong\bigoplus_{i=1}^{n}S'/\mathfrak{P}_i$, where $k_i=S/P_i=S/\mathfrak{P}_i\cap S\cong S+\mathfrak{P}_i/\mathfrak{P}_i\subset S'/\mathfrak{P}_i$.

By 2), the $A_{ij}$ are $S'$-modules and $A_{ij}A_{ji}\supset\bigcap_{i=1}^{n}\mathfrak{P}_i$.

3) $\Rightarrow$ 1) Let $I:=\bigcap_{i=1}^{n}\mathfrak{P}_i$. Since $I$ is a nonzero ideal of $S$ which is also an ideal of $L$, the conductor $D$ of $S$ in $L$ is nonzero. Since $S$ is noetherian, we obtain that $L$ is the integral closure of $S$. Since $A_{ij}A_{ji}\supset I$, we get that $A_{ij}A_{ji}$ is a product of distinct maximal ideals of $L$.

From $S/I=\bigoplus_{i=1}^{n}k_i\subset\bigoplus_{i=1}^{n}L/\mathfrak{P}_i=L/I$, we see that the maximal ideals of $S$ and of $L$ containing $I$ are in one-to-one correspondence. On the other hand, for any maximal ideal $P$ of $S$ not containing $I$, $S_P=L_P$ is a rank one discrete valuation ring. Therefore we obtain a one-to-one correspondence between all the maximal ideals of $L$ and of $S$. Since the $A_{ij}$ are $L$-modules, it follows that $O(A_{ij})=L$ for all $i\neq j$. Since $S_{ij}\supset S$, Condition 2) of Theorem 5 is satisfied for all $i\neq j$, and hence $M_i\oplus M_j$ is extending. Therefore $M$ is extending, by [6] Theorem 11.

REMARKS. (i) We note that Condition 2) of Theorem 6 mainly deals with the relationship between $S$ and its integral closure, and that further data from $M$ enter only in the last sentence. Loosely speaking, this condition says that $S$ is "almost integrally closed" and the $M_i$ are "almost isomorphic".

(ii) Condition 3) is convenient for the construction of examples, starting with an arbitrary Dedekind domain.

(iii) Even if the ring $R$ is noetherian, $S$ need not be, and then Theorem 6 does not apply. However, if $R$ is noetherian of Krull dimension one, then every overring is noetherian of Krull dimension one ([8], Theorem 13), hence in particular $S$ is noetherian and $S'$ is Dedekind ([8], Theorem 96). Thus in this case, the rest of Condition 2) yields a complete characterization of all torsion free reduced extending $R$-modules.

Conversely, again if $R$ is noetherian, and if one of the $M_i$ is finitely generated (and hence all $M_i$ are isomorphic to ideals of $R$, cf. [6], Corollary 8), then it follows that the Krull dimension is one.

**Corollary 7.** *Let $R$ be a noetherian domain. Let $M_1$, $M_2$ be finitely generated torsion free reduced uniform $R$-modules. If $M_1 \oplus M_2$ is extending, then $R$ has Krull dimension one.*

Proof. It is clear that $R \rightarrow S \rightarrow O(M_1) \cong \hom_R(M_1, M_1) \rightarrowtail \hom_R(R^n, M_1) \cong M_1^n$. Since $M_1^n$ is noetherian, it follows that $S$ is noetherian and integral over $R$. Thus $S$ and $R$ have the same Krull dimension.

Now if $M_1 \oplus M_2$ is extending, then, by Theorem 5, $S$ is one dimensional, and therefore so is $R$.

We now prove a generalization of Corollary 4, for arbitrary ideals, in case $R$ is noetherian.

**Corollary 8.** *Let $R$ be a noetherian domain, and $I$ be an ideal of $R$. Then the following statements are equivalent:*
1) *$R \oplus I$ is extending;*
2) *the integral closure $R'$ of $R$ is Dedekind. There is a one-to-one correspondence, via contraction (and extension), between the maximal ideals of $R$ and of $R'$. The conductor $D$ of $R$ in $R'$ is a product of distinct maximal ideals of $R'$. $I$ is an ideal of $R'$;*
3) *$O(I)$ is Dedekind, and $(R:I)I$ is a product of distinct maximal ideals of $O(I)$. $O(I)_P$ is a discrete rank one valuation ring for all maximal ideal $P$ of $R$ containing $(R:I)I$.*

Proof. Corollary 7, and Theorem 5.

## 4. Examples

The first example shows that the condition "if $A_P \cong W \cong B_P$, then $O(A_P)$ is discrete" in Theorem 1 does not follow from the rest of Condition 2).

EXAMPLE 9. Let $V$ be a valuation ring which is not discrete, with maximal ideal $W$, and $V/W = \boldsymbol{Q}$ the field of rational numbers. Choose additive subgroups $M_1$ and $M_2$, $W \subset M_1, M_2 \subset V$, such that $M_1/W$, $M_2/W$ are of incomparable types $\mathfrak{T}_1$ and $\mathfrak{T}_2$, and such that $\mathfrak{T}_1(P)$, $\mathfrak{T}_2(P)$ are not both $\infty$ for any prime number $P$. Then $O(M_1) \cap O(M_2) = S$ is the full inverse image of $\boldsymbol{Z}$ under the natural homomorphism $V \rightarrow V/W = \boldsymbol{Q}$. One can show that $A_P = A = W = B = B_P$, for every maximal ideal $P$ of $S$. Consequently one has $O(A_P) = O(B_P) = O(W) = V$ and $W = A_P B_P$.

Our second example shows that, in contrast to Corollary 3, if $M_1 \oplus M_2$ is extending with $M_1 \ncong M_2$, then neither $O(M_2)$ nor $O(M_1) \cap O(M_2) =: S$ need be Prüfer domains.

EXAMPLE 10. Let $F[[t]]$ be the ring of formal power series over a field $F$. Let $k$ be a proper subfield of $F$. Let $M_1:=tF[[t]]$ and $M_2:=k+tF[[t]]$. By Corollary 4, $M_1 \oplus M_2$ is extending. $O(M_2)=S=k+tF[[t]]$ is local but not a valuation ring, hence not a Prüfer domain.

The following example refers to Theorem 5 (3). It shows that, if the integral closure $S'$ of $S$ is Dedekind, and there is a one-to-one correspondence between the maximal ideals of $S'$ and of $S$, via contraction and extension, then the conductor $D$ of $S$ in $S'$ need not be a product of distinct maximal ideals of $S'$.

EXAMPLE 11. Let $S':=F[t]$ be the polynomial ring over a field $F$. Let $k$ be a proper subfield of $F$ such that $F$ is a finite dimensional over $k$. Let $S:=k+kt+t^2S'$. Then the conductor $D$ of $S$ in $S'$ is $t^2S'$, and hence $S'$ is a maximal equivalent order. Since $S/t^2S' \cong k[t]/t^2k[t]$, we see that $P:=kt+t^2S'$ is the only maximal ideal of $S$ containing $D$. It is easy to show that $PS'=tS'$ and $tS' \cap S=P$. This suffices to establish the one-to-one correspondence, via contraction and extension, between all maximal ideals of $S'$ and of $S$. But $D=t^2S'$ is not a product of distinct maximal ideals of $S'$.

The next example shows that the statement "$A$ and $B$ are $S'$-modules" does not follow from the rest of condition 3) of Theorem 5.

EXAMPLE 12. Let $S':=F[t]$ and let $k$ be as in Example 11. Let $S=k+tS'$. Then $D=tS'$ is the conductor. Let $V$ be a proper $k$-subspace of $F$ such that $\dim_k V \geq 2$, and let $M_1:=Vt+t^2S'$ and $M_2:=S$.

Then $B=M_1$ and $A=(S:B)$. Since $BS'=(Vt+t^2S')\,S'=tS' \subset S$, we have $S' \subset A$. Now let $a \in A$, hence $aB \subset S$. It follows that $at^2S' \subset S$, and thus $at^2 \in D=tS'$. Then $at \in S'$, and therefore $at=x+yt$ with $x \in F$ and $y \in S'$. On the other hand, $atV=(x+yt)\,V \subset S$; it follows that $xV \subset k$. Since $\dim_k V \geq 2$, we obtain that $x=0$ and $at=yt \in tS'$. Therefore $A=S'$ and $AB=tS'=D$. The one-to-one correspondence, via contraction and extension, between the maximal ideals of $S'$ and of $S$ can be easily established. Hence all the conditions of Theorem 5 (3) are satisfied, except that $B$ is not an $S'$-module.

The last example shows that, in contrast to Theorem 5 and Corollary 7, if $S$ is not noetherian, then $S$ need not be of Krull dimension one, and if $R$ is noetherian but the $M_i$ are infinitely generated then $R$ need not be of Krull dimension one.

EXAMPLE 13. Let $R$ be a commutative noetherian domain with quotient field $K$, and with Krull $\dim(R)>1$. There exists a valuation ring $R \subset V \subset K$ such that Krull $\dim(V)>1$; hence $V$ is not noetherian (cf. [9] Chapter V Exercise 3). By Corollary 3, $V \oplus V$ is extending as an $R$-module; and obviously

$S=V$.

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## References


[1] F.W. Anderson and K.R. Fuller: Rings and Categories of Modules, Springer Verlag, New York (1973).

[2] A.W. Chatter and C.R. Hajarnavis: *Rings in which every complement right ideal is a direct summand,* Quart. J. Math. Oxford **28** (1977), 61–80.

[3] M. Harada: *On modules with extending property,* Osaka J. Math. **19** (1982), 203–215.

[4] L. Fuchs: Infinite Abelian Groups, Acad. Press, New York (1973).

[5] M. Harada and K. Oshiro: *On extending property on direct sums of uniform modules,* Osaka J. Math. **18** (1981), 767–785.

[6] M.Kamal and B.J. Müller: *Extending modules over commutative domains,* Osaka J. Math. **25** (1988), 531–538.

[7] M. Kamal and B.J. Müller: *The structure of extending modules over noetherian rings,* Osaka J. Math. **25** (1988), 539–551.

[8] I. Kaplansky: Commutative Rings, Univ. of Chicago Press, Chicago (1974).

[9] M.D. Larsen and P.J. McCarthy, Multiplicative Theory of Ideals, Acad. Press, New York (1971).



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