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THE STRUCTURE OF EXTENDING MODULES OVER NOETHERIAN RINGS

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1. Introduction

A module is said to be extending, if every closed (i.e. complement) submodule is a direct summand. This property is usually denoted by (C_1). It is, obviously, equivalent to the requirement that every submodule is essential in a direct summand; this is the origin of the name "extending". (C_1) is one of the defining conditions of continuous and quasi-continuous modules, which in turn are generalizations of injective modules. Continuous and quasi-continuous modules have been studied in great detail by many people (c.f. Y. Utumi [13], L. Jeremy [5], B.J. Müller and S. Rizvi [8], [9], and K. Oshiro [11], [12]). Although extending modules are far from injective, they behave in some ways very similar to injective modules. For instance, M. Okado [10] proved that, over a right noetherian ring, an extending module is a direct sum of uniform submodules. The aim of this paper is to determine, conversely, when direct sums of uniform modules are extending.

In §2 we study when direct sums of uniform modules over an arbitrary ring are extending. We prove, in this general setting, some preliminary results which are used later in the paper. We also show that a module over a right noetherian ring is extending if and only if it has $(1-C_1)$ and every local direct summand is a direct summand. Consequently a direct sum of uniform modules with local endomorphism rings, over a right noetherian ring, is extending if and only if it is locally semi-T-nilpotent and each pair is extending.

Then we turn to extending modules over commutative noetherian rings. §3 is a reduction to the case of modules with only one associated prime. In §4 we give a full characterization of extending modules which are direct sums of uniform modules with the same associated prime, and with local endomorphism rings. Finally we describe the structure of extending torsion modules over Dedekind domains.

DEFINITIONS and NOTATIONS: A family of submodules of a module M, whose sum in M is direct, is called a local direct summand if every finite subsum is a direct summand of M. A decomposition $\bigoplus_{i\in I} M_i$ of modules is called locally semi-T-nilpotent, if for every sequence $f_n\colon M_{i_n}\to M_{i_{n+1}}(n\in \boldsymbol{N})$ of non-isomorphisms, with all i_n distinct, and every $x\in M_{i_0}$, there exists $m\in \boldsymbol{N}$ with $f_m f_{m-1}\cdots f_0(x)=0$.

A submodule N of a module M is called closed in M, if it has no proper essential extensions in M. $X\subset' M$ and $Y\subset^{\oplus} M$ signify that X is an essential submodule, and Y is a direct summand, of M. The injective hull of a module M will be denoted by $E(M)$. The set of all associated primes of a module M will be denoted by ass(M).

A module M is called quasi-continuous if it is extending and has the following property (C_3): for all $X, Y\subset^{\oplus} M$ with $X\cap Y=0$, one has $X\oplus Y\subset^{\oplus} M$. The property $(n-C_1)$ is the special case of (C_1), which requires that every closed submodule of uniform dimension less than or equal n is a direct summand.

2. Direct sums of uniform modules

Lemma 1. *If M is an extending module [has $(n-C_1)$] then every direct summand of M is extending [has $(n-C_1)$].*

Lemma 2 ([10], Theorem 4). *A ring R is right noetherian if and only if every extending ,ight R-module is a direct sum of uniform submodules.*

Proposition 3. *Let $M=\bigoplus_{i\in I} M_i$, where the M_i are uniform. If M has $(1-C_1)$, then every closed submodule of the form $\bigoplus_{i=1}^{n} A_i$, with all A_i uniform, is a direct summand of M.*

Proof. By induction on n. Assume that the claim holds true for n, and let $A=\bigoplus_{i=0}^{n} A_i$ be a closed submodule of M, with all A_i uniform. By induction, $A'=:\bigoplus_{i=1}^{n} A_i$ is a direct summand. Write $M=A'\oplus X$.

Now let π be the projection of M onto X. Then $\pi A_0\cong A_0$ is uniform, and hence, by $(1-C_1)$ for X, $\pi A_0\subset' B\subset^{\oplus} X$. It follows that $A'\oplus A_0\subset \pi^{-1}B\subset A'\oplus B$. Since B is uniform, we obtain $A=A'\oplus A_0\subset' A'\oplus B$. Since A is closed, we have $A=A'\oplus B\subset^{\oplus} M$.

Lemma 4. *Let $\phi\colon E(M)\to E(N)$ be an arbitrary homomorphism, and let $X=\{x\in M\colon \phi(x)\in N\}$. If there exists a homomorphism $\psi\colon Y\to N$, $X\subset Y\subset M$, such that $\psi(x)=\phi(x)$ for all $x\in X$, then $X=Y$. Moreover the submodule $B=\{x+\phi(x)\colon x\in X\}$ of $M^{\oplus}N$ is closed.*

Proof. If $(\phi-\psi)\,Y\neq 0$, then $(\phi-\psi)\,Y\cap N\neq 0$. Hence $n=(\phi-\psi)\,(y)$ for some $0\neq n\in N, y\in Y$. Then $\phi(y)=n+\psi(y)\in N$, and hence $y\in X$. Therefore $(\phi-\psi)\,(y)=0$, which is a contradiction.

Now assume $B\subset' B^*\subset M\oplus N$. Since $B\cap N=0$, we have $B^*\cap N=0$. Let

π, π' be the projections of $M\oplus N$ onto M and N respectively; it follows that $\pi|B^*$ is a monomorphism. Let $f:=\pi'\pi^{-1}|B^*: \pi(B^*)\to N$. Note that $X=\pi B\subset\pi B^*$, and that for each $x\in X$, $f(x)=\pi'\pi^{-1}(x)=\pi'(x+\phi(x))=\phi(x)$. Hence by the first part of the lemma, $X=\pi B^*$. Now let $b^*\in B^*$ be arbitrary; we have $b^*=\pi b^*+\pi' b^*=\pi b^*+\pi'\pi^{-1}\pi(b^*)=\pi b^*+f(\pi b^*)=\pi b^*+\phi(\pi b^*)\in B$. Hence $B=B^*$.

Lemma 5 ([9], Lemma 5). *Let $M=\bigoplus_{i\in I} M_i$ with all M_i uniform. Suppose $X\subset M$ with $X\cap\bigoplus_{j\in J} M_j=0$, for some $J\subset I$. Then there exists $J\subset K\subset I$ with $X\oplus\bigoplus_{i\in K} M_i\subset' M$.*

Proposition 6. *Let $M=\bigoplus_{i\in I} M_i$ with all M_i uniform. If M has $(1-C_1)$, then every non-zero closed submodule of M contains a uniform summand of M.*

Proof. Let M have (C_1), and let A be a closed submodule of M. By Lemma 5, we have $A\oplus\bigoplus_{j\in J} M_j\subset' M$ for some $J\subset I$. Let $K=I\setminus J$, and let π_K, π_J be the projections onto $\bigoplus_{i\in K} M_i$ and $\bigoplus_{j\in J} M_j$ respectively. Then $\pi_{K|A}$ is a monomorphism. Let $\phi:=\pi_J\pi_{K|A}^{-1}$. It is easy to see that $A=\{b+\phi(b): b\in\pi_K(A)\}$ and that $\phi:\pi_K(A)\to\bigoplus_{j\in J} M_j$ is not extendable (i.e. if $\psi: Y\to\bigoplus_{j\in J} M_j$, $\pi_K(A)\subset Y\subset\bigoplus_{i\in K} M_i$ extends ϕ, then $\pi_K(A)=Y$).

Now let $\hat{\phi}: E(\bigoplus_K M_i)\to E(\bigoplus_J M_j)$ be an extension of ϕ, it follows that $\pi_K(A)=\{x\in\bigoplus_K M_i: \hat{\phi}(x)\in\bigoplus_J M_i\}$. For each $\alpha\in K$, let $X_\alpha=:\{x\in M_\alpha: \hat{\phi}(x_\alpha)\in\bigoplus_J M_j\}$ and $A_\alpha=\{x+\hat{\phi}(x): x\in X_\alpha\}$. By Lemma 4, A_α is closed in $M_\alpha\oplus\bigoplus_J M_j$. It is clear that $X_\alpha\cong A_\alpha$, and hence A_α is a uniform submodule of A. By $(1-C_1)$, $A_\alpha\subset^\oplus M$, and therefore $A_\alpha\subset^\oplus A$.

Corollary 7. *Let $M=\bigoplus_{i\in I} M_i$, with all M_i uniform. If M has $(1-C_1)$, then M has $(n-C_1)$.*

Proof: Propositions 6 and 3.

Theorem 8. *Let M be a module over a right noetherian ring R. Then M is an extending module if and only if M is a direct sum of uniform submodules, has $(1-C_1)$, and every local direct summand of M is a direct summand.*

Proof. Let M be extending. Obviously M has $(1-C_1)$. Now let $U=\bigoplus_{j\in J} U_j$ be a local direct summand of M. By (C_1), $U\subset^\oplus M$ holds once we show that U is closed in M. To this end let $U\subset' N\subset M$, and let $0\neq x\in N$. Consider $I_x=:\{r\in R: xr\in U\}$. As I_x is finitely generated, there exists a finite subset F of J such that $xI_x\subset\bigoplus_{j\in F} U_j:=V$. Since $\bigoplus_{j\in J} U_j$ is a local direct summand, we have $V\subset^\oplus M$, hence V is closed in M. Consider now any $0\neq xr+v$, where $r\in R$ and

$v\in V$. Certainly there exists $s\in R$ such that $0\neq(xr+v)s=u\in U$. Consequently we have $xrs=u-vs\in U$. We conclude $0\neq(xr+v)s\in V$. This shows that $xR+V'\supset V$, and hence $x\in xR+V=V\subset U$. Therefore $U=N$.

Conversely, let $M=\bigoplus_{i\in I}M_i$, with all M_i uniform, have $(1-C_1)$, and let every local direct summand be a direct summand. Let A be a closed submodule of M. By Zorn's Lemma, we can find a maximal member $\bigoplus_{\alpha\in K}A_\alpha$ of the family of submodules of A of the form $\bigoplus_{\alpha\in L}N_\alpha$ such that all N_α are uniform and that $\bigoplus_{\alpha\in L}N_\alpha$ is a local direct summand of M. By assumption $\bigoplus_{\alpha\in K}A_\alpha\subset^\oplus M$, and hence $A=\bigoplus_{\alpha\in K}A_\alpha\oplus A'$. If A' is not zero, then, by Proposition 6, A' contains a uniform direct summand, which contradicts the maximality of $\bigoplus_{\alpha\in K}A_\alpha$. Therefore $A=\bigoplus_{\alpha\in K}A_\alpha\subset^\oplus M$.

Lemma 9. *Let $M=\bigoplus_{i\in I}M_i$ with all M_i uniform. Then a submodule A of M is uniform and closed in M if and only if $A=\{\sum_{i\in I}\phi_i(b): b\in B\subset M_k\}$ for some $k\in I$, where $(\phi_i: B\rightarrow M_i)_{i\in I}$ are homomorphisms such that $\phi_k(b)=b$ for all $b\in B$, and the ϕ_i are not simultaneously extendable* (i.e. *if $\psi_i: B_i\rightarrow M_i$ extends ϕ_i, $B\subset B_i\subset M_k$, for each i, then $B=\bigcap_{i\in I}B_i$*).

Proof. Let A be a uniform closed submodule of M. Let π_i be the projection onto M_i; $i\in I$. Since A is uniform, we have $A\cap\ker\pi_k=0$ for some $k\in I$, and hence $\pi_{k|A}$ is a monomorphism. For each $i\in I$, we have that $\pi_i(\pi_{k|A})^{-1}$ is a well defined homomorphism from $\pi_k(A)$ into M_i; we denote it by ϕ_i. Observe that $\phi_k(x)=x$ for all $x\in\pi_h(A)$. It is easy to see that $A=\{\sum_{i\in I}\phi_i(x): x\in\pi_k(A)\}$.

Now let $\psi_i: B_i\rightarrow M_i$ extend ϕ_i for each $i\in I$, where $B\subset B_i\subset M_k$. It follows that $A\subset'\{\sum_{i\in I}\psi_i(y): y\in\bigcap_{i\in I}B_i\}\subset M$. Since A is closed, we obtain $\pi_k(A)=\bigcap_{i\in I}B_i$.

Conversely, let $X=\{\sum_{i\in I}f_i(b): b\in B\subset M_k\}$, where the $(f_i: B\rightarrow M_i)_{i\in I}$ are not simultaneously extendable, and $f_k(b)=b$ for all $b\in B$. It is clear that $X\cong B$, and hence X is uniform. To show that X is closed in M, let $X\subset'X^*\subset M$. It is clear that $X\cap\ker\pi_k=0$, and thus $X^*\cap\ker\pi_k=0$. Hence $B=\pi_k(X)\subset\pi_k(X^*)\subset M_k$, and $\pi_i\pi_k^{-1}|\pi_k(x^*)$ extends f_i for each $i\in I$. Then $B=\pi_k(X^*)$ and therefore $X=X^*$.

Lemma 10. *Let $M=\bigoplus_{i\in I}M_i$ with all M_i uniform and with end(M_i) local. Suppose that $M_i\oplus M_j$ has (C_1) for all $i\neq j\in I$. Let $E(M_i)\xrightarrow{f}E(M_j)$ be given, $i\neq j\in I$, and let $X=M_i\cap f^{-1}(M_j)$. Then $X=M_i$, or $f|X: X\rightarrow M_j$ is an isomorphism (and thus $f^{-1}(M_j)\subset M_i$).*

Proof. Let $X^*=:\{x-f(x); x\in X\}$. By Lemma 4, X^* is closed in $M_i\oplus$

M_j. By (C_1) and since end(M_k) is local for all k, X^* has either M_i or M_j as complementary summand in $M_i \oplus M_j$.

Now if $M_i \oplus M_j = X^* \oplus M_j$, then $\pi(x) = f(x)$ for all $x \in X$, where $\pi: X^* \oplus M_j \to M_j$ is the projection. Hence, by Lemma 4, $M_i = X$. On the other hand if $M_i \oplus M_j = X^* \oplus M_i$, then $f|X: X \to M_j$ is an isomorphism, and hence $f^{-1}(M_j) = X \subset M_i$.

Lemma 11. *Let* $M = \bigoplus_{i \in I} M_i$ *with all* M_i *uniform and with end*(M_i) *local. Then* M *has* $(1-C_1)$ *if and only if* $M_i \oplus M_j$ *has* (C_1) *for all* $i \neq j \in I$.

Proof. Let $M_i \oplus M_j$ have (C_1) for all $i \neq j$. Let A be a uniform and closed submodule of M. By Lemma 9, $A = \{\sum_{i \in I} \phi_i(x) : x \in X \subset M_k\}$ for some $k \in I$, where $(\phi_i: X \to M_i)_{i \in I}$ are homomorphisms with $\phi_k(x) = x$ for all $x \in X$, and $(\phi_i)_{i \in I}$ are not simultaneously extendable. Observe that all but a finite number of ϕ_i are non-monomorphisms (due to $X \neq 0$ and $\phi_i(x) = 0$ for all but a finite number of indices, for any $0 \neq x \in X$).

Let $\hat{\phi}_i: E(M_k) \to E(M_i)$ be extensions of ϕ_i, $i \in I$. Let $X_i =: \{m \in M_k : \hat{\phi}_i(m) \in M_i\}$, it follows that $X = \bigcap_{i \in I} X_i$. Then $X = \bigcap_{i \in F} X_i$ where $F = \{i : X_i \subsetneq M_k\}$. By Lemma 10, the $\hat{\phi}_i$ are isomorphisms for all $i \in F$, and hence F is finite. Again by Lemma 10 (taking $f = \hat{\phi}_j \hat{\phi}_i^{-1}$), it follows that $X_i (= \hat{\phi}_i^{-1}(M_i))$ and $X_j (= \hat{\phi}_j^{-1}(M_j))$ are comparable for all $i \neq j \in F$. Hence $\{X_i\}_{i \in F}$ forms a finite chain. For the smallest element X_α of this chain, one obtains $X = X_\alpha$ and hence $A \oplus \bigoplus_{i \neq \alpha} M_i = M$.

The converse is obvious.

Theorem 12. *Let* $M = \bigoplus_{i \in I} M_i$ *be a module over a right noetherian ring* R, *where the* M_i *are uniform with local endomorphism rings. Then* M *is extending if and only if the decomposition* $\bigoplus_{i \in I} M_i$ *is locally semi-T-nilpotent and* $M_i \oplus M_j$ *is extending for all* $i \neq j \in I$.

Proof: By Theorem 8 ([3], page 172) and Lemma 11.

Lemma 13. *Let* $M = M_1 \oplus M_2$ *be a module with* M_i *uniform and with end*(M_i) *local. If* M *has* (C_1) *and* M_i *cannot be embedded in* M_j *for some* $i \neq j$, *then* M_i *is* M_j*-injective.*

Proof. Let M have (C_1), and let M_1 not be embeddable in M_2. Let $\phi: X \to M_1$ be an arbitrary homomorphism, $X \subset M_2$. Consider $X' =: \{x - \phi(x) : x \in X\}$, and let X^* be a maximal essential extension of X' in M. Then $X^* \cap M_1 = 0$, and hence $\pi_2 | X^*$ is a monomorphism, where π_2 is the projection of M onto M_2. By (C_1) and since end(M_i) is local, we have $M = X^* \oplus M_1$ (if $M = X^* \oplus M_2$, then $M_1 \cong X^* \overset{\pi_2}{\rightarrowtail} M_2$ which contradicts the assumption that M_1 cannot be embedded in M_2). Therefore $\pi|_{M_2}$ extends ϕ, where $\pi\colon X^*\oplus M_1\rightarrow M_1$ is the projection.

Corollary 14. *Let $M=M_1\oplus M_2$ be a module with M_i uniform and with end(M_i) local. Let $E(M_1)\cong E(M_2)$. If M has (C_1), then M_i can be embedded in M_j for some $i\neq j$.*

Proof. If M_1 cannot be embedded in M_2, then M_1 is M_2-injective, and hence for any isomorphism $\phi\in\hom_R(E(M_2), E(M_1))$, $\phi|M_2$ is an embedding of M_2 into M_1.

3. Extending modules over commutative noetherian rings

Lemma 15. *Let M_1 and M_2 be uniform modules over a commutative noetherian R. Let ass$(M_i)=P_i$. If $P_i\not\subset P_j$, then $\hom_R(E(M_i), E(M_j))=0$.*

Proof. By [7], Theorem 3.4.

Lemma 16. *Let $M=M_1\oplus M_2$, with M_i uniform, be a module over a commutative noetherian ring R. Let ass$(M_i)=P_i$, where $P_1\neq P_2$. Then M has (C_1) if and only if M_i is M_j-injective, $i\neq j(=1, 2)$.*

Proof. Let M have (C_1). If $P_1\not\subset P_2$, then $\hom_R(E(M_1), E(M_2))=0$ and hence M_2 is M_1-injective.

Now let $P_1\subsetneqq P_2$ and let $\phi\in\hom_R(E(M_1), E(M_2))$ be arbitrary. By Lemma 4, the submodule $B:=\{x+\phi(x)\colon x\in X\}$ of M is closed, where $X=\{x\in M_1\colon \phi(x)\in M_2\}$. By (C_1), $M=B\oplus N$. Since $B\cong X\subset M_1$, it follows that ass$(B)=P_1$ and ass$(N)=P_2$. Since $\hom_R(E(N), E(M_1))=0$, we have $\pi_1(N)=0$, and hence $N=M_2$, where π_1 is the projection of $M_1\oplus M_2$ onto M_1. Let $\pi\colon B\oplus M_2\rightarrow M_2$ be the projection, it follows that $\pi(x)=\phi(x)$ for all $x\in X$. By Lemma 4, we have $X=M_1$, and hence $\phi(M_1)\subset M_2$ for all $\phi\in\hom_R(E(M_1), E(M_2))$. Therefore M_2 is M_1-injective.

Conversely, let M_i be M_j-injective. Since every uniform module is quasi-continuous, it follows by [8] Theorem 12, that M is quasi-continuous hence has (C_1).

Lemma 17. *Let $M=X\oplus Y$ be a module over an arbitrary ring, where Y is X-injective. Let N be a submodule of M with $N\cap Y=0$. Then there exists a homomorphism $f\colon X\rightarrow Y$ such that $N\subset X^*:=\{x+f(x)\colon x\in X\}\cong X$, and that $M=X^*\oplus Y$.*

Proof. Let π_X, π_Y be the projections onto X and Y respectively. Since Y is X-injective and π_X is a monomorphism on N, there exists $f\colon X\rightarrow Y$ such that $f\pi_X(n)=\pi_Y(n)$ for all $n\in N$. Let $X^*=:\{x+f(x)\colon x\in X\}$. It is clear that $X\cong X^*$, and that $M=X^*\oplus Y$. Now $N\subset\pi_X(N)+\pi_Y(N)=\pi_X(N)+f\pi_X(N)\subset X^*$.

Theorem 18. *A module M over a commutative noetherian ring R is extending if and only if $M=\bigoplus_P M(P)$ (unique up to isomorphism) where $M(P)$ has associated prime P, is extending, and is $M(Q)$-injective for all $P\neq Q$.*

Proof. Let M be extending. By Lemma 2, $M=\bigoplus_{i\in I} M_i$ with all M_i uniform. Let $M(P)=\oplus\{M_i: \mathrm{ass}(M_i)=P\}$. By Lemma 1, we have that $M(P)$ is extending for all P. By Lemma 15 and since $M_i\oplus M_j$ has (C_1), we have that M_i is M_j-injective whenever $\mathrm{ass}(M_i)\neq\mathrm{ass}(M_j)$. Since R is noetherian, by [2] Theorem 2.5, it follows that $M(P)$ is $M(Q)$-injective for all $P\neq Q$; furthermore $M(P)$ is $\bigoplus_{Q\neq P} M(Q)$-injective.

Uniqueness: Let $\bigoplus_P M(P)=\bigoplus_P N(P)$. It is clear that $N(P)\cap\bigoplus_{P\neq Q} M(Q)=0$. Since $M(P)$ is $\bigoplus_{Q\neq P} M(Q)$-injective, by Lemma 17, we have $N(P)\subset M^*(P)\cong M(P)$. By the modular law and since $M^*(P)\cap\bigoplus_{Q\neq P} N(Q)=0$, we get $N(P)=M^*(P)\cong M(P)$ for all P.

Conversely, let $M=\bigoplus_P M(P)$ where $M(P)$ is extending and is $M(Q)$-injective for all $P\neq Q\in\mathrm{ass}(M)$.

First we show that M has $(1-C_1)$. Let N be a uniform and closed submodule of M. Let $\mathrm{ass}(N)=P$; it is clear that $N\cap\bigoplus_{Q\neq P} M(Q)=0$. By Lemma 17, there exists a submodule $M^*(P)$ of M such that $N\subset M^*(P)\cong M(P)$ and $M=M^*(P)\oplus\bigoplus_{Q\neq P} M(Q)$. By (C_1) for $M^*(P)$, we have $N\subset^{\oplus} M^*(P)\subset^{\oplus} M$.

Secondly we show that every direct sum of uniform submodules, which is a local direct summand, is a direct summand. Let $U=\bigoplus_J U_j$ with all U_j uniform be a local direct summand of M. Let $U(P)=:\oplus\{U_j: \mathrm{ass}(U_j)=P\}$. Then $U(P)\cap\bigoplus_{Q\neq P} M(Q)=0$, and hence by Lemma 17, $U(P)\subset M^*(P)=\{x+f_P(x): x\in M(P)\}\cong M(P)$ for each P, where $f_P: M(P)\to\bigoplus_{Q\neq P} M(Q)$. By (C_1) for $M^*(P)$ and since $U(P)$ is a local direct summand of $M^*(P)$, by Theorem 8, we have $U(P)\subset^{\oplus} M^*(P)$ for all P. We show that $\sum_P M^*(P)$ is direct. Suppose $x_1^*+x_2^*+\cdots+x_n^*=0$ with $0\neq x_i^*=x_i+f_{P_i}(x_i)\in M^*(P_i)$, $x_i\in M(P_i)$. Let P_j be a minimal member of $\{P_i\}_{i=1}^n$. It follows that $-x_j=f_{P_j}(x_j)+\sum_{i\neq j}^{n}(x_i+f_{P_i}(x_i))\in M(P_j)\cap\bigoplus_{Q\neq P_j} M(Q)=0$ hence $x_j^*=0$, which is a contradiction. Therefore $U=\bigoplus_P U(P)\subset^{\oplus}\bigoplus_P M^*(P)\subset M$.

We claim that $M=\bigoplus_P M^*(P)$. Let $\mathcal{P}=\mathrm{ass}(M)$, and define inductively $\mathcal{P}_\alpha=\bigcup_{\beta<\alpha}\mathcal{L}_\beta$, and $\mathcal{L}_\alpha$ the set of maximal members of $\mathcal{P}\setminus\mathcal{P}_\alpha$, for all ordinals α. Then $\mathcal{P}_{\alpha+1}=\mathcal{P}_\alpha\cup\mathcal{L}_\alpha$, and $\mathcal{P}_\lambda=\bigcup_{\alpha<\lambda}\mathcal{P}_\alpha$ for limit ordinals λ. By transfinite induction we show $\bigoplus_{P\in\mathcal{P}_\alpha} M(P)\subset\bigoplus_{P\in\mathcal{P}_\alpha} M^*(P)$. The case of a limit ordinal is obvious. For a non-limit ordinal $\alpha+1$, if $Q\in\mathcal{L}_\alpha=\mathcal{P}_{\alpha+1}\setminus\mathcal{P}_\alpha$ and $x\in M(Q)$, then $f_Q(x)\in\bigoplus_{P\in\mathcal{P}_\alpha} M(P)\subset\bigoplus_{P\in\mathcal{P}_\alpha} M^*(P)$, and therefore $x=x+f_Q(x)-f_Q(x)\in M^*(Q)\oplus\bigoplus_{P\in\mathcal{P}_\alpha} M^*(P)\subset\bigoplus_{P\in\mathcal{P}_{\alpha+1}} M^*(P)$.

Now let A be a non-zero closed submodule of M. By Zorn's Lemma, we can find a maximal submodule $\bigoplus_{\alpha\in K} A_\alpha$ of A with the property that $\bigoplus_{\alpha\in K} A_\alpha$ is a local direct summand in M and that A_α are uniform. By the second part of the proof, $\bigoplus_{\alpha\in K} A_\alpha\subset^\oplus M$ hence $A=\bigoplus_{\alpha\in K} A_\alpha\oplus A'$. If $A'\neq 0$, then, by Proposition 6, and since M has $(1-C_1)$, A' contains a uniform summand U of A'. By Proposition 3, $\bigoplus_{\alpha\in K} A_\alpha\oplus U$ is a local direct summand in M, which contradicts the maximality of $\bigoplus_{\alpha\in K} A_\alpha$. Therefore $A=\bigoplus_{\alpha\in K} A_\alpha\subset^\oplus M$.

REMARKS. 1) In order to test the relative injectivity between $M(P)$ and $M(Q)$, in Theorem 18, it suffices to check that $M(P)$ is $M(Q)$-injective whenever $P\subsetneq Q$. The latter can be done by checking that M_i is M_j-injective for the uniform direct summands of $M(P)=\oplus M_i$ and $M(Q)=\oplus M_j$ (cf. Lemma 15 and [2], Theorem 2.5).

2) If $M(P)$ is uniform for all P, then, by [8], Theorem 12, $\bigoplus_P M(P)$ is extending if and only if it is quasi-continuous.

3) In contrast to quasi-continuous modules, if a module $M=\bigoplus_{i\in I} M_i$, with all M_i uniform, is extending, then the decomposition $\bigoplus_{i\in I} M_i$ need not be unique up to isomorphism. For example, let $R=\boldsymbol{Z}[\rho]$, $\rho=\sqrt{-5}$. For any two ideals I and J of R with $I+J=R$ one can show that $I\oplus J\cong R\oplus I\cap J$ (see [1] Exercise 4). Now let $I=:\langle 3, 2+\rho\rangle$ and $J=:\langle 3, 2-\rho\rangle$. I and J are not principal, but $I\cap J=3R\cong R$. Hence $R\oplus R\cong I\oplus J$. As R is Dedekind, $R\oplus R$ is extending, by [6]. (This can also be verified directly).

4. Direct sums of uniform modules with local endomorphism rings and with the same associated prime, over commutative noetherian rings

Lemma 19 [7]. *Let R be a commutative noetherian ring, and $E\cong E(R/P)$ be an indecomposable injective R-module. Then E is an R_P-as well as an $\hat{R}_P$-module, where $\hat{R}_P$ is the completion of R_P. Furthermore $\hat{R}_P\cong end_R(E)$.*

In the situation of Corollary 14 and Lemma 19, we introduce now some notations: We can assume, without loss of generality, that $M_1\subset M_2\subset E(R/P)$, where P is the associated prime of M_i, and that $\mathrm{end}_R(E)=\hat{R}_P$.

If it happens that $M_iPR_P\subset M_j (i\neq j=1, 2)$, then we write $A=\{x\in R_P: xM_1\subset M_2\}/PR_P$ and $B=\{x\in R_P: xM_2\subset M_1\}/PR_P$ and $S=\{x\in R_P: xM_i\subset M_i\}/PR_P$.

In this case A and B are S-submodules of $R_P/PR_P=:K$ (the quotient field of R/P).

For any S-submodule L of K, we denote $\{x\in K: xL\subset L\}$ by $O(L)$ and $\{x\in K: xL\subset S\}$ by $(S:L)$. $O(L)$ is an overring of S and $(S:L)$ is an S-submodule of K. If $L\neq 0$, then $O(L)\cong \mathrm{end}_S(L)$.

Theorem 20. *Let $M=M_1\oplus M_2$ with end(M_i) local, be a module over a commutative noetherian ring R. Let $M_1\subset M_2\subset E(R/P)$. Then the following are equivalent*:

1) *M is an extending module,*
2) *$M_2PR_P\subset M_1$, $O(A)=O(B)=:O$ is a valuation ring with maximal ideal $W\subset AB$. If $A\cong B\cong W$, then O is discrete. [The condition $W\subset AB$ means $AB=W$ or $AB=O$; the latter holds precisely if $M_1\cong M_2$.]*

Proof. 1)$\Rightarrow$2): By Lemma 10, $M_2PR_P=M_2P\hat{R}_P\subset M_1$ and xM_1, M_2 are comparable for all $x\in\hat{R}_P/P\hat{R}_P$, and hence $x\in A$ or $x^{-1}\in B$ for all $x\in\hat{R}_P/P\hat{R}_P\cong R_P/PR_P=K$.

If $B=O$, then $A=K$ and 2) follows. Now assume $B\neq O$. We show that $O(A)$ and $O(B)$ are valuations rings. Let $y\in K$, $y\notin A$; then $y^{-1}\in B$ and hence $y^{-1}A\subset S$ (due to $AB\subset S$); hence $A\subset yS$. This shows that A is comparable with all S-submodules of K, and thus $O(A)$ is a valuation ring. Similarly we can show that $O(B)$ is a valuation ring. We prove the rest of condition 2) in all the different cases which can occur.

Case 1. $AB=S$. Since AB is an ideal of $O(A)$ and $O(B)$, it follows that $O(A)=S=O(B)$. Since A, B are invertible, as fractional ideals of the valuation ring S, they are principal. Hence if $A\cong B\cong W$, then S is discrete.

Case 2. $AB\subsetneq S$. Claim: $A\subsetneq(S:B)$ if and only if B is principal. If $x\in(S:B)\backslash A$, then $x^{-1}\in B$ and hence $B=x^{-1}S$. Conversely, if B is principal, then $B(S:B)=S$, and hence $A\subsetneq(S:B)$. (Similarly we can show $B\subsetneq(S:A)$ if and only if A is principal).

Subcase 2*a*. A or B is principal as S-module. Let B be principal. Then $O(B)=S$ is a valuation ring. By the claim, $A\subsetneq(S:B)$. It is clear that $(S:B)=yS$ for any $y\in(S:B)\backslash A$, and hence $(S:B)/A$ is a simple S-module. Since $AB\subset W$ and B is principal, we have $A\subset W(S:B)\subsetneq(S:B)$. Therefore $A=W(S:B)$ and thus $AB=W$.

To verify condition 2) in this subcase, it remains to show that $O(A)\subset O(B)$ (since $O(B)=S\subset O(A)$). If A is also principal, then $O(A)=S=O(B)$. On the other hand if A is not principal as S-module, then $B=(S:A)$. Now let $x\in O(A)$ be arbitrary. We have $xbA\subset bA\subset S$ for all $b\in B$. Hence $xB\subset(S:A)=B$, i.e. $x\in O(B)$. Therefore $O(A)\subset O(B)$.

A similar argument works if A is principal as S-module.

Subcase 2*b*. A and B are not principal as S-modules. Then $A=(S:B)$ and

$B=(S:A)$. By the same argument as in Subcase 2a, we can show that $O(A)=O(B)=:O$. Now let W be the maximal ideal of O, and let $x\in W$ be arbitrary. Since $x^{-1}\notin C$, it follows that $B\subsetneqq x^{-1}B$. Hence $x^{-1}b\notin B=(S:A)$ for some $b\in B$. Thus $S\subset x^{-1}bA$ and hence $x\in xS\subset bA\subset AB$. Therefore $W\subset AB$. It is clear that $AB\subset W$, and hence $AB=W$.

Now if $A\cong B\equiv W$, then $yA=W$ for some $0\neq y\in K$. Since $W=AB\subset S$, we have $y\in(S:A)=B$, i.e. $yO\subset B$. On the other hand $A=y^{-1}W=y^{-1}AB$, and hence $y^{-1}B\subset O(A)=O$. Therefore $B=yO$ and thus O is discrete.

2)$\Rightarrow$1) We first show that $q\in A$ or $q^{-1}\in B$ for all $q\in K$.

Case 1. $AB=S$. Then $S=O$ is a valuation ring, and hence $A=(S:B)$. It follows that $q\in A$ or $q^{-1}\in B$ for all $q\in K$.

Case 2. $AB\subsetneqq S$ and A or B is principal as S-module. If, for instance, B is principal, we have that $S=O$ is a valuation ring and $AB=W$ is the maximal ideal of S. Hence $W(S:B)=A$ is the unique maximal S-submodule of $(S:B)$. It follows that $(S:B)/A$ is a simple S-module. Now if $q\notin A$ and $q\in(S:B)$, then $A\subsetneqq qS\subset(S:B)$. Thus $(S:B)=qS$, and hence $q^{-1}\in B$. On the other hand it is clear that $q\notin(S:B)$ implies $q^{-1}\in B$.

Case 3. $AB\subsetneqq S$ and both A and B are not principal as S-module. We show that A is comparable with all S-submodules of K. Let $q\notin A$; then $A\subsetneqq qQ$. It follows that $q^{-1}A\subset W=AB\subset S$ and hence $A\subset qS$. Similarly we can show that B is comparable with all S-submodule of K.

Claim: If $A\subsetneqq(S:B)$, then $A\cong W\cong B$ and O is not discrete. Let $x\in(S:B)\backslash A$, we have $xB\subsetneqq S$, and hence $xB\subset W$. On the other hand $A\subsetneqq xS$, therefore $W=AB\subset xB$. Then $W=xB$. Certainly $x^{-1}\in(S:A)\backslash B$. By the same argument we can show that $x^{-1}A=W$.

Now if O is discrete, then B is a principal O-module, and hence $xB=W=AB$ yields $x\in xO\subset A$, which contradicts the choice of x.

By the claim and condition 2), we have $A=(S:B)$. Therefore $q\in A$ or $q^{-1}\in B$ for all $q\in K$.

These three cases together show that $q\in A$ or $q^{-1}\in B$ for all $q\in K$. Now let N be a closed uniform submodule of M. Without loss of generality assume that $N=\{y+\theta(y): y\in Y\subset M_1\}$ where $\theta: Y\rightarrow M_2$ is a non-extendable homomorphism. If $Y=M_1$, then $M=N\oplus M_2$. On the other hand if $Y\subsetneqq M$, then, by Lemma 4 and since $P\hat{R}_PM_1=PR_PM_1\subset M_2$, we obtain $\hat{\theta}\in\hat{R}_P\backslash P\hat{R}_P$, where $\hat{\theta}$ is an extension of θ to $\text{end}(E(M_i))=\hat{R}_P$. Hence $\bar{\hat{\theta}}\in A$ or $\bar{\hat{\theta}}^{-1}\in B, \bar{\hat{\theta}}\in K$. Then $\hat{\theta}M_1\subset M_2$ or $\hat{\theta}M_1\supset M_2$. Again by Lemma 4 and since $Y\subsetneqq M_1$, we have $M_2\subset\hat{\theta}M_1$. Therefore $Y=\hat{\theta}^{-1}(M_2)\cap M_1=\hat{\theta}^{-1}(M_2)$. Hence $\hat{\theta}(Y)=\theta(Y)=M_2$, and thus $M=N\oplus M_1$.

Corollary 21. *Let $M=N\oplus N$ with end(N) local and with $N\subset E(R/P)$. Then M is extending if and only if $NPR_P\subset N$ and $S=:\{x\in R_P: xN\subset N\}/PR_P$ is a valuation ring.*

Theorem 22. *Let M be a direct sum of uniform R-submodules with local endomorphism rings, and with the same associated prime P, where R is a commutative noetherian ring. Then the following are equivalent:*

1) *M is extending,*

2) *$M\cong \bigoplus_{i=1}^{n} M_i^{(\alpha_i)}$ where the M_i are pairwise non-isomorphic, and* i) *$PR_PM_n\subset M_1\subset M_2\subset\cdots\subset M_n\subset E(R/P)$,* ii) *$M_i$ is quasi-injective whenever α_i is infinite, and* iii) *each pair in $\bigoplus_{i=1}^{n} M_i^{(\alpha_i)}$ is extending* [(c.f. Theorem 20), *it suffices to check* iii) *if both summands of the pair are not quasi-injective*].

Proof. 1)$\Rightarrow$2): Let $M=\bigoplus_{i\in 1} M_i$ be extending, where the M_i are uniform. Partition the index set I into $I=\bigcup_{i\in F} I_i$, where $M_\alpha\cong M_\beta$ if and only if α, β belong to the same I_i. Hence $M=\bigoplus_{i\in F} M(I_i)$ $(M(I_i)=\bigoplus_{\alpha\in I_i} M_\alpha)$. We claim that F is finite.

Suppose F is infinite. For each $i\in F$ pick a representative $M_i\subset E(R/P)$ from $M(I_i)$ and consider $\bigoplus_{i\in F} M_i$. Since $\bigoplus_{i\in F} M_i$ is extending, by Theorem 12, we have that $\{M_i\}_{i\in F}$ is locally semi-T-nilpotent. Therefore, starting from any $M_\alpha(\alpha\in F)$ there exists a finite sequence of monomorphisms $M_\alpha\rightarrowtail M_{\alpha_1}\rightarrowtail M_{\alpha_2}\cdots\rightarrowtail M_{\alpha_t}$ such that all indices α_k are distinct ($\alpha=\alpha_0$) and that M_{α_t} cannot be embedded into any M_j, for $j\in F^{(1)}=: F\backslash\{\alpha, \alpha_1, \alpha_2, \cdots, \alpha_t\}$. By Lemma 13, M_{α_t} is M_j-injective, and hence $M_{\alpha_t}\supset R_PM_j$. Write $\beta_1=\alpha_t$.

Iterating this procedure we obtain a descending sequence of infinite subsets $F^{(n)}$ of F and of indices $\beta_n\in F^{(n-1)}\backslash F^{(n)}$ such that $M_{\beta_1}\supset R_PM_{\beta_2}\supset M_{\beta_2}\supset\cdots R_PM_{\beta_n}\supset M_{\beta_n}\supset R_PM_{\beta_{n+1}}\supset M_{\beta_{n+1}}\supset R_PM_{\beta_{n+2}}\supset\cdots$. Since $E(R/P)$ is an artinian R_P-module, we have $R_PM_{\beta_n}=R_PM_{\beta_m}$ for some n and for all $m\geq n$. Therefore $M_{\beta_n}=M_{\beta_m}$ for all $m\geq n$, which contradicts the choice of M_{β_n}, and establishes our claim.

Now, using Corollary 14 and Theorem 20, we obtain (up to isomorphism) $M=\bigoplus_{i=1}^{n} M_i^{(\alpha_i)}$, where $\alpha_i=|I_i|$ and $PR_PM_n\subset M_1\subset\cdots\subset M_n\subset E(R/P)$. It remains to show that M_i is quasi-injective whenever α_i is infinite. In this case, by (C_1) for $M_i\oplus M_i$ and by Corollary 21, we have that $PR_PM_i\subset M_i$ and $S_i=: \{x\in R_P: xM_i\subset M_i\}/PR_P$ is a valuation ring. Now let $x\in R_P$ be arbitrary. If $xM_i\not\subset M_i$, then $x\in R_P\backslash PR_P$ and $M_i\subsetneq xM_i$. Therefore $M_i\overset{x^{-1}}{\rightarrowtail} M_i\overset{x^{-1}}{\rightarrowtail} M_i\rightarrowtail\cdots$, which contradicts the locally semi-T-nilpotency of $M_i^{(\alpha_i)}$. Thus $R_PM_i\subset M_i$, i.e., M_i is an R_P-module and hence quasi-injective.

2)$\Rightarrow$1) Let $M=\bigoplus_{i=1}^{n} M_i^{(\alpha_i)}$ with $PR_PM_n\subset M_1\subset\cdots\subset M_n\subset E(R/P)$. It is clear that $PR_PM_i\subset M_j$ for all $i, j\ (=1, 2, \cdots, n)$. By Theorem 20, it is easy to see that $M_i\oplus M_j$ is automatically extending, whenever one of M_i or M_j is quasi-injective. Hence once (iii) holds whenever neither M_i nor M_j are quasi-injective, then every pair in $\bigoplus_{i=1}^{n} M_i^{(\alpha_i)}$ is extending.

To show that M is extending, by Theorem 12, it suffices to verify that $M_i^{(\alpha_i)}$ is locally-semi-T-nilpotent whenever α_i is infinite. Since, by assumption, M_i is quasi-injective, we have that every monomorphism from M_i to M_i is an isomorphism. Now let $M_i \xrightarrow{x_1} M_i \xrightarrow{x_2} M_i \xrightarrow{x_3} \cdots \xrightarrow{x_n} M_i \rightarrow \cdots$ be a sequence of non-monomorphisms. It follows that $\hat{x}_j \in P\hat{R}_P$ where $\hat{x}_j$ is an extension of x_j to $\operatorname{end}_R(E(M_i)) = \hat{R}_P (j=1, 2, \cdots)$. Let $y \in M_i$ be arbitrary. It follows that $P^n y = 0$ for some n, and hence $o = \hat{x}_1 \cdots \hat{x}_n(y) = x_1 \cdots x_n(y)$. Therefore $M_i^{(\alpha_i)}$ is indeed locally semi-T-nilpotent.

Any torsion module over a Dedekind domain R can be writetn as $M = \bigoplus_P M(P)$, where P runs over all non-zero prime ideals of R and $M(P)$ is the P-primary component of M. Moreover any uniform torsion R-module is isomorphic to either $E(R/P)$ or R/P^n for some prime P, and hence has a local endomorphism ring. Thus, as an immediate consequence of Theorems 18, 20 and 21, we retrieve the characterization of extending torsion modules over Dedekind domains obtained by Harada ([4], Theorem 7 (1)):

Corollary 23. *Let M be a torsion module over a Dedekind domain R. Then M is extending if and only if for each non-zero prime ideal P of R, either $M(P)$ is injective, or $M(P)$ is a direct sum of copies of R/P^n or R/P^{n+1} for some $n=n(P)$.*

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