Kamal, M.A. and Müller, B.J.
Osaka J. Math.
25 (1988), 531-538

EXTENDING MODULES OVER COMMUTATIVE DOMAINS

MAHMOUD A. KAMAL AND BRUNO J. MÜLLER

(Received June 15, 1987)

1. Introduction

A module is extending (or has the property (C_1)) if every complement submodule is a direct summand. We prove that a module over a commutative domain has this property, if and only if it is either torsion with (C_1), or the direct sum of a torsion free reduced module with (C_1) and an arbitrary injective module. The torsion case is dealt with in [6], where we also give some background and references. Here we show that a torsion free reduced module is extending if and only if it is a finite direct sum of uniform submodules, each pair of which is extending. As an application we obtain a description of all extending modules over Dedekind domains. In a subsequent paper [7] we shall discuss the extending property for direct sums of pairs of uniform modules in general.

Throughout this paper R will be a commutative domain with quotient field K. $X\subset' M$ and $Y\subset^{\oplus} M$ denote that X is an essential submodule, and Y is a direct summand, of M.

A submodule N of a module M has no proper essential extension in M, if and only if there is another submodule N' such that N is maximal with respect to $N\cap N'=0$. Such submodules N are called closed, or complements.

2. Reduction to Torsionfree Reduced Modules

Theorem 1. *Let M be a right module over an arbitrary ring R, and let $Z_2(M)$ denote its second singular submodule. Then M is extending if and only if $M=Z_2(M)\oplus N$, where $Z_2(M)$ and N are extending and $Z_2(M)$ is N-injective.*

Proof. Since $Z_2(M)$ is closed in M, by (C_1), we have $M=Z_2(M)\oplus N$, where N is non-singular. Since (C_1) is inherited by direct summands, $Z_2(M)$ and N have (C_1).

To show that $Z_2(M)$ is N-injective, let $\phi: X\rightarrow Z_2(M)$ be a homomorphism from a submodule X of N. Consider $X':=\{x-\phi(x): x\in X\}$. By (C_1), there exists $X'\subset' X^*\subset^{\oplus} M$. Write $M=X^*\oplus Y$. Since $X'\cap Z_2(M)=0$ and since $X'\subset' X^*$, it follows that X^* is non-singular and that $Z_2(M)=Z_2(Y)$. Hence, by

(C_1) for Y, $Z_2(M)\subset^{\oplus}Y$, say $Y=Y'\oplus Z_2(M)$. Let $\pi: X^*\oplus Y'\oplus Z_2(M)\to Z_2(M)$ be the projection. It is easy to see that $\pi|N$ extends ϕ.

Conversely, let $M=Z_2(M)\oplus N$, where $Z_2(M)$ and N have (C_1), and $Z_2(M)$ is N-injective. Let A be a closed submodule of M. By a straightforward calculation one can show that $Z_2(A)$ has no proper essential extensions in $Z_2(M)$. By (C_1) for $Z_2(M)$, we have $Z_2(A)\subset^{\oplus}Z_2(M)$, and hence $Z_2(A)\subset^{\oplus}A$. Write $A=Z_2(A)\oplus B$, where B is a non-singular submodule of A. Since $B\cap Z_2(M)=0$ and $Z_2(M)$ is N-injective, there exists a homomorphism $\psi: N\to Z_2(M)$ such that $\psi\pi_{2|B}=\pi_{1|B}$, where π_1, π_2 are the projections of M onto $Z_2(M)$ and N respectively. Consider $N^*:=\{n+\psi(n): n\in N\}$. If follows that B is contained in N^*, and hence B is closed in N^*. Since $N^*\cong N$ has (C_1), we have $B\subset^{\oplus}N^*$. It is clear that $M=Z_2(M)\oplus N^*$; therefore $A\subset^{\oplus}M$.

Corollary 2. *Let R be a commutative integral domain, and let M be an R-module which is not torsion. Then M is extending, if and only if its torsion submodule $t(M)$ is injective and the factor module $M/t(M)$ is extending.*

Proposition 3. *Let M be a torsion free R-module, and let $D(M)$ be its largest divisible (injective) submodule. Then M has (C_1) if and only if $M/D(M)$ has (C_1).*

Proof. Let M have (C_1), and write $M=D(M)\oplus C$, where C is reduced. Hence $M/D(M)\cong C$ has (C_1).

Conversely, let $C\cong M/D(M)$ have (C_1). Let A be a closed submodule of M. Let $D(A)$ be the largest injective submodule of A, and write $A=D(A)\oplus B$ with B reduced. It is clear that $B\cap D(M)=0$.

Now let π, π' be the projections of M onto C and $D(M)$ respectively. There exists a homomorphism $\phi: C\to D(M)$ such that $\phi\pi(b)=\pi'(b)$ for all $b\in B$. Let $C^*:=\{\phi(c)+c: c\in C\}$. Then $C^*\cong C$ has (C_1), and $M=C^*\oplus D(M)$. Since B is closed in C^*, we have $B\subset^{\oplus}C^*$. Since $D(A)\subset^{\oplus}D(M)$, we conclude $A\subset^{\oplus}M$.

3. Decomposition into Uniform Submodules

Lemma 4. *Let $M=\bigoplus_{i\in I}M_i$, with all M_i being R-submodules of the quotient field K of R. Then A is a closed submodule of M if and only if $A=[\bigoplus_{j\in J}a_jK]\cap M$, for some K-linearly independent subset $\{a_j\}_{j\in J}$ of $\bigoplus_I K$. In particular A is a uniform and closed submodule of M if and only if $A=\{(q_ix)_{i\in I}: x\in K, q_ix\in M_i$ for all $i\}$ for some $0\neq(q_i)_{i\in I}\in\bigoplus_I K$.*

Theorem 5. *Let M be a torsion free reduced module over a commutative integral domain R. If M is extending, then M is a finite direct sum of uniform submodules.*

Proof. By (C_1), if $M\neq 0$, then $M=M_o\oplus U_o$ with M_o uniform. Again by

(C_1) for U_o, if $U_o \neq 0$, we have $U_o = M_1 \oplus U_1$ with M_1 uniform, and hence $M = M_o \oplus M_1 \oplus U_1$. Continuing in this manner we get $M = \bigoplus_{i=0}^{n} M_i \oplus U_n$ as long as U_{n-1} is non-zero. If M is finite dimensional, then $U_n = 0$ for some n and $M = \bigoplus_{i=0}^{n} M_i$, as claimed.

If M is infinite dimensional, we shall derive a contradiction. In this case U_n is infinite dimensional for all n, and hence $M \supset \bigoplus_{i=0}^{\infty} M_i$. We first show that $\bigoplus_{i=0}^{\infty} M_i$ is closed in M (and hence is a direct summand of M). Let $\bigoplus_{i=0}^{\infty} M_i \subset' M^* \subset M$; then $M^* = \bigoplus_{i=1}^{n} M_i \oplus (U_n \cap M^*)$. By a straightforward calculation one can show that $U_n \cap M^*$ is essential over $\bigoplus_{i=n+1}^{\infty} M_i$. Since, in the case of torsion free modules, injective hulls are unique, and direct sums of injective modules are injective, we have $E(M^*) = \bigoplus_{i=0}^{\infty} E(M_i)$. Now let $\pi_i : \bigoplus_{i=0}^{\infty} E(M_i) \to E(M_i)$ be the projections. For each $n \geqq 0$ we have $\pi_n(M^*) = M_n + \pi_n(U_n \cap M^*)$. Since $\bigoplus_{i=n+1}^{\infty} M_i \subset' U_n \cap M^*$, it follows that $\pi_n(U_n \cap M^*) = 0$, and hence $M^* = \bigoplus_{i=0}^{\infty} M_i$.

Since the quotient field K of R is divisible hence injective, we have $E(M_i) \cong K$ for all i. Since $M_i \supset y_i R \cong R$ for $0 \neq y_i \in M_i$, without loss of generality, we may assume $R \subset M_i \subset K$ for all i, and therefore $\bigoplus_{i=0}^{\infty} R \subset M = \bigoplus_{i=0}^{\infty} M_i \subset \bigoplus_{i=0}^{\infty} K$.

Now let $0 \neq r \in R$ be an arbitrary element. Let $a_n := e_o - e_n r^n (n \geq 1)$, where $e_n = (\delta_{ni})_{i=0}^{\infty} \in \bigoplus_{i=0}^{\infty} K$. It is easy to see that $\{a_n\}_{n=1}^{\infty}$ is a linearly independent subset of $\bigoplus_{i=0}^{\infty} K$. By Lemma 4, $A := \bigoplus_{n=0}^{\infty} a_n K \cap M$ is a closed submodule of M. By (C_1), $M = A \oplus B$. Let f be the restriction to M of the homomorphism: $\bigoplus_{i=0}^{\infty} K \ni (k_i)_{i=0}^{\infty} \to \sum_{i=0}^{\infty} \frac{k_i}{r^i} \in K$. It follows that $\ker f = A$, hence f embeds B into K. Since $e_o \notin A$, B is non zero and thus uniform. As B is a direct summand and hence closed in M, $B = bK \cap M$ for some $0 \neq b = (b_i)_{i=0}^{\infty} \in \bigoplus_{i=0}^{\infty} K$, by Lemma 4. Since $e_m \in M$ for all $m \geq 0$, we have $e_m = \sum_{n=1}^{\infty} a_n k_{nm} + bk_m = \sum_{n=1}^{\infty} (e_o - e_n r^n) k_{nm} + \sum_{i=0}^{\infty} e_i b_i k_m$. where k_{nm}, $k_m \in K$ and $\sum_{n=1}^{\infty} a_n k_{nm} \in A$, $bk_m \in B$. Comparing components, and using the abbreviation $D = \sum_{i=0}^{\infty} \frac{b_i}{r^i}$, we deduce $Dk_m r^m = 1$ for all $m \geq 0$. Since $b_i k_m \in M_i$ for all i, m, we obtain, for $m = i+1$, that $\frac{b_i}{Dr^{i+1}} = b_i k_{i+1} \in M_i$. It follows that $\frac{1}{r} = \frac{1}{rD} \sum_{i=0}^{\infty} \frac{b_i}{r^i} \in \sum_{i=0}^{\infty} M_i$. Since $0 \neq r$ was arbitrary in R, we get $K = \sum_{i=0}^{\infty} M_i$.

Now let $g : \bigoplus_{i=0}^{\infty} M_i \ni (m_i)_{i=0}^{\infty} \to \sum_{i=0}^{\infty} m_i \in K$. It is easy to see that $\ker g$ is closed in M. Thus, by (C_1), $M = \ker g \oplus X$. Therefore $K \cong X \subset M$, which contradicts the fact that M is reduced.

4. Reduction to Pairs

Proposition 6. *A torsion free reduced module, over a commutative domain R, has (C_1) if and only if it has $(1-C_1)$ and is finite dimensional.*

Proof. Let M have (C_1). By Theorem 5, M is finite dimensional. Obviously M has $(1-C_1)$. We show the converse by induction over the dimension of M. Assume that it holds true for dimension $<n$, and let M be a module with $(1-C_1)$ of dimension n. Then $M=\bigoplus_{i=1}^{n} M_i$ with all M_i uniform. Let A be a closed submodule of M with $1<\dim(A)<n$. It follows that $A\cap\bigoplus_{i=1}^{n-1} M_i\neq 0$ is closed in $\bigoplus_{i=1}^{n-1} M_i$. By induction $\bigoplus_{i=1}^{n-1} M_i=A\cap\bigoplus_{i=1}^{n-1} M_i\oplus X$, where $\dim(X)\leq n-2$. Then $M=A\cap\bigoplus_{i=1}^{n-1} M_i\oplus X\oplus M_n$, and hence $A=[A\cap\bigoplus_{i+1}^{n-1} M_i]\oplus[A\cap(X\oplus M_n)]$. Since $A\cap(X\oplus M_n)$ is closed in $X\oplus M_n$, again by induction $A\cap(X\oplus M_n)\subset^{\oplus} X\oplus M_n$, and therefore $A\subset^{\oplus} M$.

From now on we consider each torsion free uniform module over a commutative integral domain R as an R-submodule of K (the quotient field of R) containing R.

Let $M_i (i=1, 2, \cdots, n)$ be R-submodules of K. By $O(M_i)$ we mean the set of all $x\in K$ such that $xM_i\subset M_i$. If $M_i\neq 0$, then $O(M_i)$ is an over ring of R isomorphic to $\mathrm{end}_R(M_i)$.

Theorem 7. *Let M be a torsion free reduced R-module. Then the following are equivalent:*

1) *M is extending*

2) *$M=\bigoplus_{i+1}^{n} M_i$ with all M_i uniform, and for all $q_1, q_2, \cdots, q_n\in K$ (not all zero) there exist $\alpha_1, \alpha_2, \cdots, \alpha_n\in K$ such that $\sum_{k=1}^{n}\alpha_k=1$ and $\alpha_k q_i M_k\subset q_k M_i$ for all k, i.*

Proof. (1)⇒(2): Let M have (C_1). Then by Theorem 5, $M=\bigoplus_{i=1}^{n} M_i$ with all M_i uniform. Now let $q_1, q_2, \cdots, q_n$ be arbitrary in K, not all zero. Then, by Lemma 4, $A:=\{(q_i x)_{i=1}^{n}: x\in K \text{ and } q_i x\in M_i \text{ for all } i\}$ is a uniform and closed submodule of M. By (C_1), $M=A\oplus B$ where B is an $(n-1)$-dimensional submodule with (C_1). Hence $B=\bigoplus_{j=1}^{n-1} B_j$ where B_j are uniform. By Lemma 4, $B_j=\{(t_{ij}x_j)_{i=1}^{n}: x_j\in K \text{ and } t_{ij}x_j\in M_i\ (i=1, 2, \cdots, n)\}$ for some $t_{ij}\in K$ not all zero.

Now $A\oplus B=M$ implies that for each $c\in M_k$ the system of equations $\sum_{j=1}^{n-1} t_{ij}x_j$

$+q_i x_n = c\,\delta_{ik}(i=1, 2, \cdots, n)$ has a unique solution, with $t_{ij}x_j \in M_i$ and $q_i x_n \in M_i$. Therefore the determinant Δ of the system is non-zero. Then by Cramer's Rule, $x_n = \frac{(-1)^{n+k}\Delta_{kn}}{\Delta}c$, where Δ_{kn} is the (k, n) minor of Δ. If we write $\alpha_k = (-1)^{k+n}q_k\Delta_{kn}/\Delta$, we have $\sum_{k=1}^{n}\alpha_k = 1$. Moreover since $q_i x_n \in M_i$, we obtain $\alpha_k q_i c = q_k q_i x_n \in q_k M_i$, thus $\alpha_k q_i M_k \subset q_k M_i$.

2)$\Rightarrow$1): The proof will be by induction on n. Assume that $\bigoplus_{i\in L} M_i$ is extending for all proper subsets L of $\{1, 2, \cdots, n\}$. By Proposition 6, it is enough to show that each uniform closed submodule A of M is a direct summand. By Lemma 4, $A = \{(q_i x)_{i=1}^n : x \in K \text{ and } q_i x \in M_i \text{ for all } i\}$.

Let $F := \{i : q_i \neq 0\}$. If $|F| < n$, then $A \subset \bigoplus_{i\in F} M_i$ and hence, by induction, $A \subset^{\oplus} \bigoplus_{i\in F} M_i \subset^{\oplus} M$. If $|F| = n$, then by condition 2), there exist $\alpha_1, \alpha_2, \cdots, \alpha_n \in K$ such $\sum_{i=1}^{n}\alpha_i = 1$ and $\alpha_i q_i^{-1}M_i \subset q_j^{-1}M_j$. Let $\Delta_{i1} := \alpha_i q_i^{-1}$; then $\sum_{i=1}^{n} q_i \Delta_{i1} = 1$. It is clear that not all Δ_{i1} are zero. Without loss of generality assume that $\Delta_{11} \neq 0$. Let $B := \{(\Delta_{21}y_2 + \sum_{i=3}^{n}\frac{\Delta_{j1}}{\Delta_{11}}y_j, \Delta_{11}y_2, y_3, \cdots, y_n) : y_j \in K \text{ and } \Delta_{21}y_2 + \sum_{j=3}^{n}\frac{\Delta_{j1}}{\Delta_{11}}y_j \in M_1, \Delta_{11}y_2 \in M_2 \text{ and } y_i \in M_i (i \geq 3)\}$. We have:

$$\begin{vmatrix} q_1 & -\Delta_{21} & -\Delta_{31}/\Delta_{11} & & -\Delta_{n1}/\Delta_{11} \\ q_2 & \Delta_{11} & 0 & & 0 \\ q_3 & 0 & 1 & & 0 \\ \cdots & \cdots & \cdots & & \cdots \\ \cdots & \cdots & \cdots & & \cdots \\ q_n & 0 & 0 & 0 & 1 \end{vmatrix} = \sum_{i=1}^{n} q_i\Delta_{i1} = 1$$

Then, for each k, the following system of equations has a unique solution, for all $m_i \in M_i$:

$$\begin{aligned} q_1 x - \Delta_{21}y_2 - \cdots \quad -(\Delta_{n1}/\Delta_{11})y_n &= \delta_{k1}m_1 \\ q_2 x + \Delta_{11}y_2 + 0 + \cdots + \quad 0 &= \delta_{k2}m_2 \\ q_3 x + 0 + y_3 + 0 + \cdots + \quad 0 &= \delta_{k3}m_3 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ q_n x + 0 + \quad \cdots \quad + 0 + y_n &= \delta_{kn}m_n \end{aligned}$$

Let $\{x_k, y_{2k}, \cdots, y_{nk}\}$ be the solution set of the k^{th} system. Since, by Cramer's Rule, $x_k = \Delta_{k1}m_k = q_k^{-1}\alpha_k m_k \in q_i^{-1}M_i$, we have $q_i x_k \in M_i$ for all k, i. It follows that $\Delta_{21}y_{2k} + \sum_{j=4}^{n}(\Delta_{j1}/\Delta_{11})y_{jk} \in M_1$, $\Delta_{11}y_{2k} \in M_2$ and $y_{ik} \in M_i (i \geq 3)$ for all k. Then

$M=A+B$. Since the above determinant is non zero, we have that each $m\in M$ has a unique representation $m=a+b$ with $a\in A$, $b\in B$. Therefore $M=A\oplus B$.

Corollary 8. *If $\bigoplus_{i=1}^{n} M_i$ is extending and reduced, then each M_i can be embedded into every M_j.*

Proof. Each pair $M_i\oplus M_j(i\neq j)$ is extending and reduced. Therefore, by Theorem 7, for each $0\neq q\in K$, there exists $\alpha_1, \alpha_2\in K$ such that $\alpha_1 qM_1\subset M_2$ and $\alpha_2 M_2\subset qM_1$. If M_1 is not embedded into M_2, then we obtain $\alpha_1=0$, hence $\alpha_2=1$, for every $0\neq q\in K$. Then $M_1=K$, in contradiction to reduceness.

Lemma 9. *Let $M_i(i=1, 2, 3)$ be R-submodvles of K. If $M_i\oplus M_j$ has (C_1) for all $i\neq j$, then $M_3\oplus q_1M_1\cap q_2M_2$ has (C_1) for all $q_1, q_2\in K$.*

Proof. Without loss of generality assume $q_1\neq 0$, $q_2\neq 0$. Let $0\neq k\in K$ be given arbitrarily. Since $M_1\oplus M_2$ has (C_1), by Theorem 7, there exist $\alpha_{12}, \alpha_{21}\in O(M_1)\cap O(M_2)$ with $\alpha_{12}+\alpha_{21}=1$ such that $\alpha_{12}q_1M_1\subset q_2M_2$ and $\alpha_{21}q_2M_2\subset q_1M_1$.

Similarly, since $M_3\oplus M_i$ has (C_1), there exist $\alpha_{i3}, \alpha_{3i}\in O(M_i)\cap O(M_3)$ with $\alpha_{i3}+\alpha_{3i}=1$ such that $\alpha_{3i}kM_3\subset q_iM_i$ and $\alpha_{i3}q_iM_i\subset kM_3$ $(i=1, 2)$.

Now let $\gamma_1=\alpha_{12}\alpha_{31}+\alpha_{21}\alpha_{32}$ *and* $\gamma_2=\alpha_{12}\alpha_{13}+\alpha_{21}\alpha_{23}$. It follows that $\gamma_1+\gamma_2=1$.

We show that $\gamma_1\gamma_2\in O(M_3)\cap O(q_1M_1\cap q_2M_2)$: $\gamma_2=\alpha_{12}\alpha_{13}(\alpha_{23}+\alpha_{32})+\alpha_{21}\alpha_{23}(\alpha_{13}+\alpha_{31})=(\alpha_{12}+\alpha_{21})\alpha_{13}\alpha_{23}+\alpha_{12}\alpha_{13}\alpha_{32}+\alpha_{21}\alpha_{23}\alpha_{31}=\alpha_{13}\alpha_{23}+\alpha_{12}\alpha_{13}\alpha_{32}+\alpha_{21}\alpha_{23}\alpha_{31}$.

$\gamma_2M_3\subset\alpha_{13}\alpha_{23}M_3+\alpha_{12}\alpha_{13}\alpha_{32}M_3+\alpha_{21}\alpha_{23}\alpha_{31}M_3\subset M_3+(1-\alpha_{21})\alpha_{13}\alpha_{32}M_3+(1-\alpha_{12})\alpha_{23}\alpha_{31}M_3\subset M_3+\alpha_{21}\alpha_{13}\alpha_{32}M_3+\alpha_{12}\alpha_{23}\alpha_{31}M_3\subset M_3+\alpha_{13}\alpha_{21}(k^{-1}q_2M_2)+\alpha_{12}\alpha_{23}(k^{-1}q_1M_1)\subset M_3+\alpha_{13}k^{-1}q_1M_1+\alpha_{23}k^{-1}q_2M_2\subset M_3$.

$\gamma_1(q_1M_1\cap q_2M_2)\subset\alpha_{12}\alpha_{31}(q_1M_1)+\alpha_{21}\alpha_{32}(q_2M_2)\subset q_1M_1\cap q_2M_2$. Since $\gamma_1+\gamma_2=1$, we have $\gamma_1, \gamma_2\in O(M_3)\cap O(q_1M_1\cap q_2M_2)$. We show that $\gamma_1kM_3\subset q_1M_1\cap q_2M_2$ and $\gamma_2(q_1M_1\cap q_2M_2)\subset kM_3$: $\gamma_1kM_3\subset\alpha_{12}\alpha_{31}(kM_3)+\alpha_{21}\alpha_{32}(kM_3)\subset\alpha_{12}(q_1M_1)+\alpha_{21}(q_2M_2)\subset q_1M_1\cap q_2M_2$.

$\gamma_2(q_1M_1\cap q_2M_2)\subset\alpha_{12}\alpha_{13}(q_1M_1\cap q_2M_2)+\alpha_{21}\alpha_{23}(q_1M_1\cap q_2M_2)\subset\alpha_{13}(q_1M_1\cap q_2M_2)+\alpha_{23}(q_1M_1\cap q_2M_2)\subset kM_3$.

Therefore, by Theorem 7, $M_3\oplus q_1M_1\cap q_2M_2$ has (C_1).

Corollary 10. *Let $M_i(i=1, 2, \cdots, n)$ be R-submodules of K. If $M_i\oplus M_j$ has (C_1) for all $i\neq j$, then $M_n\oplus\bigcap_{i=1}^{n-1} q_iM_i$ has (C_1) for all $q_1, q_2, \cdots, q_{n-1}\in K$.*

Proof. We proceed by induction over n. Since $M_n\oplus\bigcap_{i=1}^{n-2} q_iM_i$, $M_n\oplus q_{n-1}M_{n+1}\cong M_n\oplus M_{n-1}$, $\bigcap_{i=1}^{n-2} q_iM_i\oplus q_{n-1}M_{n-1}\cong\bigcap_{i=1}^{n-2} q_iM_i\oplus M_{n-1}$ all have (C_1), by assumption of induction, Lemma 9 implies that $M_n\oplus\bigcap_{i=1}^{n-2} q_iM_i\cap q_{n-1}M_{n-1}$ has (C_1).

Theorem 11. *Let M be a torsion free reduced R-module. Then M is ex-*

tending if and only if $M=\bigoplus_{i=1}^{n} M_i$, *where the* M_i *are uniform and* $M_i\oplus M_j$ *is extending for all* $i\neq j$.

Proof. Let $M=\bigoplus_{i=1}^{n} M_i$ with M_i uniform and with $M_i\oplus M_j$ extending. By induction on n, let $\bigoplus_{i\in L} M_i$ be extending for all proper subsets L of $\{1, 2, \cdots, n\}$. Let A be a closed and uniform submodule of $\bigoplus_{i=1}^{n} M_i$. By Lemma 4, $A=\{(q_i x)_{i=1}^{n};$ $x\in K$, $q_i x\in M_i$ for all $i\}$. Let $F:=\{i: q_i\neq 0\}$. By induction $A\subset^{\oplus}\bigoplus_{i\in F} M_i\subset^{\oplus} M$, if $|F|<n$. Now let $|F|=n$; it follows that $A=\{(q_i x)_{i=1}^{n}: x\in \bigcap_{i=1}^{n-1} q_i^{-1}M_i\}$. Let $\pi: M\to \bigoplus_{i=1}^{n-1} M_i$ be the projection. By Lemma 4, $B:=\{(q_1 x, q_2 x, \cdots, q_{n-1}x, 0):$ $x\in \bigcap_{i=1}^{n-1} q_i^{-1}M_i\}$ is a closed uniform submodule of $\bigoplus_{i=1}^{n-1} M_i$ containing $\pi(A)$. By induction, $B\subset^{\oplus}\bigoplus_{i=1}^{n-1} M_i$, and hence $M_n\oplus B\subset^{\oplus} M$. Since $B\cong \bigcap_{i=1}^{n-1} q_i^{-1}M_i$, we have, by Corollary 10, that $M_n\oplus B$ is extending. As A is closed in $M_n\oplus B$, $A\subset^{\oplus} M_n \oplus B\subset^{\oplus} M$. Therefore M is extending, by Proposition 6.

5. Dedekind Domains

Lemma 12. *Let* $M=M_1\oplus M_2$ *be a torsionfree reduced module over a Dedekind domain* R, *where the* M_i *are uniform. Then the following are equivalent:*
1) *M is extending,*
2) *M_i can be imbedded $M_j (i\neq j)$,*
3) *there is a fractional ideal I of R such that $M_2 I=M_1$.*

Proof. 1)⇒2) clear by Corollary 8.
2)⇒3): Without loss of generality assume that $R\subset M_1\subset M_2\subset K$. Let $B:=\{x\in K: M_2 x\subset M_1\}$ and $S=O(M_1)\cap O(M_2)$. By assumption B is a non-zero ideal of S. Now if $M_2 B\subsetneq M_1$, then $(M_2 B)_P\subsetneq M_{1P}$ for some prime ideal P of S. Since S_P is discrete rank one valuation ıing, it follows that $(M_2 B)_P\subset M_{1P}P_P=(M_1 P)_P$. For each prime ideal Q of S, $Q\neq P$, we have $(M_2 B)_Q\subset M_{1Q}=(M_1 P)_Q$. Hence $M_2 B=\bigcap (M_2 B)_Q\subset \bigcap_Q (M_1 P)_Q=M_1 P$, where Q runs over all prime ideals of S. It follows that $M_2 BP^{-1}\subset M_1$, i.e., $BP^{-1}=B$ which is a contradiction. Therefore $M_2 B=M_1$. Since any overring of R is a localization R_* of R a set of prime ideals of R, we have $S=R_*$. It follows that $B=I_*$ for some ideal I of R. Now $M_2 B=M_2 I_*=M_2 IR_*=M_2 IS=M_2 I$, and hence $M_2 I=M_1$.
3)⇒1): First we show that $J\cap R+J^{-1}\cap R=R$ for any fractional ideal J or R. If $J_P, J_P^{-1}\subsetneq R_P$ for some prime ideal P of R, then $R_P=J_P J_P^{-1}\subset J_P\subsetneq R_P$ which is a contradiction. It follows that $J_P\cap R_P=R_P$ or $J_P^{-1}\cap R_P=R_P$, and hence $(J\cap R)_P+(J^{-1}\cap R)_P=R_P$, for all prime ideals P of R. Therefore $J\cap R+J^{-1}\cap R=R$.

Now let $M_1=M_2 I$ where I is a fractional ideal of R. Let $0\neq q\in K$ be arbitrary, and $J:=q^{-1}I^{-1}$. Since $J\cap R+J^{-1}\cap R=R$, there exist $\alpha_1\in J\cap R$, $\alpha_2\in$

$J^{-1} \cap R$ such that $\alpha_1 + \alpha_2 = 1$, and that $\alpha_1 q M_1 \subset M_2$, $\alpha_2 M_2 \subset q M_1$. Therefore, by Theorem 7, M is extending.

Corollary 13. *If R is a principal ideal domain and M_1, M_2 are uniform torsion free reduced R-modules, then $M_1 \oplus M_2$ is extending if and only if M_1 is isomorphic to M_2.*

Proof. R is a Dedekind domain, and every fractional ideal of R is principal.

The following is an immediate consequence of Corollary 2, Proposition 3, Theorem 5, and Lemma 12.

Theorem 14. *Let M be a module over a Dedekind domain R. Then M is extending if and only if either:*

i) *M is torsion and has the structure described in* ([6], Corollary 23); *or*

ii) *M is non-torsion and $M = F \oplus E$, where E is injective and $F \cong \bigoplus_{i=1}^{n} N I_i$, where N is a proper R-submodule of the quotient field K and the I_i are fractional ideals of R.*

References

[1] F.W. Anderson and K.R. Fuller: Rings and Categories of Modules, Springer Verlag, New York (1973).

[2] W. Brandal: *Commutative Rings whose Finitely Generated Modules Decompose,* Springer Lecture Notes in Math. **723** (1979).

[3] A.W. Chatter and C.R. Hajarnavis: *Rings in which every complement right ideal is a direct summand,* Quart. J. Math. Oxford **28** (1977), 61–80.

[4] L. Fuchs, A. Kertesz and T. Szele: *Abelian groups in which every serving subgroup is a direct summand,* Publ. Math. Debrecen **3** (1953), 95–105.

[5] M. Harada: *On modules with extending property,* Osaka J. Math. **19** (1982), 203–215.

[6] M. Kamal and B.J. Müller: *The structure of extending modules over noetherian rings,* Osaka J. Math. **25** (1988), 539–551.

[7] M. Kamal and B.J. Müller: *Torsion free extending modules,* Osaka J. Math. [to appear].

[8] I. Kaplansky: Commutative Rings, Univ. of Chicago Press, Chicago (1974).

[9] B. Stenstrom: Rings of Quotients, Springer Verlag, New York (1975).

Department of Mathematics
McMaster University
1280 Main Street West, Hamilton
Ontario, Canada L8S 4K1