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ON ARTINIAN RINGS WHOSE INDECOMPOSABLE PROJECTIVES ARE DISTRIBUTIVE

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1. Introduction

A module $L \neq 0$ is called local (or hollow) if $L=L_1+L_2$ implies $L=L_1$ or $L=L_2$. Especially a noetherian module is local if and only if it has a unique maximal submodule.

A module M is called distributive if $X \cap (Y+Z)=(X \cap Y)+(X \cap Z)$ for every submodules X, Y, Z in M (cf. [2]). It is clear that any sub- (or factor) module of a distributive module is distributive.

We call a ring R right locally distributive, right LD in abbreviation, if it is right artinian and every projective indecomposable right R-module is distributive. It is evident that every local right module over a right LD-ring is distributive. The class of right LD-rings is a generalization of the class of right serial rings.

In this note right LD-rings are studied, mainly to construct a number of right LD-algebras.

2. Right LD-rings

The following lemma, shown by Fuller, is basic to study distributive modules over a semiperfect ring.

Lemma 1. *Let R be a semiperfect ring. The following conditions on a right R-module M are equivalent*:

(1) *M is distributive.*

(2) *For every primitive idempotent e of R, the set $\{xeR \mid x \in M\}$ of all homomorphic images of eR in M is linearly ordered.*

(3) *For every primitive idempotent e in R, the right eRe-module Me is uniserial.*

Proof. See Fuller [1].

Theorem 2. *The following conditions on a right artinian ring R are equivalent*:

(1) *Every projective indecomposable right R-module eR is distributive.*

(2) *For every primitive idempotents e and f of R, eRf is a uniserial right fRf-module.*

(3) *Every submodule in a projective indecomposable right R-module eR is characteristic, and the lattice of two-sided ideals in R is distributive.*

Proof. (1)⇔(2) is a special case of Lemma 1.

(1)⇒(3). Every submodule in eR is a sum of local submodules and every local submodule is characteristic in eR by Lemma 1. Hence every submodule in eR is characteristic.

Let $\{e_i\}_{i=1}^n$ be a complete set of primitive idempotents, and let I, J, K be two-sided ideals in R. Then by the distributivity of e_iR,

$$\begin{aligned} e_i(I\cap(J+K)) &= e_iI\cap(e_iJ+e_iK) \\ &= (e_iI\cap e_iJ)+(e_iI\cap e_iK) = e_i(I\cap J)+e_i(I\cap K)\,. \end{aligned}$$

Summing up each side of the equations ($i=1, \cdots, n$), we have $I\cap(J+K)=(I\cap J)+(I\cap K)$.

(3)⇒(1). Let A be a submodule in a projective indecomposable submodule eR. Since A is characteristic in eR, $eReA=A$. We notice that the two-sided ideal $A':=RA=ReA$ satisfies the equation $eA'=A$.

If X, Y, Z are any submodules in eR, then

$$\begin{aligned} eX'\cap(eY'+eZ') &= e(X'\cap(Y'+Z')) \\ &= e((X'\cap Y')+(X'\cap Z')) = (eX'\cap eY')+(eX'\cap eZ')\,. \end{aligned}$$

Hence eR is distributive.

A right artinian ring is called right LD if it satisfies the equivalent conditions in Theorem 2.

3. Construction of right LD-algebras

We begin with a general remark on modules. For a module M we denote by $H(M)$ the inclusion-ordered set of all local submodules in M. A homomorphism $f\colon M\to N$ of modules induces a correspondence: $H(M)\to H(N)$, $X\mapsto f(X)$. This correspondence is not a mapping in general (the image of some local submodule $\leq M$ by f may be 0). If M is a module of finite length and f is an epimorphism, then there is a natural surjection

$$(*)\qquad \{X\in H(M)\,|\,X\nleq \mathrm{Ker}(f)\}\to H(N)\,.$$

In fact, there exist $X_1, \cdots, X_n\in H(M)$ such that $f^{-1}(Y)=X_1+\cdots+X_n$ for every $Y\in H(N)$, and $f(X_i)=Y$ for some $i\in\{1, \cdots, n\}$. Moreover if M is distributive, i is unique by Lemma 1, and (*) is bijective.

In this section a method to construct some right LD-algebras is presented. We introduce some terminology.

Suppose C is a fixed set. A pair (P, t) of a set P and a mapping $t: P\to C$ is called a C-set. When (P, t) and (P', t') are C-sets, a mapping $f: P\to P'$ is called a C-set homomorphism if $t=t'f$. Moreover, in case that P and P' are posets, f is called a C-poset homomorphism if f is both a C-set homomorphism and a poset homomorphism.

A subposet U of a poset P is said to be an upper part of P if $x\in U$, $y\in P$ and $x\leq y$ imply $y\in U$. In particular, when P is finite, U is an upper part of P if and only if it is of the form $\{x\in P\mid x\geq p_1\}\cup\cdots\cup\{x\in P\mid x\geq p_n\}$ $(p_1, \cdots, p_n\in P)$.

DEFINITION. *Let C be a set. A family of C-posets $\{(P_1, t_1), \cdots, (P_n, t_n)\}$ is called an admissible system (of C-posets) if it satisfies the following conditions* $(i=1, \cdots, n)$:

(1) *Every poset P_i has a unique maximal element m_i.*

(2) $C=\{t_1(m_1), \cdots, t_n(m_n)\}$ $(t_i(m_i)\neq t_j(m_j)$ *if* $i\neq j)$.

(3) *For every $c\in C$ the subposet $\{x\in P_i\mid t_i(x)=c\}$ is linearly ordered.*

(4) *For every $a\in P_i$, there exist $j\in\{1, \cdots, n\}$ and a C-poset homomorphism from an upper part of P_j to $\{x\in P_i\mid x\leq a\}$.*

REMARK 1. Suppose that the conditions (1), (2), (3) of the above definition are satisfied and that f is a C-poset isomorphism from an upper part of P_j to $\{x\in P_i\mid x\leq a\}$. Then j is determined by $t_j(m_j)=t_if(m_j)=t_i(a)$.

Let b_0 be any element in P_j and $b_0\leq\cdots\leq b_r=m_j$ be a chain with b_{k-1} maximal in $\{x\in P_j\mid x\lneqq b_k\}$ $(k\in\{1, \cdots, r)\}$. Then $f(b_{k-1})$ is maximal in $\{x\in P_i\mid x\lneqq f(b_k)\}$. Since $t_j(b_{k-1})=t_if(b_{k-1})$, $f(b_{k-1})$ is unique in $\{x\in P_i\mid x\lneqq f(b_k)\}$ by (3). Therefore $f(b_0)$ is determined inductively, and the isomorphism in (4) of the above definition is unique.

REMARK 2. By a similar argument we can replace (4) with (4′) If $a\in P_i$ is maximal in $P_i\setminus\{m_i\}$, there exist $j\in\{1, \cdots, n\}$ and a C-poset isomorphism from an upper part of P_j to $\{x\in P_i\mid x\leq a\}$.

If R is a right LD-ring with the Jacobson radical J, and $\{e_i\}_{i=1}^n$ is a basic set of primitive idempotents for R, then by the first remark of this section, the posets $H(e_1R), \cdots, H(e_nR)$ form an admissible system with the mapping

$$\mathrm{top}(\ \ )\,(:=(\ \ )/(\ \ )J)\colon\ H(e_iR)\to T(R) \qquad (i\in\{1, \cdots, n\}),$$

where $T(R)$ denotes the set of all isomorphism class of simple R-modules.

Theorem 3. *For any admissible system $\{(P_i, t_i)\}_{i=1}^n$ of C-posets, there exists a right LD-ring R such that $H(e_iR)$ is isomorphic to (P_i, t_i) ($T(R)$ is identified with C by a bijection β: (the isomorphism class of* $\mathrm{top}(e_iR))\mapsto t_i(m_i))$, *where $\{e_i\}_{i=1}^n$ is a basic set of primitive idempotents for R.*

Proof. Since the C-poset isomorphism of (4) in Definition is uniquely determined by an element $a\in P_i$ (Remark 1), we denote the isomorphism by $\bar{a}$. Letting any element in P_i outside the domain of definition of $\bar{a}$ correspond to no element, the isomorphism $\bar{a}$ is extended to a correspondence: $P_j\to P_i$, which operates P_j on the left. This extension is so trivial that it is also denoted by $\bar{a}$.

For two correspondences $\bar{a}_1$: $P_i\to P_j$ and $\bar{a}_2$: $P_k\to P_h$ $(a_1\in P_j,\ a_2\in P_h)$, we define $\bar{a}_1\bar{a}_2=0$ if (the composition $\bar{a}_1\circ\bar{a}_2$ of the correspondences)$=\emptyset$ or $h\neq i$, and otherwise $\bar{a}_1\bar{a}_2=\bar{a}_1\circ\bar{a}_2$ the composition of correspondences. Then the disjoint union of $\{a\,|\,a\in P_i\}_i$ and $\{0\}$ forms a semigroup S with the multiplication defined above. If $a_1\leq a_2$ in P_i, there exists $x\in S$ satisfying $\bar{a}_1=\bar{a}_2x$ by (4) in Definition.

Let $R:=KS$ be the semigroup algebra of S over a field K. Then R is an artinian algebra over K with the Jacobson radical $\{\Sigma k_a\bar{a}\,|\,a\neq m_i$ for any i, and $k_a\in K\}$ and $\{\bar{m}_i\}_{i=1}^n$ is a basic set of primitive idempotents for R.

For any element $x\neq 0$ in $\bar{m}_jR\bar{m}_i$, $x=k_1\bar{a}_1+\cdots+k_s\bar{a}_s$ with some distinct $\bar{a}_1, \cdots, \bar{a}_s$: $P_j\to P_j$ and $k_1, \cdots, k_s\in K\setminus\{0\}$. Since $a_1, \cdots, a_s\in P_i$ and $t_i(a_1)=\cdots=t_i(a_s)=t_j(m_j)$, there exists uniquely the maximal element $a(x)$ of $\{a_1, \cdots, a_s\}$ by (3) in Definition. If $a_u=a(x)$ $(u\in\{1, \cdots, s\})$,

$$x=\bar{a}_u(k_u+\text{an element of the Jacobson radical})$$

and $xR=\overline{a(x)}R$. Therefore R is right LD by Theorem 2. It is easily verified that α_1: $H(\bar{m}_iR)\to P_i$; $xR\mapsto a(x)$ is an isomorphism of poset, and that the diagram

$$\begin{array}{ccc} H(\bar{m}_iR) & \xrightarrow{\ \alpha_i\ } & P_i \\ {\scriptstyle \mathrm{top}(\ )}\downarrow & & \downarrow{\scriptstyle t_i} \\ T(R) & \xrightarrow{\ \beta\ } & C \end{array}$$

is commutative.

4. Right and left *LD*-rings

If R is a right LD-ring with a basic set $\{e_i\}_{i\in IR}$ of primitive idempotents, we construct a semigroup S_R from the admissible system $\{(H(e_iR),\ (\ \ )/(\ \ )J\}_{i\in I}$. Let $X\in H(e_iR)$ and $X/XJ\simeq e_jR/e_jJ$ $(i, j\in I)$, then the correspondence $\bar{X}$ is induced by the left multiplication of some $x\in e_iRe_j$ (cf. the first paragraph and Remark 1 in the section 3) and $X=xR$.

Symmetrically we have a semigroup ${}_RS$, the left version of S_R, from the admissible system $\{(H(Re_i),\ (\ \ )/(\ \ ))\}_{i\in I}$ if R is a left LD-ring with a basic set $\{e_i\}_{i\in I}$ of primitive idempotents, where correspondences operate $H(Re_i)$ on the right.

The semigroup algebra KS_R (resp. K_RS) over a field K is considered a model of right (resp. left) LD-ring R with respect to the submodule-lattice structure of the projective indecomposable right (resp. left) R-modules.

However, if R is a right and left LD-ring, the "one-sided model" KS_R or K_RS is two-sided (see Proposition 5 below).

Lemma 4. *Let e be an idempotent of a ring R, and suppose that every submodule in eR is characteristic. Then $Rx \leq Ry$ implies $xR \leq yR$ for any $x, y \leq eR$.*

Proof. If $Rx \leq Ry$, there is $r \in eRe$ satisfying $x = ry$. Since yR is characteristic in eR, $xR = ryR \leq yR$.

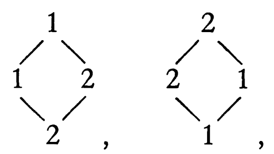
Proposition 5. *Let R be a right and left LD-ring with a basic set $\{e_i\}_{i\in I}$ of primitive idempotents. Then $S_R \cong {}_RS$, $KS_R \cong K_RS$ is a right and left LD-ring, and one of the admissible systems $\{H(e_iR)\}_{i\in I}$, $\{H(Re_i)\}_{i\in I}$ is obtained by the other.*

Proof. A bijection $S_R \to {}_RS$; $\overline{xR} \mapsto \overline{Rx}$ $(x \in e_iRe_j)$ is defined by Lemma 4, where $\overline{xR}$ (resp. $\overline{Rx}$) is the correspondence $H(e_jR)$ $H(e_iR)$ (resp. $H(Re_i) \to H(Re_j)$) associated to xR (resp. Rx) adopting the notation in the proof of Theorem 3. Since $\overline{xR}\,\overline{yR} = \overline{xyR}$ and $\overline{Rx}\,\overline{Ry} = \overline{Rxy}$ for $x \in e_iRe_j$ and $y \in e_kRe_h$ $(i, j, k, h \in I)$ (cf. Remark 1 in the section 3), this bijection is an isomorphism. The rest follows immediately.

5. Examples

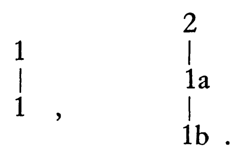
The construction of right LD-rings in the section 3 is useful especially in case that the Loewy length is small.

(1) From the admissible system of C-posets ($C = \{1, 2\}$) with the Hasse diagram;



a QF-LD-ring is given, where the numbers on the vertices are their values in C.

(2) Let R be a right LD-ring with the admissible system of $\{1, 2\}$-posets;



Then R is not left LD, since there is no element x in S_R satisfying $\bar{b} = x\bar{a}$.

References

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