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CONTINUOUS MODULES AND QUASI-CONTINUOUS MODULES

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Let M be an R-module and $\mathcal{A}$ a subfamily of the family $\mathcal{L}(M)$ of all submodules of M. M is said to have the extending property of modules for $\mathcal{A}$ provided that, for any A in $\mathcal{A}$, there exists a direct summand A^* of M which contains A as an essential submodule. For several natural subfamilies (e.g., the family of all simple submodules and that of all uniform submodules), Harada [4], [6] and Harada and Oshiro [8] have recently studied this property. In the module theory, one type of this property has also appeared Utumi's series [17]~[18]. He has showed that a von Neumann regular ring R is upper continuous if and only if R_R has the extending property of modules for the family of all right ideals of R. The reader is referred to [2] for the work of Utumi and other related results.

The notion of continuous was carried over to modules by Jeremy [10], [11]. She says that an R-module M is continuous (resp. quasi-continuous) if the conditions (1) and (2) (resp. (1) and (3)) below are satisfied:

(1) M has the extending property of modules for $\mathcal{L}(M)$.

(2) For any direct summand N of M and any monomorphism f from N to M, $f(N)$ is a direct summand of M.

(3) If N_1 and N_2 are direct summands of M with $N_1 \cap N_2=0$, then $N_1 \oplus N_2$ is also a direct summand of M.

In section 1 of this paper, we introduce $\mathcal{A}$-continuous modules and $\mathcal{A}$-quasi-continuous modules as concepts dual to $\mathcal{A}$-semiperfect modules and $\mathcal{A}$-quasi-semiperfect modules mentioned in [16], respectively. In Theorems 1.7 and 1.8 we give some characterizations of those modules. In section 2, we study continuous and quasi-continuous modules with indecomposable decompositions. In section 3, we study continuous and quasi-continuous modules and modules with the extending property of direct sums. It is shown that continuous modules over a Dedekind domain are just quasi-injective.

0. Preliminaries

Throughout this paper R is a ring with identity and all R-modules are unitary right R-modules. For a given R-module M, we denote its injective hull by $E(M)$ and the family of all submodules of M by $\mathcal{A}(M)$. We use the symbol $N\subseteq_e M$ to mean that N is essential in M. An R-module is said to have the condition $(M-I)$ if every monomorphism of the module into itself is an isomorphism ([18]). Let N be a submodule of an R-module M. N is said to be a closed submodule of M if N has no proper essential extension in M.

Let τ be a cardinal number. An R-module M is said to be τ-dimensional if it satisfies the following conditions: i) There exists an independent family of τ non-zero submodules of M. ii) For any independent family of γ non-zero submodules of M, we have $\gamma\leq\tau$. Of course, such τ is uniquely determined just as the finite dimension.

For an R-module M and a cardinal number τ, $\mathcal{A}(\tau\text{-}dim(\mathcal{L}(M)))$ denotes the family $\{A\in\mathcal{L}(M)\mid \text{dimension of } A\leq\tau\}$ and $\mathcal{A}(\tau\text{-}gen(\mathcal{L}(M)))$ denotes the family of all $A\in\mathcal{L}(M)$ which contains a submodule generated by τ elements as an essential submodule.

Let M be an R-module and $\mathcal{A}$ a subfamily of $\mathcal{L}(M)$. M is said to have the extending property of modules for $\mathcal{A}$ if, for any A in $\mathcal{A}$, there exists a direct summand A^* of M with $A\subseteq_e A^*$. In particular, M is said to have the extending property of simple (resp. uniform) modules if it has the extending property of modules for the family of all simple (resp. uniform) submodules of M. Further M is said to have the extending property of direct sums for $\mathcal{A}$ provided that it satisfies the following condition:

(*) For any independent submodules $\{M_\alpha\}_I$ with $M_\alpha\in\mathcal{A}$, there exists a decomposition $M=\sum_I\oplus M_\alpha^*\oplus M^*$ such that $M_\alpha\subseteq_e M_\alpha^*$ for all $\alpha\in I$.

If (*) holds whenever the index set I is finite, M is said to have the extending property of finite direct sums for $\mathcal{A}$. If M has the extending property of direct sums for the family of all uniform submodules of M, we simply say that M has the extending property of direct sums of uniform modules.

Let $\{M_\alpha\}_I$ be a set of R-modules. $\{M_\alpha\}_I$ is said to be locally semi-T-nilpotent ([6]) if it satisfies the following condition: Let $\{M_{\alpha_i}\}_{i=1}^{\infty}$ be a countable subset of $\{M_\alpha\}_I$ with $\alpha_n\neq\alpha_{n'}$ if $n\neq n'$. Then, for any non-isomorphisms $\{f_{\alpha_n}\colon M_{\alpha_n}\to M_{\alpha_{n+1}}\mid n\geq 1\}$ and x in M_{α_1}, there exists an integer m depending on x such that $f_{\alpha_m}f_{\alpha_{m-1}}\cdots f_{\alpha_1}(x)=0$.

Let $\{A_\alpha\}_I$ be an independent family of submodules of an R-module M. $\sum_I A_\alpha$ is said to be a locally direct summand of M if $\sum_F A_\beta$ is a direct summand of M for any finite subset F of I ([3], [9]).

1. $\mathcal{A}$-continuous modules and $\mathcal{A}$-quasi-continuous modules

Let M be an R-module, and $\mathcal{A}$ a subfamily of $\mathcal{L}(M)$. We assume that

$\mathcal{A}$ satisfies the following conditions:

(α) For $A\in\mathcal{A}$ and $N\in\mathcal{L}(M)$, $A\simeq N$ implies $N\in\mathcal{A}$.

(β) For $A\in\mathcal{A}$ and $N\in\mathcal{L}(M)$, $A\subseteq_e N$ implies $N\in\mathcal{A}$.

For examples, $\mathcal{L}(M)$ itself, the family of all uniform submodules of M, more generally $\mathcal{A}(\tau\text{-}gen(\mathcal{L}(M)))$ and $\mathcal{A}(\tau\text{-}dim(\mathcal{L}(M)))$ are such families.

For $\mathcal{A}$, we consider the following conditions:

(C_1) M has the extending property of modules for $\mathcal{A}$.

(C_2) For any $A\in\mathcal{A}$ such that $A\langle\oplus M$, any sequence $0\to A\to M$ splits.

(C_3) For $A\in\mathcal{A}$ and $N\in\mathcal{L}(M)$, if they are direct summands of M with $A\cap N=0$ then $A\oplus N$ is also a direct summand of M.

DEFINITION. We say that M is $\mathcal{A}$-continuous (resp. $\mathcal{A}$-quasi-continuous) if the conditions (C_1) and (C_2) (resp. (C_1) and (C_3)) are satisfied.

We simply say that M is continuous (resp. quasi-continuous) if it is $\mathcal{L}(M)$-continuous (resp. $\mathcal{L}(M)$-quasi-continuous) ([11]).

$\mathcal{A}$-continuous modules and $\mathcal{A}$-quasi-continuous modules are investigated in connection with the following conditions:

(C_4) For any $A\in\mathcal{A}$, $N\in\mathcal{L}(M)$ and any monomorphism f: $A\to M/N$, there exists a homomorphism h: $M/N\to M$ such that the diagram

$$\begin{array}{ccccc} 0 & \to & A & \xrightarrow{i} & M \\ & & \downarrow f & \nearrow h & \\ & & M/N & & \end{array}$$

is commutative, where i is the inclusion map.

(C_5) For any $A\in\mathcal{A}$ and $N\in\mathcal{L}(M)$ such that $N\langle\oplus M$ and $A\cap N=0$, every homomorphism of A to N is extended to a homomorphism of M to N.

(C_6) For any $A\in\mathcal{A}$, there exists a direct summand N of M such that $A\oplus N\subseteq_e M$.

DEFINITION. We say that M is $\mathcal{A}$-quasi-injective provided that M satisfies the condition (C_4).

NOTE. By virtue of Miyashita [15], we know that $\mathcal{L}(M)$-quasi-injectivity is nothing but usual quasi-injectivity.

Proposition 1.1. *The condition* (C_4) *is equivalent to the following condition*:

(C_4') *Let* $A\in\mathcal{A}$, $N\in\mathcal{L}(M)$ *and* j *a monomorphism from* A *to* M. *Then, for any monomorphism* f: $A\to M/N$, *there exists a homomorphism* h: $M/N\to M$ *such that the diagram*

$$\begin{array}{ccccc} 0 & \to & A & \xrightarrow{j} & M \\ & & f\downarrow & \nearrow h & \\ & & M/N & & \end{array}$$

is commutative.

Proof. $(C_4') \Rightarrow (C_4)$ is clear. Since $\mathcal{A}$ satisfies (α) it is also easy to see that $(C_4) \Rightarrow (C_4')$.

Theorem 1.2. *If M is $\mathcal{A}$-quasi-injective, then it is $\mathcal{A}$-continuous.*

Proof. Let $A \in \mathcal{A}$ and assume that A is a direct summand of M; write $M = A \oplus A'$. By π and i we denote the projection: $M = A \oplus A' \to A$ and the injection: $A \to M$, respectively. Now, let $f: A \to M$ be a monomorphism and put $N = f(A)$. Consider the diagram

$$\begin{array}{ccccc} 0 & \to & A & \xrightarrow{i} & M \\ & & \downarrow f & \nearrow h & \\ & & M & & \end{array}$$

By the assumption there exists $h: M \to M$ such that $hf = i$. Then $\pi hf = 1_A$; whence $N \subset_{\oplus} M$. Thus the condition (C_2) holds.

To show that (C_1) holds, let $A \in \mathcal{A}$. We can assume that A is a closed submodule in M, because $\mathcal{A}$ satisfies the condition (β). We take $E(A)$ in $E(M)$: $E(M) = E(A) \oplus T$. Let π be the projection: $E(M) = E(A) \oplus T \to E(A)$. We claim $\pi(M) \subseteq M$. Put $N = \{x \in M \mid \pi(x) \in M\}$ (cf. [12, Theorem 1.1]). Since A is a closed submodule of M, we see that

$$\begin{aligned} N &= A \oplus (T \cap M) \\ &= (E(A) \cap A) \oplus (T \cap M) . \end{aligned}$$

Put $K = T \cap M$ and consider the diagram

$$\begin{array}{ccccc} 0 & \to & A & \xrightarrow{i} & M \\ & & \downarrow f & & \\ & & (A \oplus K)/K & & \\ & & \cap & & \\ & & M/K & & \end{array}$$

where i is the identity map and f the canonical isomorphism. By the $\mathcal{A}$-quasi-injectivity of M we get $h: M/K \to M$ such that $hf = i$. Let η_K be the canonical map: $M \to M/K$ and put $\phi = h\eta_K$. Then $\phi \in \mathrm{End}_R(M)$. Consider the set $U = \{\phi(m) - \pi(m) \mid m \in M\}$. If $U \neq 0$, then there exists $m \in M$ such that $0 \neq \phi(m) - \pi(m) \in M$. Then $\pi(m) \in M$; whence $m \in N = A \oplus K$. Set $m = a + k$, where $a \in A$, $k \in K$. Then $\phi(m) = \phi(a+k) = h\eta_K(a+k) = h(a) = a$, while $\pi(m) = \tau(a+k) = a$. As a result, $\phi(m) - \pi(m) = 0$, a contradiction. Thus we have $\phi = \pi \mid M$ and hence $\pi(M) \subseteq M$. This shows that $M = A \oplus K$.

Theorem 1.3. *The condition (C_2) implies the condition (C_3). Therefore,* *if M is $\mathcal{A}$-continuous, then it is $\mathcal{A}$-quasi-continuous.*

Proof. Let A and N be direct summands of M with $A\in\mathcal{A}$ and $A\cap N=0$. Put $M=N\oplus X$ and denote the projection: $M=N\oplus X\to X$ by π. Since $N\cap A=0$, we see $A\simeq\pi(A)$. So, $\pi(A)<\oplus X$ by the assumption. If we put $X=\pi(A)\oplus Y$, we have $N\oplus A=N\oplus\pi(A)<\oplus M$.

Proposition 1.4. *If M satisfies the condition* (C_1) *for $\mathcal{A}$, then every direct summand M_1 satisfies the condition* (C_1) *for* $\{A\in\mathcal{A}\mid A\subseteq M_1\}$.

Proof. Let $M=M_1\oplus M_2$ and let A_1 be a submodule of M_1. To show the assertion, we can assume by the condition (β) that A_1 is a closed submodule of M_1. So, we want to see that A_1 is a direct summand of M_1. At any rate, there exists a direct summand A_1^* of M with $A_1\subseteq_e A_1^*$. Let π_i: $M=M_1\oplus M_2\to M_i$ be the projection for $i=1$, 2. Then $A_1\subseteq\pi_1(A_1^*)$. Moreover, we can see from $A_1\subseteq_e A_1^*$ and $A_1^*\cap M_2=0$ that $A_1\subseteq_e\pi_1(A_1^*)$; whence $A_1=\pi_1(A_1^*)$. This implies that $A_1^*=A_1\oplus\pi_2(A_1^*)$ and hence $\pi_2(A_1^*)=0$ and $A_1^*=\pi_1(A_1^*)=A$. Thus A is a direct summand of M_1.

Proposition 1.5. *Under the condition* (C_1), *the condition* (C_3) *is equivalent to* (C_5).

Proof. Assume that (C_3) holds. Let $A_1\in\mathcal{A}$ and $N_2<\oplus M$ such that $A_1\cap N_2=0$, and let f: $A_1\to N_2$ a homomorphism. Put $B_1=\{x+f(x)\mid x\in A_1\}$. Then $B_1\in\mathcal{A}$ since $A_1\simeq B_1$. Hence there exists a direct summand $B_1^*<\oplus M$ such that $B_1\subseteq_e B_1^*$. Since $N_2\cap B^*=0$, we see from (C_3) that $M=N_2\oplus B^*\oplus Y$ for some submodule Y. Let π: $M=N_2\oplus B^*\oplus Y\to N_2$ be the projection. Then Then $-\pi$ is a required extension of f.

Conversely, assume that (C_5) holds. Let A and N be direct summands of M such that $A\in\mathcal{A}$ and $A\cap N=0$. Let X be a submodule of M with $M=N\oplus X$, and let π: $M=N\oplus X\to X$ be the projection. Then $A\simeq\pi(A)$ since $A\cap N=0$. Hence $\pi(A)$ is in $\mathcal{A}$ and therefore there exists a direct summand $Y<\oplus X$ such that $\pi(A)\subseteq_e Y$ by Proposition 1.4. Then

$$M=N\oplus Y\oplus Z,\ X=Y\oplus Z$$

for some Z. Let π_1 and π_2 be the projections: $M=N\oplus Y\oplus Z\to N$ and $M=N\oplus Y\oplus Z\to Y$, respectively. Clearly, $A=\{\pi_1(a)+\pi_2(a)\mid a\in A\}$ and the mapping f: $\pi_2(A)\to N$ given by $\pi_2(a)\to\pi_1(a)$ is well defined. Applying the condition C_5), we can extend f to a homomorphism $\bar{f}$: $Y\to\pi_1(A)$. Since $\pi_2(A)\subseteq_e Y$, we see that $A\subseteq_e\{y+\bar{f}(y)\mid y\in Y\}$; whence $A=\{y+\bar{f}(y)\mid y\in Y\}$ and hence $\pi(A)=Y$. Consequently we see that $N\oplus A=N\oplus Y<\oplus M$.

Proposition 1.6. *If the condition* (C_5) *and* (C_6) *hold then the condition* (C_1) *holds.*

Proof. Let $A\in\mathcal{A}$. By the condition (C_6) there exists a direct summand $N_2\langle\oplus M$ such that $A\oplus N_2\subseteq_e M$. Put $M=N_1\oplus N_2$ and denote the projection $M=N_1\oplus N_2\to N_i$ by π_i. Then $A\simeq\pi_1(A)$ and $\pi_1(A)\subseteq_e N_1$. The mapping f: $\pi_1(A)\to\pi_2(A)$ given by $\pi_1(A)\to\pi_2(a)$ is well defined, and therefore it is extended to a mapping $\bar{f}$: $N_1\to N_2$ by the condition (C_5). Putting $A^*=\{x+\bar{f}(x)\,|\,x\in N_1\}$, we see $A\subseteq_e A^*$. Moreover $A^*\oplus N_2=N_1\oplus N_2$ and hence $A^*\langle\oplus M$.

From Propositions 1.5, 1.6 and Theorem 1.3, we have the following theorems.

Theorem 1.7. *The following conditions are equivalent*:
1) *M is $\mathcal{A}$-quasi-continuous.*
2) *M satisfies the conditions* (C_1) *and* (C_5) *for* $\mathcal{A}$.
3) *M satisfies the conditions* (C_5) *and* (C_6) *for* $\mathcal{A}$.

Theorem 1.8. *M is $\mathcal{A}$-continuous if and only if it satisfies the conditions* (C_2), (C_5) *and* (C_6) *for* $\mathcal{A}$.

From Theorems 1.2 and 1.3 and Proposition 1.5 we have

Theorem 1.9 ([11]). *The following conditions are equivalent*:
1) *M is quasi-injective.*
2) *$M\oplus M$ is continuous.*
3) *$M\oplus M$ is quasi-continuous.*

2. Continuous modules with indecomposable decompositions

In this section, we assume that M is a direct sum of uniform modules $\{M_\alpha\}_I$;

$$M=\sum_I\oplus M_\alpha\,.$$

For a subset J of I, we put

$$M(J)=\sum_J\oplus M_\beta$$

and denote its cardinal by $|J|$.

The following lemma is easily shown by Zorn's lemma.

Lemma 2.1. *M satisfies the condition* (C_6) *for* $\mathcal{L}(M)$. *More precisely, for any submodule A of M and $M(K)$ with $A\cap M(K)=0$, there exists a subset J of I with $J\supset K$ such that $A\cap M(J)=0$ and $A\oplus M(J)\subseteq_e M$.*

Let τ be a cardinal number. We shall consider the following conditions (C_3^*) and (C_5^*) instead of (C_3) and (C_5) for $\mathcal{A}(\tau\text{-}dim(\mathcal{L}(M)))$, respectively:

(C_3^*): Let $A\in\mathcal{A}(\tau\text{-}dim(\mathcal{L}(M)))$ and $J\subseteq I$. If $A\langle\oplus M$ and $A\cap M(J)=0$ then $A\oplus M(J)\langle\oplus M$, equivalently (cf. Lemma 2.1), $A\oplus M(J)\subseteq_e M$ implies that

$M=A\oplus M(J)$.

(C_5^*): For any J with $|J|\leq\tau$ and any $A\subseteq M(J)$, every homomorphism from A to $M(I-J)$ is extended to one from $M(I)$ to $M(I-J)$.

REMARKS. 1) Let N be a direct summand of M with $M=N\oplus\sum_L\oplus M_\alpha$ for some $L\subseteq I$. Then N is written as a direct sum $\sum_{I-J}\oplus N_\alpha$ of uniform submodules corresponding to $\sum_{I-J}\oplus M_\beta$. Clearly M satisfies the condition (C_5^*) for $\mathcal{A}(\tau\text{-}dim(\mathcal{L}(M)))$ with respect to $M=\sum_I\oplus M_\alpha$ if and only if M satisfies the condition (C_5^*) for $\mathcal{A}(\tau\text{-}dim(\mathcal{L}(M)))$ with respect to $M=\sum_L\oplus N_\alpha\oplus\sum_{I-L}\oplus M_\beta$.

2) By the same proof as in Proposition 1.5 we can show that under the condition (C_1) for $\mathcal{A}(\tau\text{-}dim(\mathcal{L}(M)))$, ($C_5^*$) is equivalent to ($C_3^*$) for $\mathcal{A}(\tau\text{-}dim(\mathcal{L}(M)))$, and furthermore we can see from the remark 1) that the conditions (C_1) and (C_5^*) for $\mathcal{A}(\tau\text{-}dim(\mathcal{L}(M)))$ implies the condition (C_3) for $\mathcal{A}(\tau\text{-}dim(\mathcal{L}(M)))$.

Now, by Lemma 2.1, the remarks above and Proposition 1.6 we get the following theorems:

Theorem A. *The following conditions are equivalent*:

1) *M is $\mathcal{A}(\tau\text{-}dim(\mathcal{L}(M)))$-quasi-continuous.*

2) *M satisfies the conditions* (C_1) *and* (C_3^*) *for $\mathcal{A}(\tau\text{-}dim(\mathcal{L}(M)))$.*

3) *M satisfies the condition* (C_5^*) *for $\mathcal{A}(\tau\text{-}dim(\mathcal{L}(M)))$.*

Theorem B. *M is $\mathcal{A}(\tau\text{-}dim(\mathcal{L}(M)))$-continuous if and only if M satisfies the conditions* (C_2) *and* (C_5^*) *for $\mathcal{A}(\tau\text{-}dim(\mathcal{L}(M)))$.*

Theorem 2.2. *We assume that each M_α is completely indecomposable. Then the following conditions are equivalent for a finite cardinal n.*

1) *M is $\mathcal{A}(n\text{-}dim(\mathcal{L}(M)))$-quasi-continuous.*

2) *M satisfies the condition* (C_1) *for $\mathcal{A}(n\text{-}dim(\mathcal{L}(M)))$, and for any pair α, $\beta\in I$ every monomorphism from M_α to M_β is an isomorphism.*

3) *M has the extending property of finite direct sum for $\mathcal{A}(n\text{-}dim(\mathcal{L}(M)))$.*

Proof. The implication 1)$\Rightarrow$3) is clear.

3)$\Rightarrow$2). Let α, $\beta\in I$ with $\alpha\neq\beta$, and f a monomorphism from M_α to M_β. Putting $M_\alpha^*=\{x+f(x)\,|\,x\in M_\alpha\}$, we see $M_\alpha^*\oplus M_\beta=M_\alpha\oplus M_\beta$; whence $M_\alpha^*<\oplus M$. On the other hand $M_\alpha^*+M_\alpha=M_\alpha^*\oplus M_\alpha\subseteq_e M_\alpha\oplus M_\beta$. Therefore we have $M_\alpha\oplus M_\alpha^*=M_\alpha\oplus M_\beta$ by the assumption. Thus $f(M_\alpha)=M_\beta$.

2)$\Rightarrow$1). By [8, Theorem 12], M satisfies the condition (C_5) for $\mathcal{A}(1\text{-}dim(\mathcal{L}(M)))$; whence M is $\mathcal{A}(1\text{-}dim(\mathcal{L}(M)))$-quasi-continuous. Let $A\in\mathcal{A}(n\text{-}dim(\mathcal{L}(M)))$ and $N\in\mathcal{L}(M)$ such that $A<\oplus M$, $N<\oplus M$ and $A\cap N=0$. Here, from the Azumaya's theorem ([1]), A is expressed as a finite direct sum of uniform modules. Hence we can see $A\oplus N\langle\oplus M$ since M is $\mathcal{A}(1\text{-}dim(\mathcal{L}(M)))$-quasi-continuous.

Theorem 2.3. *The following conditions are equivalent for a finite cardinal* n:

1) *M is $\mathcal{A}(n\text{-}dim(\mathcal{L}(M)))$-continuous.*

2) i) *M satisfies the condition* (C_1) *for $\mathcal{A}(n\text{-}dim(\mathcal{L}(M)))$,*

ii) *each M_α satisfies the condition* (M−I),

iii) *for any pair α, β in I, every monomorphism from M_α to M_β is an isomorphism.*

Proof. The implication 1)⇒2) is clear.

2)⇒1). By ii) each M_α is completely indecomposable. To show 1) we may show that M satisfies the condition (C_2) for $\mathcal{A}(n\text{-}dim(\mathcal{L}(M)))$. Let $A\in\mathcal{A}(n\text{-}dim(\mathcal{L}(M)))$ and $A\langle\oplus M$. Then by the Azumaya's theorem ([1]), A is expressed as a direct sum of uniform modules, each of which is isomorphic to some member in $\{M_\alpha\}_I$. Hence, for a monomorphism f from A to M, $f(M)$ is indeed a direct summand of M by [8, Theorems 13 and 16].

Theorem 2.4. *Assume that each M_α is completely indecomposable and let τ be a cardinal number. Then the following conditions are equivalent*:

1) *M is $\mathcal{A}(n\text{-}dim(\mathcal{L}(M)))$-quasi-continuous for every finite cardinal n, and $\{M_\alpha\}_I$ is a locally semi-T-nilpotent set.*

2) *M has the extending property of direct sums for $\mathcal{A}(\tau\text{-}dim(\mathcal{L}(M)))$.*

Proof. 1)⇒2). Let $\{A_\beta\}_J$ be an independent subfamily of $\mathcal{A}(\tau\text{-}dim(\mathcal{L}(M)))$. Since M satisfies the condition (C_1) for $\mathcal{A}(1\text{-}dim(\mathcal{L}(M)))$, there exists a direct summand $A_\alpha^*\langle\oplus M$ satisfying $A_\alpha\subseteq_e A_\alpha^*$ for each $\alpha\in I$. Then, the condition (C_3) for $\mathcal{A}(n\text{-}dim(\mathcal{L}(M)))$ shows that $\sum_K\oplus A_\beta^*\langle\oplus M$ for any finite subset K of J. Since $\{M_\alpha\}_I$ is locally semi-T-nilpotent, it follows that $\sum_J\oplus A_\beta^*\langle\oplus M$ by [9].

2)⇒1). By Theorem 2.2, M is $\mathcal{A}(n\text{-}dim(\mathcal{L}(M)))$-quasi-continuous for all finite cardinal n, and every monomorphism from M_α to M_β is an isomorphism for any pair α, β in I. To show that $\{M_\alpha\}_I$ is locally semi-T-nilpotent, let $\{\alpha_i\,|\,i=1, 2, \cdots\}\subseteq I$ with $\alpha_i\neq\alpha_j$ if $i\neq j$, and let f_i: $M_{\alpha_i}\to M_{\alpha_{i+1}}$ be a non-monomorphism, $i=1, 2, \cdots$. We put $M_{\alpha_i}^*=\{x+f_i(x)\,|\,x\in M_i\}$. Then $M_{\alpha_i}^*\langle\oplus M$ for each i and $\sum_{i=0}^{\infty}\oplus M_{\alpha_i}\simeq\sum_{i=1}^{\infty}\oplus M_{\alpha_i}^*$. Hence using 2) we see $\sum_{i=1}^{\infty}\oplus M_{\alpha_i}^*\langle\oplus M$ by again [9]. As a result $\sum_{i=1}^{\infty}\oplus M_{\alpha_i}=\sum_{i=1}^{\infty}\oplus M_{\alpha_i}^*$. Therefore it follows that for any x in M_{α_1} there exists n such that $f_nf_{n-1}\cdots f_1(x)=0$.

Theorem 2.5. *The following conditions are equivalent for a given cardinal*

$\tau \geq \aleph_0$:

1) *M is $\mathcal{A}(\tau$-$dim(\mathcal{L}(M)))$-continuous.*

2) *M is $\mathcal{A}(\tau$-$dim(\mathcal{L}(M)))$-quasi-continuous with the conditions:*

i) *Each M_α satisfies the condition* (M—I),

ii) *$\{M_\alpha\}_I$ is locally semi-T-nilpotent.*

3) *M has the extending property of direct sum for $\mathcal{A}(\tau$-$dim(\mathcal{L}(M)))$ and each M_α satisfies the condition* (M—I).

Proof. Every condition of 1)~3) implies that each M_α is completely indecomposable; so the implication 2)⇔3) follows from Theorem 2.4.

1)⇒2). We may only show that $\{M_\alpha\}_I$ is locally semi-T-nilpotent. In order to prove this, as in the proof of Theorem 2.4, consider $\{\alpha_i \mid i=1, 2, \cdots\} \subseteq I$ and non-monomorphisms $\{f_i\colon M_{\alpha_i} \to M_{\alpha_{i+1}} \mid i=1, 2, \cdots\}$, and put $M^*_{\alpha_i} = \{x+f_i(x) \mid x \in M_{\alpha_i}\}$, $i=1, 2, \cdots$. Then we see $\sum_{i=1}^{\infty} \oplus M^*_{\alpha_i} \subseteq_e \sum_{i=1}^{\infty} \oplus M_{\alpha_i}$; whence the condition ($C_2$) for $\mathcal{A}(\tau$-$dim(\mathcal{L}(M)))$ shows $\sum_{i=1}^{\infty} \oplus M^*_{\alpha_i} < \oplus M$. (Note $\tau \geq \aleph_0$). As a result $\sum_{i=1}^{\infty} \oplus M_{\alpha_i} = \sum_{i=1}^{\infty} \oplus M^*_{\alpha_i}$ and hence we see that for any x in M_{α_1} there exists n satisfying $f_n f_{n-1} \cdots f_1(x)=0$.

2)⇒1). We want to show that M satisfies the condition (C_2) for $\mathcal{A}(\tau$-$dim(\mathcal{L}(M)))$. More strictly we can show that M satisfies the condition (C_2) for $\mathcal{L}(M)$. Let A be a direct summand of M. Since $\{M_\alpha\}_I$ is locally semi-T-nilpotent, A is expressed as a direct sum of uniform modules $\{A_\beta\}_J$ each of which is isomorphic to some M_α (cf. [7], [13]). Therefore for any monomorphism f from A to M, $f(A)$ is a direct summand of M by [8, Theorems 13, 16 and Corollary 14].

3. Quasi-continuous modules and the extending property of direct sum

Quasi-continuous modules are characterized as follows:

Proposition 3.1 (cf. [11]). *For a given R-module M, the following conditions are equivalent:*

1) *M is quasi-continuous.*

2) *Every decomposition $E(M)=E_1 \oplus \cdots \oplus E_n$ implies $M=(E_1 \cap M) \oplus \cdots \oplus (E_n \cap M)$.*

3) *Every decomposition* $E(M) = \sum_I \oplus E_\alpha$ *implies* $M = \sum_I \oplus (E_\alpha \cap M)$.

Proof. 1)⇒2). Let $E(M)=E_1 \oplus \cdots \oplus E_n$. Then $M_e \supseteq (E_1 \cap M) \oplus \cdots \oplus (E_n \cap M)$ and each $E_i \cap M$ is a closed submodule of M; whence we see from 1) that $M=(E_1 \cap M) \oplus \cdots \oplus (E_n \cap M)$.

2)⇒1) Let A be a submodule of M. We take its injective hull $E(A)$ in

$E(M)$. Then $A\subseteq_e(\mathrm{E}(A)\cap M)$ and, by 2), $\mathrm{E}(A)\cap M\langle\oplus M$. Hence M satisfies the condition (C_1) for $\mathcal{L}(M)$. Similarly the condition (C_3) for $\mathcal{L}(M)$ is shown.

3)$\Rightarrow$2) is clear.

2)$\Rightarrow$3). Let $\mathrm{E}(M)=\sum_I \oplus E_\alpha$, and $x\in M$. Then x lies in $E_{\alpha_1}\oplus\cdots\oplus E_{\alpha_n}$ for some $\alpha_1,\cdots,\alpha_n\in I$. By 2) we see $M\cap(E_{\alpha_1}\oplus\cdots\oplus E_{\alpha_n})=(E_{\alpha_1}\cap M)\oplus\cdots\oplus(E_{\alpha_n}\cap M)$; whence $x\in(E_{\alpha_1}\cap M)\oplus\cdots\oplus(E_{\alpha_n}\cap M)\subseteq\sum_I\oplus(E_\alpha\cap M)$ and hence $M=\sum_I\oplus(E_\alpha\cap M)$.

Theorem 3.2. *Let R be a right Noetherian ring. Then the following conditions are equivalent for a given R-module M:*

1) *M is quasi-continuous.*
2) *M has the extending property of direct sum for $\mathcal{L}(M)$.*
3) *M has the extending property of direct sum of uniform modules.*

Proof. Since R is Noetherian, we know from [14] that every direct sum of injective R-modules is also injective and every injective R-module is expressed as a direct sum of indecomposable modules. Therefore the proof follows from the above proposition.

Combining Theorem 3.2 to [8, Theorem 31], we have

Corollary 3.3. *Let R be a Dedekind domain. Then an R-module M is quasi-continuous if and only if either* i) *M is quasi-injective or* ii) *$M=K\oplus E$ where E is torsion and injective and $0\neq K\subsetneqq Q$, the quotient field of R.*

Corollary 3.4. *Let R be a Dedekind domain. Then an R-module M is continuous if and only if it is quasi-injective.*

Proof. Quasi-injectives are always continuous modules by Theorem 1.1. Now assume that M is continuous. If M is not quasi-injective then M is written as $M=K\oplus E$ where E is injective and $0\neq K\subsetneqq Q$, the quotient field of R. However, inasmuch as M is continuous, K satisfies (M$-$I); whence K coincides with Q, a contradiction. Thus M must be quasi-injective.

Finally, we show the following theorem.

Theorem 3.5. *For a given R-module M the following conditions are equivalent:*

1) *M is quasi-continuous and every internal direct sum of submodules of M which is a locally direct summand of M is a direct summand of M*
2) *M has the extending property of direct sum for $\mathcal{L}(M)$.*
3) *M is written as $M=\sum_I\oplus M_\alpha$ with the following conditions:*

i) *Each M_α is uniform.*

ii) *$\{M_\alpha\}_I$ is locally semi-T-nilpotent.*

iii) *For any partition $I=I_1\cup I_2$ and any submodule A of $\sum_{I_1}\oplus M_\alpha$, every homomorphism from A to $\sum_{I_2}\oplus M_\alpha$ is extended to one from $\sum_{I_1}\oplus M_\alpha$ to $\sum_{I_2}\oplus M_\beta$.*

Proof. It is easy to see 1)$\Leftrightarrow$2).

2)$\Rightarrow$3). Take an independent family of non-zero cyclic submodules $\{x_\alpha R\}_K$ of M with $\sum_K\oplus x_\alpha R\subseteq_e M$. Then by the assumption we get a decomposition $M=\sum_K\oplus N_\alpha$ such that $x_\alpha R\subseteq_e N_\alpha$ for each α. Let $\alpha\in K$ and suppose that N_α is not finite dimensional. Then there exists a countable family $\{N_{\alpha_i}\mid i=1, 2, \cdots\}$ such that $\sum_{i=1}^{\infty}\oplus N_{\alpha_i}\subseteq_e N_\alpha$. Again by the assumption there exists a decomposition $M=(\sum_{i=1}^{\infty}\oplus N_{\alpha_i}^*)\oplus\sum_{K-\{\alpha\}}\oplus N_\beta$ with $N_{\alpha_i}\subseteq_e N_{\alpha_i}^*$ for each i. Then

$$x_\alpha R\subseteq_e N_\alpha\simeq\sum_{i=1}^{\infty}\oplus N_{\alpha_i}^*.$$

But this is impossible. Thus each N_α must be finite dimensional, and hence by the assumption, M is written as a direct sum of uniform modules; say

$$M=\sum_I\oplus M_\alpha.$$

Now, as in the proof of Theorem 2.2, we can show that, for any pair α, β in I, every monomorphism from M_α to M_β is an isomorphism. Using this fact, we can also show that $\{M_\alpha\}_I$ is locally semi-T-nilpotent as in the proof of Theorem 2.4.

Next, let $I=I_1\cup I_2$ be a partition of I, A an essential submodule of $M(I_1)=\sum_{I_1}\oplus M_\alpha$ and f a homomorphism from A to $M(I_2)=\sum_{I_2}\oplus M_\beta$. By the assumption we get $M=A^*\oplus M(I_2)$ such that $\{x+f(x)\mid x\in A\}\subseteq_e A^*$. Let π be the projection: $M=A^*\oplus M(I_2)\rightarrow M(I_2)$. Then we can see $-\pi\mid A=f$.

3)$\Rightarrow$2). Let $M=\sum_I\oplus M_\alpha$ be a decomposition with the conditions i)$\sim$iii). We first show that for any submodule A of M there exists a direct summand $A^*\langle\oplus M$ such that $A\subseteq_e A^*$ and A^* is written as a direct sum of uniform modules. By Lemma 2.1 there exists $K\subseteq I$ such that

$$M_e\supseteq A\oplus(\sum_K\oplus M_\beta).$$

Put $\mathrm{M}(K)=\sum_K\oplus M_\beta$ and $\mathrm{M}(I-K)=\sum_{I-K}\oplus M_\alpha$, and denote the projections: $M\rightarrow\mathrm{M}(I-K)$ and $M\rightarrow\mathrm{M}(K)$ by π_1 and π_2, respectively with respect to $M=\mathrm{M}(K)\oplus\mathrm{M}(I-K)$. Then the mapping f: $\pi_1(A)\rightarrow\mathrm{M}(K)$ given by $\pi_1(a)\rightarrow\pi_2(a)$ is well defined. By iii) f is extended to a homomorphism f': $\mathrm{M}(I-K)\rightarrow\mathrm{M}(K)$. Put

$$A^* = \{x+f'(x) \mid x \in \mathrm{M}(I-K)\}\ .$$

Then we see $A \subseteq_e A^* \langle \oplus M$ and $A \simeq \mathrm{M}(I-K)$ as claimed.

From this fact, our proof follows if we show the following: Let $\{A_\alpha\}_L$ be an independent family of indecomposable (uniform) direct summands of M with $M_e \supseteq \sum_L \oplus A_\alpha$. Then $M = \sum_L \oplus A_\alpha$.

We claim that $\sum_L \oplus A_\alpha$ is a locally direct summand of M. Let $\{\alpha_1, \cdots, \alpha_n, \alpha_{n+1}\} \subseteq L$ and assume $\sum_{i=1}^n \oplus A_{\alpha_i} \langle \oplus M$. Then by Lemma 2.1 we can take a subset $J \subseteq I$ such that

$$M = \mathrm{M}(I-J) \oplus \mathrm{M}(J)$$
$$_e\supseteq A_{\alpha_1} \oplus \cdots A \oplus_{\alpha_{n+1}} \oplus \mathrm{M}(J)\ .$$

Then the dimension of $\mathrm{M}(I-J)$ is equal to $n+1$; so $\mathrm{M}(I-J) = M_{\alpha_1} \oplus \cdots \oplus M_{\alpha_{n+1}}$ for some $\{\alpha_1, \cdots, \alpha_{n+1}\} \subseteq I$. Here using the condition iii) we get a homomorphism g from $\mathrm{M}(I-J)$ to $\mathrm{M}(J)$ such that $A \subseteq_e T = \{x+g(x) \mid x \in M(I-J)\} \langle \oplus M$. Then

$$M = T \oplus \mathrm{M}(J)\ .$$

Since $A_{\beta_1} \oplus \cdots \oplus A_{\beta_n} \langle \oplus T$, we have $T = A_{\beta_1} \oplus \cdots \oplus A_{\beta_n} \oplus X$ for some indecomposable module X. Since $(A_{\beta_1} \oplus \cdots \oplus A_{\beta_n}) \cap \mathrm{M}(J) = 0$ there exists $\alpha_i \in \{\alpha_1, \cdots, \alpha_{n+1}\}$ such that

$$M_e \supseteq A_{\beta_1} \oplus \cdots \oplus A_{\beta_n} \oplus M_{\alpha_i} \oplus \mathrm{M}(J)$$

(cf. Lemma 2.1). We can assume $\alpha_i = \alpha_{n+1}$ without loss of generality. Since $A_{\beta_1} \oplus \cdots \oplus A_{\beta_n} \langle \oplus M$, we see from the condition iii) that

$$M = A_{\beta_1} \oplus \cdots \oplus A_{\beta_n} \oplus \mathrm{M}_{\alpha_{n+1}} \oplus \mathrm{M}(J)\ .$$

Since $A_{\beta_{n+1}} \cap (A_{\beta_1} \oplus \cdots \oplus A_{\beta_n} \oplus \mathrm{M}(J)) = 0$ and $A_{\beta_{n+1}} \langle \oplus M$, we see again by the condition iii) that

$$M = A_{\beta_1} \oplus \cdots \oplus A_{\beta_n} \oplus A_{\beta_{n+1}} \oplus \mathrm{M}(J)\ .$$

Consequently $\sum_K \oplus A_\alpha$ is a locally direct summand of M. On the contrary to the assertion, we assume $M \supsetneqq \sum_K \oplus A_\alpha$. Then there must exist $\alpha_1 \in I$ such that $M_{\alpha_1} \not\subseteq \sum_K \oplus A_\alpha$. Pick $x_{11} \in M_{\alpha_1}$ with $x_{11} \notin \sum_K \oplus A_\alpha$. Since $M_e \supseteq \sum_K \oplus A_\alpha$ there exists $r \in R$ such that $0 \neq x_{11} r \in \sum_{F_1} \oplus A_\beta$ for some finite subset $F_1 \subseteq K$. By the above,

$$M = \sum_{F_1} \oplus A_\beta \oplus \mathrm{M}(J) \cdots (*)$$

for some J_1 of I. We express x_{11} in (*) as

$$x_{11}=y_1+x_{12}+x_{13}+\cdots+x_{1n_1}$$

where $y_1\in\sum_{F_1}\oplus A_\beta$, $x_{11}\in M_{\alpha_{12}}, \cdots, x_{1n_1}\in M_{\alpha_{1n_1}}$ $(\alpha_{12},\cdots,\alpha_{1n_1}\in J_1)$. Then $(0: x_{11})\subsetneqq(0: x_{1j})$, $j=2,\cdots,n_1$ since $0\neq x_{11}r\in\sum_{F_1}\oplus A_\beta$. Because of $x_{11}\notin\sum_K\oplus A_\beta$, one of $\{x_{12},\cdots x_{1n_1}\}$ does not lie in $\sum_K\oplus A_\beta$; say x_{12}. Then $M_{\alpha_1}\neq M_{\alpha_2}$ and $x_{12}\in M_{\alpha_2}$ and $x_{12}\notin\sum_K\oplus A_\alpha$.

Since $M_e\supseteq\sum_K\oplus A_\alpha$, there exists $r_2\in R$ such that $0\neq x_{12}r_2\in\sum_{F_2}\oplus A_\beta$ for some $F_2\supseteq F_1$. Since $\sum_{F_2}\oplus A_\beta<\oplus M$ and $F_2\supseteq F_1$ we see from (*) and the condition iii) that

$$M=(\sum_{F_2}\oplus A_\beta)\oplus\mathrm{M}(J_2)\cdots(**)$$

for some $J_2\subseteq J_1$. We express x_{12} in (**) as

$$x_{12}=y_2+x_{23}+\cdots+x_{2n_2}$$

where $y_2\in\sum_{F_2}\oplus A_\beta$, $x_{2i}\in M_{\alpha_{2i}}$, $i=3,\cdots,n_2$. Then $(0: x_{12})\subsetneqq(0: x_{2j})$, $j=3,\cdots,n_2$, and some x_{2j}, say x_{23} does not lie in $\sum_K\oplus A_\alpha$. Then $x_{23}\notin\sum_K\oplus A_\alpha$, $(0: x_{11})\subsetneqq(0: x_{12})\subsetneqq(0: x_{23})$ and $M_{\alpha_i}\neq M_{\alpha_3}$ for $i=1, 2$. Continuing this fashion we get a countable subset $\{\alpha_i\,|\,i=1,2,\cdots\}\subseteq I$ and $\{x_{i,i+1}\in M_{\alpha_i}\,|\,i=1,2,\cdots\}$ such that $\alpha_i\neq\alpha_j$ if $i\neq j$ and $(0: x_{11})\subsetneqq(0: x_{12})\subsetneqq(0: x_{23})\subsetneqq\cdots$. Hence using the condition iii), we get a homomorphism f_i: $M_{\alpha_i}\to M_{\alpha_{i+1}}$ such that $f_i(x_{i,i+1})\to x_{i+1,i+2}$ for each i, which contradicts the condition ii). Thus we have $M=\sum_K\oplus A_\alpha$.

We end this paper with the following remarks:

1) Let M be a non-singular R-module, and consider submodules A, B and C with $A\subseteq_e C<\oplus M$ and $A\subseteq_e B$. Then B is contained in C. For, set $M=C\oplus C'$ and π: $M\to C$ and π': $M\to C'$ denotes the projections with respect to $M=C\oplus C'$. Since $A\subseteq_e B$ we see $B\cap C'=0$; so the mapping f: $\pi(B)\to\pi'(B)$ given by $\pi(b)\to\pi'(b)$ is well defined. Since $A\subseteq ker(f)$, it follows $ker(f)\subseteq_e\pi(B)$. As a result, $\pi(B)/ker(f)$ is a singular module, while $f(\pi(B))=\pi'(B)$ is non-singular. Therefore $\pi'(B)=0$ and hence $B=\pi(B)\subseteq C$.

2) A von Neumann regular ring R is right $\aleph_0$-continuous in the sense of Halperin if and only if it has the extending property of module for the family of all $\aleph_0$ generated right ideals of R (see [2]). Combining this to the remark 1) we see that a von Neumann regular ring R is right $\aleph_0$-continuous if and only if it is $\mathcal{A}(\aleph_0\text{-}gen(\mathcal{L}(M)))$-continuous.

3) We also see from the remark 1) that a non-singular module with es-

sential socle has the extending property of simple module if and only if it has the extending property of uniform module.

4) The dual result of Theorem 3.5 is also shown (see, [16]).

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