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# SEMIPERFECT MODULES AND QUASI-SEMIPERFECT MODULES

KIYOICHI OSHIRO

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Recently in his study of $QF$-2 rings, M. Harada has introduced the extending property of simple modules and the lifting property of simple modules which are mutually dual notions, and he has extensively studied modules with these properties ([10]~[14]). It should be noted that the extending property of simple modules is one of considerable extending properties on modules and it has been somewhat widely studied than the lifting property of simple modules ([11], [13], [14]).

Let $M$ be an $R$-module and $\mathcal{A}$ a subfamily of the family $\mathcal{L}(M)$ of all submodules of $M$. $M$ is said to have the extending property of modules for $\mathcal{A}$ provided that every member of $\mathcal{A}$ is embedded to a direct summand of $M$ as an essential submodule. In particular, $M$ is said to have the extending property of simple modules if it has the extending property of modules for $\{A \in \mathcal{L}(M) \mid A \text{ is simple}\}$. Dually $M$ is said to have the lifting property of simple modules if every simple submodule of $M/\mathrm{J}(M)$ is induced from a direct summand of $M$, where $\mathrm{J}(M)$ is the Jacobson radical of $M$.

Under this circumstance, the following natural question immediately arises: Can we define the notion dual to 'the extending property of modules for $\mathcal{A}$'? This question seems to be interested in module theory, because this dualization leads us to the dualizations of continuous modules and quasi-continuous modules mentioned in Utumi [28]~[30] and Jeremy [16], [17] (cf. [23]).

Section 1 of this paper is concerned with this problem, and the lifting property of modules for $\mathcal{A}$ is defined as follows: $M$ is said to have the lifting property of modules for $\mathcal{A}$, provided that, for any $A$ in $\mathcal{A}$, there exists a direct summand $A^*$ of $M$ such that $A^* \subseteq A$ and $A/A^*$ is small in $M/A^*$.

Using this lifting property, in section 2, we introduce $\mathcal{A}$-semiperfect modules and $\mathcal{A}$-quasi-semiperfect modules as duals to $\mathcal{A}$-continuous modules and $\mathcal{A}$-quasi-continuous modules, respectively, which have been studied in [23]. Of course, these names follow from 'semi-perfect module' in the sense of E. Mares [20] defined on projective modules. $\mathcal{L}(M)$-semiperfect modules and $\mathcal{L}(M)$-quasi-semiperfect modules are simply called semiperfect modules

and quasi-semiperfect modules, respectively, and it is shown that a projective module is semiperfect if and only if it is semiperfect in the sense of Mares. All results in section 2 are quite duals to those in section 1 of [23].

In section 3, we devote fundamental properties of semiperfect modules and quasi-semiperfect modules. A key property of a quasi-semiperfect module $M$ is the following (Proposition 3.2): Every internal direct sum of submodules of $M$ which is a locally direct summand of $M$ is a direct summand of $M$. We derive many theorems using this result. Theorem 3.5 is one of theorems in which it is shown that every quasi-semiperfect module is expressed as a direct sum of hollow modules.

In section 4 we introduce the lifting property of direct sum for $\mathcal{A}$. Let $M$ be an $R$-module and $M/X=\sum_I \oplus(T_\alpha/X)$ a decomposition of a homomorphic image, where $X \subsetneqq T_\alpha$ for all $\alpha \in I$. We say that $M/X=\sum_I \oplus(T_\alpha/X)$ is co-essentially lifted to a decomposition of $M$ if there exists a decomposition $M=X^* \oplus \sum_I \oplus T_\alpha^*$ satisfying $X^* \subseteq X$, $T_\alpha = X+T_\alpha^*$ and $T_\alpha^* \cap X$ is small in $T_\alpha^*$. The notion of 'co-independent family' is defined as follows: A subfamily $\mathcal{N}=\{N_\alpha\}_I$ of $\mathcal{L}(M)$ is said to be co-independent if $M/\bigcap_I N_\alpha \simeq \sum_I \oplus(M/N_\alpha)$, canonically. There is a canonical one to one onto map between the family of all decompositions of all homomorphic images of $M$ and the family of all co-independent subfamily of $\mathcal{L}(M)$. We say that $M$ has the lifting property of direct sums for $\mathcal{A}$, provided that if $\mathcal{N}=\{N_\alpha\}_I$ is a co-independent subfamily of $\mathcal{A}$ then its corresponding decomposition is co-essentially lifted to a decomposition of $M$, or equivalently, there exists a co-independent family $\{N_\alpha^*\}_I$ such that $N_\alpha^* < \oplus M$, $N_\alpha^* \subseteq N_\alpha, N_\alpha/N_\alpha^*$ is small in $M/N_\alpha^*$ for all $\alpha \in I$ and $\bigcap_I N_\alpha^* < \oplus M$. In Theorem 4.9, we show that every quasi-semiperfect module $M$ has the lifting property of direct sums for $\mathcal{L}(M)$.

In the final section 5, we determine all types of quasi-semiperfect modules over Dedekind domains.

Throughout this paper, $R$ is an associative ring with unit, and all modules considered are unitary right $R$-modules.

Let $M$ be an $R$-module. We denote its Jacobson radical by $\mathrm{J}(M)$, and the family of all submodules of $M$ by $\mathcal{L}(M)$. $M$ is said to be hollow if every proper submodule of $M$ is small in $M$. For a submodule $N$ of $M$, we use the symbol $N \subseteq_e M$ to mean that $N$ is essential in $M$. $M$ is said to be completely indecomposable if $\mathrm{End}_R(M)$ is a local ring. Let $\{N_\alpha\}_I$ be an independent set of submodules of $M$. $\sum_I \oplus N_\alpha$ is called a locally direct summand ([15]) if $\sum_F \oplus N_\alpha$ is a direct summand of $M$ for any finite subset $F$ of $I$. Finally, $M$ is said to satisfy (E–I) if every epimorphism of $M$ to $M$ is an isomorphism ([10]).

## 1. Co-closed submodules

Let $M$ be an $R$-module and $N$ a submodule of $M$. We consider the following condition:

(*) *There exists a direct summand* $N^* <\oplus M$ *which is an essential extension of* $N$.

For a subfamily $\mathcal{A}$ of the family $\mathcal{L}(M)$ of all submodules of $M$, $M$ is said to have the extending property of modules for $\mathcal{A}$ if every member in $\mathcal{A}$ satisfies the condition (*). Therefore, in order to dualize this extending property, we must first study the problem; How do we define the condition dual to (*)?

Now, the condition (*) can be re-phrased as follows;

(**) *There exists a closed submodule* $N^*$ *of* $M$ *which is just a direct summand of* $M$ *and is an essential extension of* $N$.

Therefore, for our purpose, we require to obtain the concepts 'co-essential extension' and 'co-closed submodule' in $M$ which correspond to 'essential extension' and 'closed submodule' in $M$, respectively. We define these concepts quite naturally as follows:

DEFINITION. Let $N_1$ and $N_2$ be submodules of an $R$-module $M$ with $N_1 \subseteq N_2$. We say that $N_1$ is a co-essential submodule of $N_2$ in $M$ if the kernel of the canonical map

$$M/N_1 \to M/N_2 \to 0$$

is small in $M/N_1$, or equivalently, if $M=N_2+X$ with $X \supseteq N_1$, then $M=X$. We use the symbol $N_1 \subseteq_c N_2$ in $M$ to mean this situation.

**Proposition 1.1.** *Let* $N$, $N'$, $N_1$, $N_2$, $N_3$ *be submodules of an* $R$*-module* $M$ *with* $N \subseteq N'$ *and* $N_1 \subseteq N_2 \subseteq N_3$. *Then*

a) $N \subseteq_c N$ *in* $M$.

b) $0 \subseteq_c N$ *in* $M$ *iff* $N$ *is small in* $M$.

c) $N_1 \subseteq_c N_2$ *in* $M$ *and* $N_2 \subseteq_c N_3$ *in* $M$ *iff* $N_1 \subseteq_c N_3$ *in* $M$.

d) $N'$ *is small in* $M$ *iff* $N \subseteq_c N'$ *in* $M$ *and* $N$ *is small in* $M$.

Proof. The proofs of a) and b) are trivial, and d) follows from b) and c). We show c). ($\Rightarrow$) Consider $M=N_3+X$ with $X \supseteq N_1$. Since $M=N_3+(N_2+X)$ and $N_2+X \supseteq N_2$, we get $M=N_2+X$ by $N_2 \subseteq_c N_3$ in $M$. Hence noting $N_1 \subseteq X$, we see that $M=X$. Hence $N_1 \subseteq_c N_3$ in $M$. ($\Leftarrow$) Let $X$ be a submodule of $M$ such that $M=N_2+X$ and $X \supseteq N_1$. Then $M=N_3+X$ and hence it follows from $N_1 \subseteq_c N_3$ in $M$ that $M=X$. As a result, $N_1 \subseteq_c N_2$ in $M$. Next, if $M=N_3+X$ with $X \supseteq N_2$, then surely $M=X$ since $X \supseteq N_1$ and $N_1 \subseteq_c N_3$ in $M$. Therefore $N_2 \subseteq_c N_3$ in $M$.

DEFINITION. Let $N$ be a submodule of an $R$-module $M$. We say that $N$ is a co-closed submodule of $M$ provided that $N$ has no proper co-essential

submodule in $M$, i.e., $N' \subseteq_c N$ in $M$ implies $N=N'$.

DEFINITION. Let $N$ and $N'$ be submodules of an $R$-module $M$. $N'$ is said to be a relative supplement for $N$ in $M$ provided that $M=N+N'$ but $M \neq N+X$ for any $X \subsetneq N'$.

We note that there are deep relations between co-closed submodules and relative supplements, as the following result shows:

**Proposition 1.2.** *Let $N$ be a submodule of an $R$-module $M$. If $N$ has a relative supplement $N'$ in $M$, then the following are equivalent:*

a) *$N$ is a co-closed submodule in $M$.*

b) *$N$ is a relative supplement for some submodule in $M$.*

c) *$N$ is a relative supplement for $N'$ in $M$.*

Proof. a)⇒c). Let $X$ be a submodule of $N$ such that $M=X+N'$. To show that $X \subseteq_c N$ in $M$, consider $M=N+Y$ with $Y \supseteq X$. Then $Y=X+(Y \cap N')$ and hence $M=N+Y=N+X+(Y \cap N')=N+(Y \cap N')$. Accordingly the minimality of $N'$ shows $N'=Y \cap N'$; so $N' \subseteq Y$ and hence $M=Y$. Therefore $X \subseteq_c N$ in $M$ and hence a) says that $X=N$ as desired.

c)⇒b) is clear.

b)⇒a). Let $N'$ be a submodule of $M$ for which $N$ is a relative supplement in $M$. If $T$ is a submodule of $M$ with $T \subseteq_c N$ in $M$, then $M=N+N'=N+T+N'$; hence it follows from $T \subseteq_c N$ in $M$ that $M=T+N'$. Therefore we have $N=T$ by the minimality of $N$.

Now, about our co-essential submodules and co-closed submodules, we should observe that, for a given submodule $N$ of $M$, whether there exist a co-essential submodule $N^*$ of $N$ in $M$ which is a co-closed submodule of $M$. Although we do not know whether such $N^*$ always exist or not in general, the following theorem holds.

**Theorem 1.3.** *Let $M$ be an $R$-module and $N$ a submodule of $M$. If $M$ and $M/N$ have projective covers, then there exists a co-closed submodule $N^*$ of $M$ with $N^* \subseteq_c N$ in $M$.*

Proof. Let $(P, f)$ be a projective cover of $M$, i.e., $P$ is projective and $f$: $P \to M$ is an epimorphism with $0 \subseteq_c \ker(f)$ in $P$. Since $M/N$ has a projective cover, by virtue of Bass's lemma ([2, Lemma 2.3]), we have $P=P_1 \oplus P_2$ such that $f(P_2) \subseteq N$ and

$$P_1 \xrightarrow{\eta f} M/N \longrightarrow 0$$

is a projective cover, where $\eta$ is the canonical map: $M \to M/N \to 0$. We claim that $f(P_2)$ is a co-closed submodule of $M$ with $f(P_2) \subseteq_c N$ in $M$. Consider a

submodule $X$ of $M$ such that $M=N+X$ and $X\supseteq f(P_2)$. Putting $Q=ker(\eta(f|P_1))+f^{-1}(X)$, we show $P=Q$. $f^{-1}(X)\supseteq P_2$ is clear and so we may show that $P_1\subseteq Q$. Let $p_1\in P_1$. Then $f(p_1)=n+x$ for some $n\in N$ and $x\in X$. Express $x$ as $x=f(q_1)+f(q_2)$ where $q_1\in P_1$ and $q_2\in P_2$. Since $f(q_1)=x-f(q_2)\in X$, we see that $q_1\in f^{-1}(X)$. Since $\eta f(p_1-q_1)=\eta(n)+\eta(x)=\eta f(q_2)+\eta f(q_1)-\eta f(q_1)=0$, we have $p_1-q_1\in ker(\eta(f|P_1))$ and hence $p_1\in Q$ as required.

Inasmuch as $ker(\eta(f|P_1))$ is small in $P_1$ (and so is in $P$), we infer from $P=ker(\eta(f|P_1))+f^{-1}(X)$ that $P=f^{-1}(X)$. As a result, $M=X$ and hence $f(P_2)\subseteq_c N$ in $M$.

Next, we show that $f(P_2)$ is a co-closed submodule. Let $T$ be a submodule of $f(P_2)$ with $T\subseteq_c f(P_2)$ in $M$. Taking a submodule $P_2'$ of $P_2$ with $f(P_2')=T$, we have $M=f(P_1)+f(P_2')$ since $M=N+f(P_1)+f(P_2')$ and $f(P_2)\subseteq_c N$ in $M$. This implies that $P=P_1+P_2'+ker(f)$ and hence $P=P_1\oplus P_2'$. Thus $P_2=P_2'$ and this finishes the proof.

The following lemma plays an important role in this paper.

**Lemma 1.4.** *Let $N$, $N^*$ and $N^{**}$ be submodules of an $R$-module $M$ with $M=N^*\oplus N^{**}$ and $N^*\subseteq N$. Then the following are equivalent:*

1) *$N^*$ is a co-closed submodule of $M$ with $N^*\subseteq_c N$ in $M$.*
2) *$N\cap N^{**}$ is small in $M$.*

Proof. The proof easily follows from the fact that $N/N^*\simeq N^{**}\cap N$ and $N/N^*\simeq N^{**}$, canonically.

Now, let $M$ be an $R$-module and $\mathcal{A}$ a subfamily of the family $\mathcal{L}(M)$ of all submodules of $M$. $M$ is said to have the extending property of modules for $\mathcal{A}$ provided that if $A\in\mathcal{A}$, then there exists $A^*<\oplus M$ such that $A\subseteq_e A^*$ ([14], [23]). We define a dual notion of this extending property as follows:

DEFINITION. We say that $M$ has the lifting property of modules for $\mathcal{A}$ provided that, for any $A\in\mathcal{A}$, there exists a direct summand $A^*$ of $M$ such that $A^*\subseteq_c A$ in $M$.

REMARK. (1) We note that one type of the lifting property had appeared in Nicholson [22]. He says that an $R$-module $M$ is semi-regular if it satisfies the following condition for $\mathcal{A}=\{xR|x\in M\}$: For any $A\in\mathcal{A}$, there exists a decomposition $M=A^*\oplus A^{**}$ such that $A^*$ is projective with $A^*\subseteq A$ and $A\cap A^{**}$ is small in $A^{**}$. So, in view of Lemma1.4, a projective module $P$ is semi-regular if and only if it has the lifting property of modules for the family of all cyclic submodules of $P$.

(2) The lifting property of simple modules mentioned by Harada [10] is also one of our considerable lifting properties. Harada says that an $R$-module has the lifting property of simple modules if it satisfies the following condition:

(*) Every simple submodule of $M/\mathrm{J}(M)$ is induced from a direct summand of $M$.

As is easily seen, this condition is equivalent to the condition that, for any submodule $A$ of $M$ such that $A$ contains $\mathrm{J}(M)$ and $A/\mathrm{J}(M)$ is simple, then there exists a direct summand $A^*$ of $M$ with $A^* \subseteq_c A$ in $M$. It should be noted that the condition (*) has been defined as a dual condition of the extending property of simple modules defined as follows: $M$ is said to have the extending property of simple modules provided that if $A$ is simple submodule of $M$ then there exists $A^* < \oplus M$ such that $A \subseteq_e A^*$. The lifting property of simple modules is surely a dual notion of the extending property of simple module. However the lifting property of module for the family of all maximal submodule of $M$ is also seems to be a dual notion of the extending property of simple modules (see [24]).

(3) The lifting property had also appeared in Bass's article [2]. Indeed we look at his well known lemma:

(Bass's lemma) *The following conditions are equivalent for a given projective R-module M*:

1) *M is semiperfect in the sense of Mares, i.e., every homomorphic image fo M has a projective cover.*

2) *For any submodule N of M, there exists a decomposition* $M=N^* \oplus N^{**}$ *such that* $N^* \subseteq N$ *and* $N \cap N^{**}$ *is small in* $N^{**}$.

The condition 2) is nothing but 2) in Lemma 1.4, so we can re-phrase the Bass's lemma as follows; A projective module $M$ is semiperfect if and only if it has the lifting property of module for $\mathcal{L}(M)$, the family of all submodules of $M$. This fact suggests that there is a dual relation between the works of Bass [2] and Mares [20] and those of Utumi [28]~[30] and Jeremy [16], [17].

## 2. $\mathcal{A}$-semiperfect modules and $\mathcal{A}$-quasi-semiperfect modules

Let $M$ be an $R$-module and we denote the family of all submodules of $M$ by $\mathcal{L}(M)$. In [23] we studied those subfamilies $\mathcal{A}^*$ satisfying the conditions:

($\alpha^*$) For $A \in \mathcal{A}^*$ and $N \in \mathcal{L}(M)$, $A \approx N$ implies $N \in \mathcal{A}^*$.

($\beta^*$) For $A \in \mathcal{A}^*$ and $N \in \mathcal{L}(M)$, $A \subseteq_e N$ implies $N \in \mathcal{A}^*$.

For such $\mathcal{A}^*$, we introduced $\mathcal{A}^*$-continuous modules, $\mathcal{A}^*$-quasi-continuous modules and $\mathcal{A}^*$-quasi-injective modules and devoted some fundamental results of such modules in [23].

In this section we intend to give dualizations of these concepts.

Let $\mathcal{A}$ be a subfamily of $\mathcal{L}(M)$ and assume that $\mathcal{A}$ satisfies the following conditions ($\alpha$) and ($\beta$):

($\alpha$) For $A \in \mathcal{A}$ and $N \in \mathcal{L}(M)$, $M/A \approx M/N$ implies $N \in \mathcal{A}$.

($\beta$) For $A \in \mathcal{A}$ and $N \in \mathcal{L}(M)$, $N \subseteq_c A$ in $M$ implies $N \in \mathcal{A}$.

Of course, these ($\alpha$) and ($\beta$) correspond to above ($\alpha^*$) and ($\beta^*$), respectively. For examples, $\mathcal{L}(M)$ and $\{A \in \mathcal{L}(M) \mid M/A$ is hollow$\}$ are good examples of such $\mathcal{A}$ (cf. [24]).

Now, for such $\mathcal{A}$ we shall introduce the concepts of $\mathcal{A}$-semiperfect modules, $\mathcal{A}$-quasi-semiperfect modules and $\mathcal{A}$-quasi-injective modules as notions dual to those of $\mathcal{A}^*$-continuous modules, $\mathcal{A}^*$-quasi-continuous modules and $\mathcal{A}^*$-quasi-injective modules, respectively.

DEFINITION. $M$ is $\mathcal{A}$-semiperfect (resp. $\mathcal{A}$-quasi-semiperfect) if the conditions ($C_1$) and ($C_2$) (resp. ($C_1$) and ($C_3$)) below are satisfied:

($C_1$) *$M$ has the lifting property of modules for $\mathcal{A}$.*

($C_2$) *For any $A \in \mathcal{A}$ such that $A < \oplus M$, any sequence*

$$M \to M/A \to 0$$

*splits.*

($C_3$) *Let $A \in \mathcal{A}$ and $N \in \mathcal{L}(M)$ which are direct summands of $M$. If $X = A \cap N$ is small in $M$ and $A/X \oplus N/X < \oplus M/X$, then $X = 0$.*

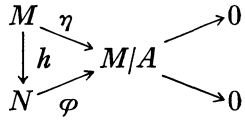
In particular, we simply say that $M$ is semiperfect (resp. quasi-semiperfect) when it is $\mathcal{L}(M)$-semiperfect (resp. $\mathcal{L}(M)$-quasi-semiperfect). We note that there are no confusions between our name of semiperfect modules and that of Mares's semiperfect modules on projective modules. Because, if $M$ is projective then clearly it satisfies the condition ($C_2$) for $\mathcal{L}(M)$, and hence it is semiperfect if and only if it is semiperfect in the sense of Mares by Lemma 1.4. Needless to say, we must take a serious attitude when we name new concepts. In view of the results in the later sections, the author believe that our name of semiperfect modules will be justified in module thoery.

DEFINITION. $M$ is $\mathcal{A}$-quasi-projective if it satisfies the condition:

($C_4$) *For any $A \in \mathcal{A}$, $N \in \mathcal{L}(M)$ and any sequence*

$$N \xrightarrow{\varphi} M/A \to 0$$

*there exists a homomorphism $h$: $M \to N$ which makes the diagram*




*commute, where $\eta$ is the canonical map.*

We note that $\mathcal{L}(M)$-quasi-projectivity is nothing but the usual quasi-projectivity by virtue of Miyashita [21, Proposition 2.1].

Now, in this section we devote fundamental properties of $\mathcal{A}$-quasi-projective modules, $\mathcal{A}$-semiperfect modules and $\mathcal{A}$-quasi-semiperfect modules. All

results are obtained as results dual to those in section 1 of [23].

**Theorem 2.1.** *Assume that $M$ and $M/A$ have projective covers for all $A$ in $\mathcal{A}$. Then if $M$ is $\mathcal{A}$-quasi-projective, then it is $\mathcal{A}$-semiperfect.*

Proof. The condition $(C_2)$ is easily verified. Let $A$ be in $\mathcal{A}$. Consider a projective cover of $M$;

$$P \xrightarrow{g} M \to 0 .$$

Since $M/A$ has a projective cover, by the Bass's lemma, we have a decomposition $P=P_1\oplus P_2$ such that $f(P_2)\subseteq A$ and

$$P_1 \xrightarrow{g} M/A \to 0$$

is a projective cover of $M/A$, where $g=\eta(f\,|\,P_1)$ and $\eta$ is the canonical map: $M\to M/A\to 0$.

Let $\pi_i$: $P=P_1\oplus P_2\to P_i$ be the projections, $i=1, 2$. By the proof of Theorem 1.3 we see that $f(P_2)\subseteq_c A$ in $M$. We claim that

$$M = f(P_1)\oplus f(P_2) .$$

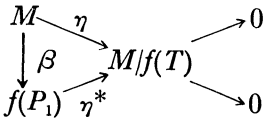
and show this by a slight modification of the proof of Wu-Jans [31, Theorem].

Putting

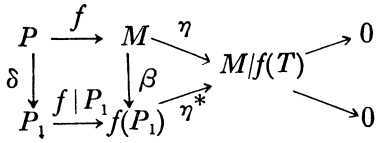
$$T = \pi_1(ker(f))+P_2 \cdots (*) ,$$

we can see that $f(P_2)\subseteq f(T)\subset A$. In fact, $f(P_2)\subseteq f(T)$ is clear. Next, let $t\in T$ and express it in $(*)$ as $t=\pi_1(q)+p_2$, where $q\in ker(f)$ and $p_2\in P_2$. Then, it follows from $f(q)=0$ that $f\pi_1(q)=-f\pi_2(q)\in f(P_2)\subseteq A$. Hence $f(t)=f\pi_1(q)+f(p_2)=f\pi_1(q)+f(p_2)\in A$; so $f(t)\in A$. Since $f(P_2)\subseteq f(T)\subseteq A$ and $f(P_2)\subseteq_c A$ in $M$, we see that $f(T)\subseteq_c A$ in $M$ by Proposition 1.1. Thus $f(T)\in\mathcal{A}$.

Now, let $\eta=\eta_{f(T)}$ be the canonical map: $M\to M/f(T)\to 0$, and let $\eta^*$ be the restriction map $\eta\,|\,f(P_1)$. Then, by the condition $(C_4)$, there exists a homomorphism $\beta$: $M\to f(P_1)$ which makes the diagram




commute. Since $P$ is projective, there exists a homomorphism $\delta$: $P\to P_1$ for which the diagram



is commutative. Putting $X=\{p\in P\mid \pi_1(p)-\delta(p)\in P_1\cap ker(f)\}$ we show that

$$P = ker(f)+X \cdots (**)$$

Let $p\in P$. Since $g(f\mid P_1)\pi_1=\eta f=g(f\mid P_1)\delta$, we see that $f((\pi_1-\delta)(p))=f(t)$ for some $t\in T$. As a result, $(\pi_1-\delta)(p)-t\in ker(f)\subseteq\pi_1(ker(f))+P_2\subseteq T$ and hence $(\pi_1-\delta)(p)\in T$. We express it in $(*)$ as

$$\pi_1(p)-\delta(p) = \pi_1(k)+p_2$$

where $k\in ker(f)$ and $p_2\in P_2$. Then noting that all $\pi_1(p)$, $\delta(p)$ and $\pi_1(k)$ are in $P_1$, we see that $p_2=0$. Therefore $\pi_1(p-k)-\delta(p-k)=\delta(k)\in P_1\cap ker(f)$; whence $p-k\in X$ and hence $p\in X+ker(f)$. Thus $(**)$ holds as claimed. Since $ker(f)$ is small in $P$, $(**)$ shows that $P=X$ and it follows that

$$\pi_1(ker(f))\subseteq P_1\cap ker(f) \cdots (***)$$

Now, let $x\in f(P_1)\cap f(P_2)$. Then $x=f(p_1)=f(p_2)$ for some $p_i\in P_i$, $i=1, 2$. Putting $p=p_1-p_2$, $p\in ker(f)$. Hence $p_1=\pi_1(p)\in ker(f)$ by $(***)$ and hence $0=f(p_2)=x$. Therefore $M=f(P_1)\oplus f(P_2)$, and consequently we get that $f(P_1)$ is a direct summand of $M$ with $f(P_1)\subseteq_e A$ in $M$.

**Proposition 2.2.** *The condition* $(C_3)$ *is equivalent to the following* $(C_3')$:

$(C_3')$ *For any* $A\in\mathcal{A}$ *and* $N\in\mathcal{L}(M)$ *such that* $A<\oplus M$, $N<\oplus M$ *and* $M=A+N$, *if* $X=A\cap N$ *is small in* $M$, *then* $X=0$.

Proof. $(C_3)\Rightarrow(C_3')$ is trivial. Assume that $(C_3')$ holds, and let $A\in\mathcal{A}$, $N$, $Y\in\mathcal{L}(M)$ such that $A<\oplus M$, $N<\oplus M$ and

$$M/X = A/X\oplus N/X\oplus(Y+X)/X$$

where $X=A\cap N$. Assuming that $X$ is samll in $M$, we show $X=0$.

Since $N<\oplus M$, $N+Y=N\oplus Y'$ for some $Y'\subseteq M$. Then $(A+N)\cap Y'\subseteq X\subseteq N$ and hence we see that $(A+N)\cap Y'=0$. Furthermore, it follows from $N<\oplus M$ that $A+N=A'\oplus N$ for some $A'$. As a result, $N\oplus Y'<\oplus M$. Putting $N'=N\oplus Y'$, we see that $A\cap N'=X$ and $M=A+N'$. Therefore $(C_3')$ says that $X=0$.

**Theorem 2.3.** *The condition* $(C_2)$ *implies* $(C_3)$. *Therefore if* $M$ *is* $\mathcal{A}$-*semiperfect then it is* $\mathcal{A}$-*quasi-semiperfect.*

Proof. By Proposition 2.2, we may show that $(C_2)$ implies $(C_3')$.

Let $A\in\mathcal{A}$ and $N\in\mathcal{L}(M)$ such that $A<\oplus M$, $N<\oplus M$, $M=A+N$ and $X=A\cap N$ is small in $M$. We say $M=A\oplus A^*=N\oplus N^*$ and put $\bar{M}=M/X$. Then

$$\begin{aligned}\bar{M} &= \bar{A}\oplus\bar{N}\\ &= \bar{A}\oplus\bar{A}^*\\ &= \bar{N}\oplus\bar{N}^*.\end{aligned}$$

By $\pi$ we denote the projection: $\bar{M}=\bar{A}\oplus\bar{A}^*\to\bar{A}^*$. Then $\bar{N}\simeq\bar{A}^*$ by the restriction map $\pi|\bar{N}$. Let $\alpha$ be the canonical isomorphism from $\bar{A}^*$ to $M/A$ and $g$ be the canonical map: $M\to\bar{N}$. Then $\alpha\pi g$ is an epimorphism from $M$ to $M/A$ and its kernel coincides with $N^*\oplus X$. Since $M$ satisfies the condition $(C_2)$, it follows that $N^*\oplus X<\oplus M$ and hence $0\subseteq_c X$ in $M$ says that $X=0$.

Here we further consider the following condition which is closely related to the condition $(C_3)$:

$(C_5)$ *For any $A\in\mathcal{A}$ and $N<\oplus M$ such that $0\subseteq_c X=A\cap N$ in $M$ and $M/X\oplus\rangle A/X\oplus N/X$, every homomorphism from $A/X$ to $M/X$ is induced from a homomorphism from $A$ to $N$, i.e., for $f$: $A/X\to N/X$ there exists $f'$: $A\to N$ which makes the diagram*

$$\begin{array}{ccc} A/X & \xrightarrow{f} & N/X \\ {\scriptstyle\phi_A}\uparrow & & \uparrow{\scriptstyle\phi_N} \\ A & \xrightarrow{f'} & N \end{array}$$

*commute, where $\phi_A$ and $\phi_N$ are canonical maps.*

**Proposition 2.4.** *The condition* $(C_5)$ *is equivalent to the condition*:

$(C_5')$ *For any $A\in\mathcal{A}$ and $N<\oplus M$ such that $M=A+N$ and $0\subseteq_c X=A\cap N$ in $M$, every homomorphism from $A/X$ to $N/X$ is induced from a homomorphism from $A$ to $N$.*

Proof. We may only show $(C_5')\Rightarrow(C_5)$. Let $A\in\mathcal{A}$ and $N<\oplus M$ such that $X=A\cap N$ is small in $M$ and $M/X\oplus\rangle A/X\oplus N/X$, and let $f$ be a homomorphism from $A/X$ to $N/X$. Since $N<\oplus M$ and $A/X\oplus N/X<\oplus M/X$, we see from the proof of Proposition 2.2 that there exists $Y<\oplus M$ such that $N\oplus Y<\oplus M$ and $(N\oplus Y)\cap A=X$. Hence, applying the condition $(C_5)$ here, we have a homomorphism $f'$: $A\to N\oplus Y$ for which the diagram

$$\begin{array}{ccc} A/X & \xrightarrow{f} & N/X\oplus Y/X=(N\oplus Y)/X \\ \downarrow{\scriptstyle\phi_1} & & \downarrow{\scriptstyle\phi_2} \\ A & \xrightarrow{f'} & N\oplus Y \end{array}$$

is commutative, where $\phi_i$ is the canonical map, $i=1, 2$. If we denote the projection: $N\oplus Y\to N$ by $\pi$, then as is easily seen, $f$ is induced from $\pi f'$.

**Proposition 2.5.** *Under the condition* $(C_1)$, $(C_3)$ *is equivalent to* $(C_5)$.

Proof. By Propositions 2.2 and 2.4, we may show that $(C_3')$ is equivalent to $(C_5')$.

$(C_3')\Rightarrow(C_5')$: Let $A\in\mathcal{A}$ and $N<\oplus M$ such that $M=A+N$ and $0\subseteq_e(A\cap N)$ in $M$, Put $X=A\cap N$, and let $f$ be a homomorphism from $A/X$ to $N/X$. We put $\bar{M}=M/X$ and

$$B=\{x\in M\mid \bar{x}=\bar{a}+f(\bar{a})\} \text{ for some } a \text{ in } A\} .$$

Then $B$ is a submodule of $M$ containing $X$, and we see that

$$\bar{M}=\bar{B}\oplus\bar{N} .$$

Since $M/B\approx\bar{N}\approx M/A$, $B$ lies in $\mathcal{A}$ by the assumption for $\mathcal{A}$. Using the condition $(C_1)$, there exists $B^*<\oplus M$ such that $B^*\subseteq_e B$ in $M$. Then we also see that $B^*\in\mathcal{A}$ by the assumption for $\mathcal{A}$. Since $M=B+B^*+N$ and $B^*\subseteq_e B$ in $M$, we get $M=B^*+N$; whence

$$\begin{aligned}\bar{M}&=\bar{B}^*\oplus\bar{N} ,\\ \bar{B}&=\bar{B}^* .\end{aligned}$$

Inasmuch as $0\subseteq_e X$ in $M$ and $B^*\cap N\subseteq X$ we see that $0\subseteq_e(B^*\cap N)$ in $M$. Therefore $(C_3')$ says that

$$M=B^*\oplus N .$$

Let $\pi$ be the projection: $M=B^*\oplus N\to N$ and put $f'=-\pi\mid A$. Then, as is easily seen, the diagram

$$\begin{array}{ccc} A/X & \xrightarrow{f} & N/X \\ {\scriptstyle\phi_A}\downarrow & & \downarrow{\scriptstyle\phi_N} \\ A & \xrightarrow{f'} & N \end{array}$$

is commutative, where $\phi_A$ and $\phi_N$ are canonical maps.

$(C_5')\Rightarrow(C_3')$: Let $A$ and $N$ be direct summands of $M$ such that $M=A+N$, $A\in\mathcal{A}$ and $0\subseteq_e X=A\cap N$ in $M$. Write $M=A\oplus A^*=N\oplus N^*$ and put $\bar{M}=M/X$. Then

$$\begin{aligned}\bar{M}&=\bar{A}\oplus\bar{N}\\ &=\bar{A}\oplus\bar{A}^*\\ &=\bar{N}\oplus\bar{N}^* .\end{aligned}$$

Let $\pi$ and $\pi^*$ be the projections: $\bar{M}=\bar{N}\oplus\bar{N}^*\to\bar{N}$ and $\bar{M}=\bar{N}\oplus\bar{N}^*\to\bar{N}^*$, respectively. Then we see that

$$\begin{aligned}\bar{A}&=\{\pi(\bar{a})+\pi^*(\bar{a})\mid a\in A\} ,\\ \pi^*(\bar{A})&=\bar{N}^*\end{aligned}$$

and the map $f$: $\bar{N}^* \to \bar{N}$ given by $\pi^*(\bar{a}) \to \pi(\bar{a})$ is well defined.

Now, we see that $X=(N^*+X)\cap N$, and it follows from $M/(N^*+X) \simeq \bar{N} \simeq M/A$ that $N^*+X \in \mathcal{A}$. Here, by $(C_5')$, we have a homomorphism $f'$: $N^*+X \to N$ which makes the diagram

$$\begin{array}{ccc} \overline{N^*+X} & \xrightarrow{f} & \bar{N} \\ {\scriptstyle \phi_1}\downarrow & & \uparrow{\scriptstyle \phi_2} \\ N^*+X & \xrightarrow{f'} & N \end{array}$$

commute, where $\phi_1$ and $\phi_2$ are canonical maps. We put $P=\{n+f'(n) \mid n \in N^*\}$, and claim that $A=P+X$.

In fact, if $n \in N^*$ then $\overline{f'(n)}=f(\bar{n})$ and hence $\bar{n}+\overline{f'(n)}=\bar{a}$ for some $a \in A$; so $n+f'(n)-a=x$ for some $x$ in $X$. This implies that $P+X \subseteq A$. Conversely if $a \in A$ then $\bar{a}=\overline{\pi(a)}+\overline{\pi^*(a)}=\overline{\pi(a)}+\overline{f(\pi^*(a))}$ and hence the above commutative diagram shows that $\bar{a}=\bar{n}+f'(\bar{n})$ for some $n \in N^*$. Hence $a \in P+X$ and therefore $A \subseteq P+X$. Thus we get $A=P+X$. Furthermore we can see that $A=P \oplus X$. Since $A < \oplus M$ and $0 \subseteq_c X$ in $M$, we see that $0 \subseteq_c X$ in $A$. Accordingly $X=0$ as required.

**Proposition 2.6.** $(C_1)$ *implies the following condition*: i

$(C_6)$ *For any* $A \in \mathcal{A}$, *there exists a direct summand* $N < \oplus M$ *such that* $M=A+N$ *and* $0 \subseteq_c (A \cap N)$ *in* $M$.

Proof. Use Lemma 1.4.

**Proposition 2.7.** $(C_5)$ *and* $(C_6)$ *imply* $(C_1)$.

Proof. Let $A \in \mathcal{A}$. $(C_6)$ says that there exists a direct summand $N_2$, say $M=N_1 \oplus N_2$, such that $M=A+N_2$ and $0 \subseteq_c (A \cap N_2)=X$ in $M$. Putting $\bar{M}=M/X$ we see that

$$\begin{aligned} \bar{M} &= \bar{N}_1 \oplus \bar{N}_2 \\ &= \bar{A} \oplus \bar{N}_2 . \end{aligned}$$

Let $\pi_i$ be the projection: $\bar{M}=\bar{N}_1 \oplus \bar{N}_2 \to \bar{N}_i$, $i=1, 2$. Then $\pi_1(\bar{A})=\bar{N}_1$ and the map $f$: $\bar{N}_1 \to \bar{N}_2$ given by $\pi_1(\bar{a}) \to \pi_2(\bar{a})$, $a \in A$, is well defined. Since $M/(N_1+X) \approx \bar{N}_2 \approx M/A$, $N_1+X$ lies in $\mathcal{A}$ and moreover $(N_1+X) \cap N_2=X$. Thus by $(C_5)$ there exists $f'$: $N_1+X \to N_2$ such that the diagram

$$\begin{array}{ccc} \bar{N}_1 & \xrightarrow{f} & \bar{N}_2 \\ {\scriptstyle \phi_1}\uparrow & & \uparrow{\scriptstyle \phi_2} \\ N_1+X & \xrightarrow{f'} & N_2 \end{array}$$

is commutative, where $\phi_1$ and $\phi_1$ are canonical maps. We put $A=\{x+f'(x) \mid$

$x \in N_1\}$. Then $M=A^*\oplus N_2$ and $\bar{A}^*=\bar{A}$. Nothing $X\subseteq A$, we see from $\bar{A}^*=\bar{A}$ that $A^*\subseteq A$; whence $A^*\subseteq_e A$ in $M$ by Lemma 1.4. This completes the proof.

By Theorem 2.3 and Propositions 2.5, 2.6 and 2.7 we have the following two theorems.

**Theorem 2.8.** *The following conditions are equivalent*:

1) *$M$ is $\mathcal{A}$-quasi-semiperfect.*
2) *$M$ satisfies* $(C_1)$ *and* $(C_5)$.
3) *$M$ satisfies* $(C_5)$ *and* $(C_6)$.

**Theorem 2.9.** *$M$ is $\mathcal{A}$-semiperfect if and only if it satisfies* $(C_2)$, $(C_5)$ *and* $(C_6)$.

**Proposition 2.10.** *$M$ is $\mathcal{A}$-quasi-projective if and only if it satisfies the condition*:

$(C_4')$ *For any $A\in\mathcal{A}$, $N\in\mathcal{L}(M)$ and epimorphisms $f$: $M\to M/A$ and $g$: $N\to M/A$ there exists a homomorphism $h$: $M\to N$ satisfying $gh=f$.*

Proof. 'If' part is obvious. To show 'Only if' part, let $A\in\mathcal{A}$, $N\in\mathcal{L}(M)$ and consider epimorphisms: $M\xrightarrow{f}M/A\to 0$ and $N\xrightarrow{g}M/A\to 0$. Since $\mathcal{A}$ satisfies the condition $(\alpha)$, $ker(f)$ lies in $\mathcal{A}$. Let $\phi$ be the canonical isomorphism: $M/A\approx M/ker(f)$ such that

$$\begin{array}{ccc} & \overset{f}{\nearrow} & M/A \\ M & & \| \phi \\ & \underset{\eta}{\searrow} & M/ker(f) \end{array}$$

is commutative, where $\eta$ is the canonical map. Here using the condition $(C_4)$, we have a homomorphism $h$: $M\to N$ satisfying $\phi gh=\eta$. Since $gh=\phi^{-1}\phi gh=\phi^{-1}\eta=\phi^{-1}(\phi f)=f$; whence we have $gh=f$.

**Proposition 2.11.** *Assume that $M$ is $\mathcal{A}$-quasi-projective. Let $A\in\mathcal{A}$ and $N\in\mathcal{L}(M)$ such that $M=A+N$. Then for any homomorphism $f$ from $A/X$ to $N/X$ is induced from a homomorphism $f'$ from $A$ to $N$ where $X=A\cap N$, that is, the diagram*

$$\begin{array}{ccc} A/X & \xrightarrow{f} & N/X \\ {\scriptstyle\alpha}\uparrow & & \downarrow{\scriptstyle\beta} \\ A & \xrightarrow{f'} & N \end{array}$$

*is commutative, where $\alpha$ and $\beta$ are canonical maps.*

Proof. Inasmuch as $M=A+N$, there are canonical isomorphisms:

$A/X \overset{\phi_1}{\simeq} M/N$ and $N/X \overset{\phi_2}{\simeq} M/A$.

Consider the diagram:

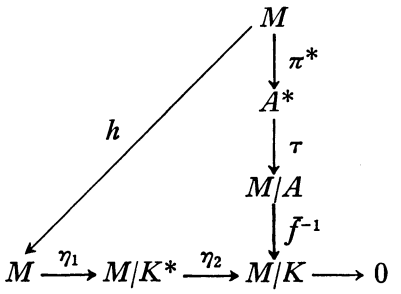
$$\begin{array}{ccc} A/X & \xrightarrow{f} & N/X \\ {\scriptstyle\phi_1}\downarrow & & \downarrow{\scriptstyle\phi_2} \\ M/N & \xrightarrow{\sigma} & M/A \\ {\scriptstyle\eta_N}\downarrow & & \downarrow{\scriptstyle\eta_A} \\ M & & N \end{array} \qquad (\sigma = \phi_1^{-1} f \phi_2)$$

where $\eta_N$, $\eta_A$ are canonical maps. Applying Proposition 2.10 we have a homomorphism $h$: $M \to N$ satisfying $\phi_N = \phi_A h$. Then the restriction map $f' = h|A$ is a required map.

**Theorem 2.12.** *If $M$ satisfies the conditions* $(C_4)$, $(C_5)$ *and* $(C_6)$ *then it is $\mathcal{A}$-semiperfect.*

Proof. By Theorem 2.7 $M$ satisfies the condition $(C_1)$. In order to show that the condition $(C_2)$ is satisfied, let $A \in \mathcal{A}$ such that $A < \oplus M$; put $M = A \oplus A^*$, and let $f$ be an epimorphism from $M$ to $M/A$. Put $K = ker(f)$. Then $M/K \overset{f}{\simeq} M/A$ shows $K \in \mathcal{A}$. Hence, by $(C_1)$, there exists $K^* < \oplus K$ with $K^* \subseteq_c K$ in $M$. We wish to get $K = K^*$.

Let $\pi^*$ be the projection: $M = A \oplus A^* \to A^*$, and let $\eta_1$, $\eta_2$ and $\tau$ be the canonical maps: $M \overset{\eta_1}{\to} M/K$, $M/K^* \overset{\eta_2}{\to} M/K$ and $A^* \overset{\tau}{\approx} M/A$. Applying $(C_4')$ for $\mathcal{A}$ we have a homomorphism $h$: $M \to M$ which makes the diagram below commute:




(cf. Proposition 2.10). If we denote the injection: $A^* \to M$ by $i$. Then $\eta_2\eta_1 hi = \bar{f}^{-1}\tau$. Putting $h' = hi$ we obtain

$$M = h'(A^*) + K. \qquad \cdots\cdots\cdots\cdots\cdots\cdots (*)$$

Since $K^* < \oplus M$, $K = K^* \oplus T$ for some $T$. Assume $T \neq 0$, and pick a non-zero

element $t$ in $T$ and express it in $(*)$ as $t=h'(a^*)+k^*$ where $a^*\in A^*$ and $k^*\in K^*$. Since $t\in K$, $\eta_2(t+K^*)=0$. Hence $\eta_2(h'(a^*)+K^*)=\eta_2(t-k^*+K^*)=\eta_2(t+K^*)=0$ and hence $h'(a^*)\in K$. This implies $\bar{f}^{-1}\tau(a^*)=0$ and hence $a^*=0$; so $t=k^*\in K^*$, a contradiction. Accordingly $ker(f)=K=K^*<\oplus M$.

## 3. Semiperfect modules and quasi-semiperfect modules

The following theorem due to Mares [20] is well known:

(*Mares's theorem*): *A projective $R$-module $P$ is semiperfect if and only if it satisfies the following conditions*:

1) $\mathrm{J}(P)$ *is small in* $P$.

2) $P/\mathrm{J}(P)$ *is completely reducible*.

3) *Every direct decomposition of* $P/\mathrm{J}(P)$ *is induced from* a *decomposition of* $P$.

In this section we study the fundamental properties of quasi-semiperfect modules and semiperfect modules, and investigate the Mares's theorem from our point of view.

**Proposition 3.1.** *Let $M$ be an $R$-module which satisfies the condition* $(\mathrm{C}_1)$ *for* $\mathcal{L}(M)$ *then the following statement hold.*

1) *There is a decomposition* $M=H\oplus K$ *such that* $0\subseteq_c \mathrm{J}(H)$ *in* $M$ *and* $\mathrm{J}(K)=K$.

2) *Every submodule of* $M/\mathrm{J}(M)$ *is induced by a direct summand of* $M$; *so* $M/\mathrm{J}(M)$ *is completely reducible.*

Proof. 1) By the condition $(\mathrm{C}_1)$ for $\mathcal{L}(M)$ and Lemma 1.4 we have a decomposition $M=H\oplus K$ with $K\subseteq \mathrm{J}(M)$ and $0\subseteq_c(\mathrm{J}(M)\cap H)$ in $H$. Then $\mathrm{J}(K)=K$ and $\mathrm{J}(H)=\mathrm{J}(M)\cap H$.

2) Let $A$ be a submodule of $M$. Again using $(\mathrm{C}_1)$ for $\mathcal{L}(M)$ and Lemma 1.4 we have a decomposition $M=A^*\oplus A^{**}$ such that $A^*\subseteq A$ and $0\subseteq_c(A\cap A^{**})$ in $M$. Then clearly $\bar{A}^*=\bar{A}$ in $M=M/\mathrm{J}(M)$.

**Proposition 3.2.** *If $M$ is a quasi-semiperfect $R$-module then every internal direct sum of submodules of $M$ which is a locally direct summand of $M$ is a direct summand of $M$.*

Proof. Let $\{N_\alpha\}_I$ be an independent family of submodules of $M$ such that $N=\sum_I \oplus N_\alpha$ is a locally direct summand of $M$. By the condition $(\mathrm{C}_1)$ for $\mathcal{L}(M)$ and Lemma 1.4 there exists a decomposition $M=N^*\oplus N^{**}$ such that $N^*\subseteq N$ and $0\subseteq_c(N\cap N^{**})$ in $M$. We claim that $M=N\oplus N^{**}$.

Clearly $M=N+N^{**}$. Let $\pi^*$ and $\pi^{**}$ be the projections: $M=N^*\oplus N^{**}\to N^*$ and $M=N^*\oplus N^{**}\to N^{**}$, respectively. Since $N^*\subseteq N$, we see $N\cap N^{**}=\pi^{**}(N)$; whence

$$0\subseteq_c \pi^{**}(N) \quad \text{in} \quad M. \qquad \cdots\cdots\cdots\cdots\cdots\cdots(1)$$

Now consider a finite subset $\{N_{\alpha_1}, \cdots, N_{\alpha_n}\}$ of $\{N_\alpha\}_I$ and put $T=N_{\alpha_1}\oplus\cdots\oplus N_{\alpha_n}$. Then $T$ is a direct summand of $M$ by the assumption. Put $S=\pi^*(T)$. Again by $(C_1)$ for $\mathcal{L}(M)$ and Lemma 1.4 we have a decomposition $N^*=S^*\oplus S^{**}$ such that $S^*\subseteq S$ and $0\subseteq_c(S\cap S^{**})$ in $M$. We denote the projection: $M=S^*\oplus S^{**}\oplus N^{**}\to S^{**}$ by $\pi$. Then $\pi(T)=\pi(\pi^*(T))=\pi(S)=S\cap S^{**}$ and hence we see

$$0\subseteq_c \pi(T) \quad \text{in} \quad M. \qquad \cdots\cdots\cdots\cdots\cdots\cdots(2)$$

Put $X=T\cap(S^{**}\oplus N^{**})$. Then $X\subseteq\pi(T)+\pi^{**}(T)\subseteq\pi(T)+\pi^{**}(N)$ and therefore $0\subseteq_c X$ in $M$ by (1) and (2). As a result we have the situation $M=T+(S^{**}\oplus N^{**})$, $T<\oplus M$, $(S^{**}\oplus N^{**})<\oplus M$ and $0\subseteq_c(T\cap(S^{**}\oplus N^{**}))$ in $M$. Thus the condition $(C_3)$ for $\mathcal{L}(M)$ says that $T\cap(S^{**}\oplus N^{**})=0$. Hence $T\cap N^{**}=0$ and we have $M=N\oplus N^{**}$.

**Theorem 3.3** *If $M$ is a quasi-semiperfect $R$-module then every direct decomposition of $M/\mathrm{J}(M)$ is induced from a decomposition of $M$.*

Proof. By Proposition 3.1 we can assume that $\mathrm{J}(M)$ is small in $M$. Put $\bar{M}=M/\mathrm{J}(M)$ and consider a decomposition $\bar{M}=\sum_I\oplus\bar{A}_\alpha$ with each $A_\alpha$ a submodule of $M$. We may assume $I$ is a well ordered set. Let $\gamma$ be its ordinal number, and consider $\sigma\leq\gamma$ and assume for each $\omega<\sigma$ there exists a direct summand $A^*_\omega$ of $M$ such that $\bar{A}^*_\omega=\bar{A}_\omega$ and $\sum_{\omega<\sigma}\oplus A^*_\omega$ is a locally direct summand of $M$. Put $N=\sum_{\omega<\sigma}\oplus A^*_\omega$. Then $N<\oplus M$ by Proposition 3.2. Put $T=\sum_{\omega<\tau}A_\nu$. By the condition $(C_1)$ for $\mathcal{L}(M)$ there exists a decomposition $M=T^*\oplus T^{**}$ with $T^*\subseteq_c T$ in $M$. Inasmuch as $\bar{M}=\sum_{\omega<\sigma}\bar{A}_\omega\oplus\sum_{\sigma\leq\nu}\bar{A}_\nu=\bar{N}\oplus\bar{T}^*$, we see $M=N+T^*$ with $0\subseteq_c(N\cap T^*)$ in $M$. Hence by $(C_3)$ for $\mathcal{L}(M)$ we get that $M=N\oplus T^*$. Since $\bar{A}_\sigma\subseteq\bar{T}^*$, we can take a direct summand $A^*_\sigma<\oplus T$ such that $\bar{A}^*_\sigma=\bar{A}_\sigma$ by Proposition 3.1. Thus, by the transfinite induction, there exists a direct summand $A^*_\sigma$ of $M$ for each $\sigma\leq\gamma$ such that $\bar{A}^*_\sigma=\bar{A}_\sigma$ and $\sum_{\sigma\leq\gamma}\oplus A^*_\sigma$ is a locally direct summand of $M$. Then we have $M=\sum_{\sigma\leq\gamma}\oplus A^*_\sigma$ since $0\subseteq_c\mathrm{J}(M)$ in $M$.

**Lemma 3.4.** *If $M$ is a non-zero quasi-semiperfect $R$-module, then there exists a non-zero direct summand of $M$ which is hollow.*

Proof. On the contrary, we assume that every non-zero direct summand

of $M$ is not hollow. First, we show that $M$ is written as a direct sum of countably infinite non-zero submodules of $M$. Since $M$ is not hollow, we can take a proper submodule $N$ of $M$ which is not small in $M$. Then, by the condition $(C_1)$ for $\mathcal{L}(M)$, we have a decomposition $M=N_1\oplus N_1'$ such that $N_1\subseteq_c N$ in $M$. Then both $N_1$ and $N_1'$ are non-zero. Using the same argument on $N_1'$, we have a decomposition $N_1'=N_2\oplus N_2'$ with $N_2\neq 0$ and $N_2'\neq 0$. Continuing this procedure, we get an independent family $\{N_i \mid i=1, 2, \cdots\}$ of non-zero submodules of $M$ such that $\sum_{i=1}^{\infty}\oplus N_i$ is a locally direct summand of $M$. Then $\sum_{i=1}^{\infty}\oplus N_i$ is just a direct summand of $M$ by Proposition 3.2. Hence $M=(\sum_{i=1}^{\infty}\oplus N_i)\oplus V$ for some submodule $V$ as desired.

Now, pick $0\neq x$ in $N_1$. By Zorn's lemma, we can take a maximal independent family $\{M_\alpha\}_I$ such that $x\notin M'=\sum_I\oplus M_\alpha$ and $M'=\sum_I\oplus M_\alpha$ is a locally direct summand of $M$. By Proposition 3.2, we see $M'<\oplus M$; write $M=M'\oplus M''$. Then $M''\neq 0$ since $x\notin M'$. Therefore $M''$ is written as a direct sum of countably infinite non-zero submodules of $M$; $M''=\sum_J\oplus T_\beta$. Since $x\in M=M'\oplus M''=M'\oplus\sum_J\oplus T_\beta$, there exists a finite subset $J_0$ of $J$ with $x\in M'\oplus\sum_J\oplus T_\gamma$. If $x\in M'\oplus\sum_{J-J_0}\oplus T_\delta$ then we see $x\in M'$, a contradiction. Therefore $x\notin M'\oplus\sum_{J-J_0}\oplus T_\delta$ and this contradicts the maximality of $M$. Thus the lemma follows.

**Theorem 3.5.** *Every quasi-semiperfect R-modules is expressed as a direct sum of hollow modules.*

Proof. This is immediate from Proposition 3.2, Lemma 3.4 and Zorn's lemma.

**Lemma 3.6.** *Let $M$ be a quasi-semiperfect R-module, and let $M=\sum_I\oplus M_\alpha$ be a decomposition with each $M_\alpha$ a hollow module. Then for any $M=H\oplus T$ with $H$ a hollow module there exists $\alpha\in I$ satisfying $M=M_\alpha\oplus T$.*

Proof. Let $\pi_H$ be the projection: $M=H\oplus T\to H$. If we can choose $\alpha\in I$ such that $\pi_H(M_\alpha)=H$ then $M=M_\alpha+T$ and $0\subseteq_c(M_\alpha\cap T)$ in $M$; so $M=M_\alpha\oplus T$ by the condition $(C_3)$ for $\mathcal{L}(M)$. Now assume $\pi_H(M_\alpha)\subsetneqq H$ for all $\alpha\in I$.We take $\{\alpha_1, \cdots, \alpha_n\}\subseteq I$ such that $(M_{\alpha_1}\oplus\cdots\oplus M_{\alpha_n})\cap H\neq 0$ and put $N=M_{\alpha_1}\oplus\cdots\oplus M_{\alpha_n}$. Since $0\subseteq_c\pi_H(M_{\alpha_i})$ in $M$ for all $i=1, \cdots, n$, we see $0\subseteq_c\pi_H(N)$ in $M$. Hence it follows that $\pi_T(N)$ is not small in $M$ where $\pi_T$ denotes the projection: $M=H\oplus T\to T$. Using $(C_1)$ for $\mathcal{L}(M)$ there exists a decomposition $T=T^*\oplus T^{**}$ such that $T^*\subseteq\pi_T(N)$ and $0\subseteq_c(T^{**}\cap\pi_T(N))$ in

$M$ (cf. Lemma 1.4). By $\pi_{T^*}$ and $\pi_{T^{**}}$ we denote the projections: $M=H\oplus T^*\oplus T^{**}\to T^*$ and $M=H\oplus T^*\oplus T^{**}\to T^{**}$, respectively. Then clearly $\pi_{T^{**}}(N)=T^{**}\cap\pi_T(N)$ and $\pi_{T^*}(N)=T^*$, so that $M=N+(H\oplus T^{**})$ and $N\cap(H\oplus T^{**})\subseteq\pi_H(N)\oplus\pi_{T^{**}}(N)=\pi_H(N)+(T^{**}\cap\pi_T(N))$. Since both $\pi_H(N)$ and $T^{**}\cap\pi_T(N)$ are small in $M$ we see that $N\cap(H\oplus T^{**})$ is small in $M$. Thus by $(C_3)$ for $\mathcal{L}(M)$ we get $0=N\cap(H\oplus T^{**})$; whence $N\cap H=0$, a contradiction. This completes the proof.

**Proposition 3.7.** *Let $M$ be a quasi-semiperfect $R$-module, and let $M=\sum_I\oplus M_\alpha$ be a decomposition with each $M_\alpha$ hollow. Then a submodule $A$ of $M$ is small in $M$ if and only if $\pi_\alpha(A)\neq M_\alpha$ for all $\alpha\in I$, where $\pi_\alpha$ denotes the projection: $M=\sum_I\oplus M_\alpha\to M_\alpha$.*

Proof. 'If' part: If $A$ is not small in $M$ then by the condition $(C_1)$ for $\mathcal{L}(M)$ and Lemma 3.4 we can take a direct summand $H$ $(\neq 0)$ of $M$ which is hollow and contained in $A$. Then Theorem 3.5 and Lemma 3.6 show that $M=H\oplus\sum_{I-\{\alpha\}}\oplus M_\beta$ for some $\alpha$ in $I$. We denote the projection: $M=H\oplus\sum_{I-\{\alpha\}}\oplus M_\beta\to H$ by $\pi_H$. Then the restriction map $\pi_H|M_\alpha$ is an isomorphism. On the other hand, we see from $H\oplus\sum_{I-\{\alpha\}}\oplus\pi_\beta(A)\subseteq\sum_I\oplus\pi_\alpha(A)$ that $\pi_H\pi_\alpha(A)=H$ and hence $\pi_\alpha(A)=M_\alpha$, a contradiction. Thus $A$ must be small in $M$.

'Only if' part: If there exists $\alpha$ in $I$ such that $\pi_\alpha(A)=M_\alpha$ then $M=A+\sum_{I-\{\alpha\}}\oplus M_\beta$. Since $0\subseteq_c A$ in $M$, it follows that $M=\sum_{I-\{\alpha\}}\oplus M_\beta$, a contradiction. Hence $\pi_\alpha(A)\neq M_\alpha$ for all $\alpha\in I$.

REMARK. Let $\{M_\alpha\}_I$ be a set of completely indecomposable $R$-modules. $\{M_\alpha\}_I$ is said to be locally semi-$T$-nilpotent ([3]) if it satisfies the following condition: Let $\{M_{\alpha_i}\}_{i=1}^\infty$ be a countable subset of $\{M_\alpha\}_I$ with $\alpha_n\neq\alpha_{n'}$ if $n\neq n'$. Then, for any non-isomorphism $\{f_{\alpha_n}: M_{\alpha_n}\to M_{\alpha_{n+1}}\mid n\geq 1\}$ and any $x$ in $M_{\alpha_1}$, there exists a number $m$ depending on $x$ such that $f_{\alpha_m}f_{\alpha_{m-1}}\cdots f_{\alpha_1}(x)=0$. Yamagata pointed out in [33] that it follows from [3], [4], [5], [18] and [33] that $\{M_\alpha\}_I$ is locally semi-$T$-nilpotent if and only if $M=\sum_I\oplus M_\alpha$ satisfies the finite exchange property. On the other hand, it is known in [6] or [15] that $\{M_\alpha\}_I$ is locally semi-$T$-nilpotent if and only if $M=\sum_I\oplus M_\alpha$ satisfies the following conditions: For any independent family $\{T_\beta\}_J$ of submodules of $M$, if $\sum_J T_\beta$ is a locally direct summand of $M$ then $\sum_J T_\beta$ is a direct summand of $M$.

By this remark and Lemma 3.2 we have the following lemma.

**Lemma 3.8.** *Let $M$ be a quasi-semiperfect $R$-module, and let $M=\sum_I\oplus M_\alpha$ be a decomposiiion of $M$ with each $M_\alpha$ a hollow module. Put $J=\{\beta\in I\mid M_\beta$ is*

*completely indecomposable*$\}$. *Then* $\sum_J \oplus M_\beta$ *has the finite exchange property.*

**Lemma 3.9.** *Let $M$ be a quasi-semiperfect $R$-module, and let $N_1$, $N_2$ direct summands of $M$ with $N_1 \oplus N_2 < \oplus M$. Then every sequence $N_1 \to N_2 \to 0$ splits. In particular, for a decomposition $M=\sum_I \oplus M_\alpha$ with each $M_\alpha$ hollow, every epimorphism from $M_\alpha$ to $M_\beta$ is an isomorphism for any pair $\alpha$, $\beta$ in $I$.*

Proof. Put $M=N_1 \oplus N_2 \oplus Y$, and let $f$ an epimorphism from $N_1$ to $N_2$. We put $X=ker(f)$ and $A=N_2 \oplus X$. Then $M/X=A/X \oplus N_1/X \oplus (Y+X)/X$ and $f$ induces a canonical isomorphism $h$ from $A$ to $N_1/X$; whence by Proposition 2.5 there exists a homomorphism $k$ from $A$ to $N_1$ satisfying $\eta_X k=h$, where $\eta_X$ denotes the canonical map: $N_1 \to N_1/X$. Then we can verify that $f(k \mid N_2)$ is the identity map of $N_2$, and hence the sequence $N_1 \xrightarrow{f} N_2 \to 0$ splits.

**Theorem 3.10.** *Let $M$ be a quasi-semiperfect $R$-module, and let $M=\sum_I \oplus M_\alpha$ be a decomposition with each $M_\alpha$ hollow. Then for another decomposition $M=\sum_J \oplus N_\beta$ with each $N_\beta$ indecomposable the following statements hold:*

1) *There exists a one to one onto map $\sigma: I \to J$ such that $M_\alpha \simeq N_{\sigma(\alpha)}$ for all $\alpha \in I$.*

2) *For any subset $K$ of $I$ there exists a one to one into map $\delta: K \to J$ for which $M_\beta \simeq N_{\delta(\beta)}$ for all $\beta \in K$ and $M=\sum_K \oplus M_\alpha \oplus \sum_{J-\delta(K)} \oplus N_\beta$.*

Proof. First we note that if $\alpha \neq \beta$ and $M_\alpha \simeq M_\beta$ then $M_\alpha$ and $M_\beta$ are completely indecomposable since they satisfy $(E-I)$ by Lemma 3.8. Let $I=I_1 \cup I_2$ be the partition of $I$ such that if $\alpha \in I_1$ then $M_\alpha$ is completely indecomposable, while $M_\alpha$ is non-completely indecomposable if $\alpha \in I_2$. We also consider the similar partition $J=J_1 \cup J_2$.

Now we may show the following:

1) $\sum_{I_1} \oplus M_\alpha \simeq \sum_{J_1} \oplus N_\beta$, $\sum_{I_2} \oplus M_\alpha \simeq \sum_{J_2} \oplus N_\beta$,

2) For any $K_1 \subseteq J_1$ and $K_2 \subseteq J_2$ there exist a one to one map: $K_1 \xrightarrow{f_1} I_1$ and a one to one map: $K_2 \xrightarrow{f_2} I_2$ such that

$$\begin{aligned} M &= \sum_{K_1} \oplus N_\beta \oplus \sum_{I_1 - \delta_1(K_1)} \oplus M_\alpha \oplus \sum_{I_2} \oplus M_\alpha \\ &= \sum_{I_1} \oplus M_\alpha \oplus \sum_{K_2} \oplus N_\beta \oplus \sum_{I_2 - \delta_2(K_2)} \oplus M_\alpha . \end{aligned}$$

For a convenience we put $M(L)=\sum_L \oplus M_\alpha$ (resp. $N(L)=\sum_L \oplus N_\beta$) for a subset $L$ of $I$ (resp. $J$).

Let $K_1$ be a subset of $J_1$. Then $N(K_1)$ satisfies the finite exchange property by Lemma 3.9. Hence

$$M = N(K_1)\oplus M(V_1)\oplus M(V_2)$$

for some $V_1 \subseteq I_1$ and $V_2 \subseteq I_2$. If $V_2 \neq I_2$ then $M(I_2 - V_2)$ is isomorphic to a non-zero direct summand of $\mathrm{N}(K_1)\oplus \mathrm{M}(V_1)$. But by the remark above there exists $\alpha \in I_2 - V_2$ such that $M_\alpha$ is completely indecomposable, a contradiction. Therefore we get

$$\mathrm{M} = \mathrm{N}(K_1)\oplus \mathrm{M}(V_1)\oplus \mathrm{M}(I_2)\,.$$

In particular if we take $J_1$ as $K_1$ then

$$M = \mathrm{N}(J_1)\oplus \mathrm{M}(V_1)\oplus \mathrm{M}(I_2)\,.$$

But in this case we can obtain $V_1 = \emptyset$ by a similar argument to the above. Thus we get

$$M = \mathrm{N}(J_1)\oplus \mathrm{M}(I_2)\,.$$

Accordingly $\mathrm{M}(I_1) \simeq \mathrm{N}(J_1)$ and hence by the Krull-Remak-Schmidt-Azumaya's theorem ([1]) there exists a one to one onto map $\delta_1 : I_1 \to J_1$ such that $M_\alpha \simeq N_{\delta_1(\alpha)}$ for all $\alpha \in I_1$.

Next if $\beta$ is an element in $J_2$ then we see from Lemma 3.6 that there exists $\beta'$ in $I_2$ such that $M = \mathrm{M}(I_1)\oplus N_{\beta'}\oplus \mathrm{M}(I_2 - \{\beta\})$. Then such $\beta'$ is uniquely determined by the fact noted above. As a result, $I_2 \simeq J_2$ and for any $L_2 \subseteq J_2$ there exists a one to one into map $\delta_2 : L_2 \to I_2$ such that

$$M = \mathrm{M}(I_1)\oplus \mathrm{N}(L_2)\oplus \mathrm{M}(I - \delta_2(L_2))$$

The proof is now complete.

**Corollary 3.11.** *Let $M$ be a quasi-semiperfect $R$-module, and let $M = \sum_I \oplus M_\alpha$ be a decomposition of $M$ with each $M_\alpha$ hollow. Then the following statements hold.*

1) *If $N$ is a direct summand of $M$ then there exists a subset $J$ of $I$ satisfying $M = N \oplus \sum_J \oplus M_\beta$.*

2) *If $A$ is a submodule of $M$ then there exists a subset $J$ of $I$ such that $M = A + \sum_J \oplus M_\beta$ and $A \cap \sum_J \oplus M_\beta$ is small in $M$.*

Proof. For a subset $K$ of $I$ we put $\mathrm{M}(K) = \sum_K \oplus M_\alpha$. 1) is immediate from Theorem 3.5 and 3.10.

2) Let $A$ be a submodule of $M$. Then by the condition $(\mathrm{C}_1)$ for $\mathcal{L}(M)$ there exists a direct summand $A^*$ of $M$ with $A^* \subseteq_e A$ in $M$. Here using 1) we get $M = A^* \oplus \mathrm{M}(J)$ for some subset $J$ of $I$. Then we can see that $M = A + M(J)$ and $0 \subseteq_e (A \cap M(J))$ in $M$.

**Lemma 3.12.** *Let $M$ be a quasi-semiperfect $R$-module and let $N_1$, $N_2$ direct summands of $M$ with $M = N_1 \oplus N_2$. Then for any submodule $A$ of $N_2$ and*

*a homomorphism $f$ from $N_1$ to $\bar{N}_2=N_2/A$ there exists a homomorphism $f'$ from $N_1$ to $N_2$ satisfying $\phi_A f'=f$ where $\phi_A$ is the canonical map: $N_2\to\bar{N}_2$*

Proof. Let $A$ be a submodule of $N_2$ and $f$ a homomorphism from $N_1$ to $\bar{N}_2=N_2/A$. By the condition $(C_1)$ for $\mathcal{L}(M)$ there exists a direct summand $A^*$ of $M$ with $A^*\subseteq_c A$ in $M$. Put $N_2=A^*\oplus N_2'$. Then it follows from $A^*\subseteq_c A$ in $M$ that $0\subseteq_c(A\cap N_2')$ in $M$. Now $\sigma$ denotes the canonical isomorphism from $N_2/A$ onto $N_2'(A\cap N_2')$. Since $0\subseteq_c(A\cap N_2')$ in $M$, by Proposition 2.5 there exists a homomorphism $g$ from $N_1$ to $N_2'$ satisfying $\eta_A g=\sigma f$ where $\eta_A$ denotes the canonical map: $N_2'\to N_2'/(A\cap N_2')$. Then $\eta_A ig=f$ where $i$ denotes the inclusion map: $N_2'\to N_2$. Thus $ig$ is a required homomorphism from $N_1$ to $N_2$.

**Corollary 3.13.** *Let $M$ be an $R$-module. If $M\oplus M$ is quasi-semiperfect $R$-module then $M$ is quasi-projective.*

**Theorem 3.14.** *Let $M$ be an $R$-module such that every homomorphic image of $M\oplus M$ has a projective cover. Then the following conditions are equivalent:*

1) *$M$ is quasi-projective.*
2) *$M\oplus M$ is semiperfect.*
3) *$M\oplus M$ is quasi-semiperfect.*

Proof. 2)$\Rightarrow$3) follows from Thoerem 2.3, and 3)$\Rightarrow$1) is Corollary 3.13.

1)$\Rightarrow$2). Since $M$ is quasi-projective and every homomorphic image has a projective cover, $M\oplus M$ is also quasi-projective (see [21] or [31]). Thus $M\oplus M$ is semiperfect.

**Theorem 3.15.** *Let $M$ be a quasi-projective $R$-module with the property that $J(N)\neq N$ for any direct summand $N$ of $M$. Then the following conditions are equivalent:*

1) *$M$ is semiperfect.*
2) *$M$ is quasi-semiperfect.*
3) i) *$J(M)$ is small in $M$,*
   ii) *$M/J(M)$ is completely reducible,*
   iii) *Every decomposition of $M/J(M)$ is induced from a decomposition of $M$.*

Proof. Since $M$ is quasi-projective, clearly it satisfies the condition $(C_2)$ for $\mathcal{L}(M)$; whence 1)$\Leftrightarrow$2) is evident.

2)$\Rightarrow$3). i) and ii) follow from Proposition 3.1 and iii) follows from Theorem 3.3.

3)$\Rightarrow$1). By i)$\sim$iii) it is easy to see that $M$ satisfies the condition $(C_1)$ for $\mathcal{L}(M)$. Since $M$ is quasi-projective it satisfies the condition $(C_2)$. Hence $M$ is semiperfect.

REMARKS. 1) If $M$ is a quasi-projective $R$-module whose homomor-

phic images have projective covers then $M$ is semiperfect by Theorem 2.1 and moreover $J(N) \neq N$ for every direct summand $N$ of $M$. So $M$ satisfies the conditions i)~iii) in Theorem 3.15.

2) If $M$ is a projective $R$-module then $J(N) \neq N$ for every direct summand $N$ of $M$. Thus Theorem 3.15 is an extension of the Mares's theorem.

**Theorem 3.16.** *An $R$-module $M$ is quasi-semiperfect if and only if $M$ is written as a direct sum of hollow modules $\{M_\alpha\}_I$ and satisfies the following conditions*:

1) *For any submodule $A$ of $M$ there exists a subset $J$ of $I$ such that $M=A+\sum_J \oplus M_\beta$ and $A \cap \sum_J \oplus M_\beta$ is small in $M$.*

2) *Let $J_1$, $J_2$ be subsets of $I$ with $J_1 \cap J_2 = \emptyset$ and let $X$ a small submodule of $\sum_{J_2} \oplus M_\beta$. Then for every homomorphism $f$ from $\sum_{J_1} \oplus M_\alpha$ to $(\sum_{J_2} \oplus M_\beta)/X$ there exists a homomorphism $f'$ from $\sum_{J_1} \oplus M_\alpha$ to $\sum_{J_2} \oplus M_\beta$ satisfying $\eta_A f' = f$ where $\eta_A$ is the canonical map: $\sum_{J_2} \oplus M_\beta \to (\sum_{J_2} \oplus M_\beta)/X$.*

Proof. If part: 1) follows from Corollary 3.11, and 2) is clear.

Only if part: For a subset $K$ of $I$ we set $\mathrm{M}(K)=\sum_K \oplus M_\alpha$. We first show the condition $(C_1)$ for $\mathcal{L}(M)$ is satisfied. Let $A$ be a submodule of $M$. Then by 1) there exists a subset $J$ of $I$ satisfying $M=A+\mathrm{M}(J)$ and $0 \subseteq_c (A \cap \mathrm{M}(J))$ in $M$. Put $X=A \cap \mathrm{M}(J)$ and $\bar{M}=M/X$. Then

$$\begin{aligned} \bar{M} &= \bar{A} \oplus \overline{\mathrm{M}(J)} \\ &= \overline{\mathrm{M}(I-J)} \oplus \overline{\mathrm{M}(J)} . \end{aligned}$$

By $\pi_1$ and $\pi_2$ we denote the projections: $\bar{M} \to \overline{\mathrm{M}(I-J)}$ and $\bar{M} \to \overline{\mathrm{M}(J)}$, respectively, with respect to $\bar{M}=\overline{\mathrm{M}(I-J)} \oplus \overline{\mathrm{M}(I)}$. Then $\bar{A} \simeq \overline{\mathrm{M}(I-J)}$ by $\pi_1$ and moreover $f$: $\overline{\mathrm{M}(I-J)} \to \overline{\mathrm{M}(J)}$ given by $f(\pi_1(\bar{a}))=\pi_2(\bar{a})$, $a \in A$, is a homomorphism. Here by 2) $f$ is induced from a homomorphism $f'$: $\mathrm{M}(I-J) \to \mathrm{M}(J)$. Put $A^*=\{x+f'(x) \mid x \in \mathrm{M}(I-J)\}$. Then $M=A^* \oplus \mathrm{M}(J)$ and $\bar{A}^*=\bar{A}$, moreover it follows from $X \subseteq A$ that $A^* \subseteq A$ and $X=A \cap \mathrm{M}(J)$. Thus $A^* \subseteq_c A$ in $M$ by Lemma 1.4.

Next we wish to show that $M$ satisfies $(C_5')$ for $\mathcal{L}(M)$ (cf. Propositions 2.4 and 2.5) in order to show that $M$ satisfies the condition $(C_3)$ for $\mathcal{L}(M)$. But in view of the proof of Proposition 2.5 it is further enough to show that every homomorphism from $\bar{A}$ to $\overline{\mathrm{M}(J)}$ is induced from one from $A$ to $\mathrm{M}(J)$.

Let $h$ be a homomorphism from $\bar{A}$ to $\overline{\mathrm{M}(J)}$. Then by 2) there exists a homomorphism $g$ from $\mathrm{M}(I-J)$ to $\mathrm{M}(J)$ for which the diagram

$$\begin{array}{ccccc} \overline{\mathrm{M}(I-J)} & \xrightarrow{\pi_1^{-1}} & \bar{A} & \xrightarrow{h} & \overline{\mathrm{M}(J)} \\ \downarrow{\scriptstyle \eta_1} & & & & \downarrow{\scriptstyle \eta_2} \\ \mathrm{M}(I-J) & & \xrightarrow{\quad g \quad} & & \mathrm{M}(J) \end{array}$$

is commutative, where $\eta_i$ is the canonical map, $i=1, 2$. Let $a\in A$ and express it in $M=A^*\oplus \mathrm{M}(J)$ as $a=(x_a+f(x_a))+y_a$ where $x_a\in M(I-J)$ and $y_a\in \mathrm{M}(J)$. Then the mapping $l$: $A\to \mathrm{M}(I-J)$ given by $l(a)=x_a$ is a homomorphism. Put $h'=gl$. Then $h\eta_A(a)=h(\bar{a})=h\pi_1^{-1}(\pi_1(a))=h\pi_1^{-1}(\bar{x}_a)=\eta_2 g(x_a)=\eta_2 gl(a)=\eta_2 h'(a)$; whence $h\eta_A=\eta_2 h'$ where $\eta_A$ denotes the canonical map: $A\to\bar{A}$. The proof is now complete.

**Theorem 3.17.** *Let $M$ be a quasi-semiperfect $R$-module, and let $M=\sum_I\oplus M_\alpha$ a decomposition with each $M_\alpha$ hollow. Then the following conditions are equivalent:*

1) *$M$ is semiperfect.*
2) *Every $M_\alpha$ satisfies the condition* (E−I).

Proof. 1)⇒2) is clear by the condition $(C_2)$ for $\mathcal{L}(M)$.

2)⇒1). By 2) and Lemma 3.9 we see that for any two indecomposable direct summands $N_1$, $N_2$ of $M$ every epimorphism from $N_1$ to $N_2$ is an isomorphism. We wish to show that $M$ satisfies the condition $(C_2)$ for $\mathcal{L}(M)$. Further to verify this it is enough by Corollary 3.11 (1) to show the following: Let $J$ be a subset of $I$ and $f$ an epimorphism from $M$ to $\sum_J\oplus M_\alpha$. Then the sequence $M\to\sum_J\oplus M_\alpha\to 0$ splits.

Put $N=\sum_J\oplus M_\alpha$. For $\alpha\in J$ we show that there exists a direct summand $N_\alpha$ of $M$ such that $f(N_\alpha)=M_\alpha$. Put $A=f^{-1}(M_\alpha)$. Since $A$ is not small in $M$, there exists a decomposition $M=A^*\oplus A^{**}$ with $0\neq A^*\subseteq A$ and $0\subseteq_c(A\cap A^{**})$ in M (cf. Lemma 1.4). Clearly $f(A^*)=M_\alpha$. First we consider the case: $(A^*\oplus M_\alpha)<\oplus M$. Then put $X=f^{-1}(M_\alpha)\cap A^*$ and denote the induced isomorphism: $A^*/X\simeq M_\alpha$ by $g$. Here applying Lemma 1.12, we have a homomorphism $h$ from $M_\alpha$ to $A^*$ satisfying $\eta_X h=g$ where $\eta_X$ is the canonical map: $A^*\to A^*/X$. Then we see that $fh$ is the identity map of $M_\alpha$; whence we may take $h(M_\alpha)$ as $N_\alpha$.

Next we consider the case: $\not((A^*\oplus M_\alpha)<\oplus M)$. Then by Theorem 3.5 and Lemma 3.6 we can get a decomposition $A^*=B\oplus C$ such that $B\simeq M_\alpha$ and $(C\oplus M_\alpha)<\oplus M$. If $f(B)=M_\alpha$ then (2) says $B\simeq M_\alpha$ by $f$. In the case of $f(B)\subsetneqq M_\alpha$ we see $f(C)=M_\alpha$. Then by the same argument above there exists a direct summand $D$ of $C$ with $D\simeq M_\alpha$ by $f$.

Thus at any rate for each $\alpha\in J$ there exists a direct summand $N_\alpha$ of $M$ such that the restriction map $f|N_\alpha$ is an isomorphism from $N_\alpha$ to $M_\alpha$. So if we put $p=\sum_J(f|N_\alpha)$ then $fp$ is the identity map of $N=\sum_J\oplus M_\alpha$ as required.

**Theorem 3.18.** *Let $M$ be a quasi-projective $R$-module satisfying the condition* $(C_6)$ *for* $\mathcal{L}(M)$. *Then $M$ is semiperfect; so it is written as a direct sum of hollow modules by Theorem* 3.5.

Proof. Since $M$ is quasi-projective, clearly it satisfies the condition $(C_2)$ for $\mathcal{L}(M)$. To show the condition $(C_1)$ for $\mathcal{L}(M)$ let $A$ be a submodule of $M$. By the assumption there exists a decomposition $M=A^*\oplus A^{**}$ such that $M=A+A^{**}$ and $A\cap A^{**}$ is small in $M$. Put $X=A\cap A^{**}$ and $\bar{M}=M/X$. Then

$$\begin{aligned}\bar{M} &= \bar{A}\oplus\bar{A}^{**}\\ &= \bar{A}^*\oplus\bar{A}^{**}.\end{aligned}$$

$\bar{\pi}^*$ and $\bar{\pi}^{**}$ denote the projections: $\bar{M}\to\bar{A}^*$ and $\bar{M}\to\bar{A}^{**}$, respectively, with respect to $\bar{M}=\bar{A}^*\oplus\bar{A}^{**}$. Then $\bar{\pi}^*(\bar{A})=\bar{A}^*$ and the map $f$: $\bar{A}^*\to\bar{A}^{**}$ given by $f(\bar{\pi}^*(\bar{a}))=\bar{\pi}^{**}(\bar{a})$ for $a\in A$ is a homomorphism. Consider the map $g=f\pi^*\eta$, where $\eta$ is the canonical map: $M\to\bar{M}$;

$$\begin{array}{ccc}\bar{M} & \xrightarrow{g} & \bar{M}\\ {\scriptstyle\eta}\uparrow & & \uparrow{\scriptstyle\eta}\\ M & & M\end{array}$$

Now by the quasi-projectivity of $M$ there exists $h$: $M\to M$ satisfying $\eta h=g\eta$. Put $f'=\pi^{**}(h\,|\,A^*)$. Then the diagram

$$\begin{array}{ccc}\bar{A}^* & \xrightarrow{f} & \bar{A}^{**}\\ {\scriptstyle\eta_1}\downarrow & & \downarrow{\scriptstyle\eta_2}\\ A^* & \xrightarrow{f'} & A^*\end{array}\qquad (\eta_1=\eta\,|\,A^*,\ \eta_2=\eta\,|\,A)$$

is commutative. We set $A^0=\{x+f'(x)\,|\,x\in{}^*\}$. Then $M=A^0\oplus A^{**}$ and it follows from $\bar{A}^0=\bar{A}$ that $A^0\subseteq A$. Thus by Lemma 1.4, $A^0\subseteq_c A$ in $M$ as desired.

## 4. Lifting property of direct sum for $\mathcal{A}$

In this section we introduce the notion of the lifting property of direct sum for $\mathcal{A}$, from which it will be clarified why we say $M$ to have the lifting property of module for $\mathcal{A}$ if it satisfies the condition $(C_1)$ for $\mathcal{A}$.

Let $M$ be an $R$-module and $\mathcal{N}=\{N_\alpha\}_I$ a subfamily of $\mathcal{L}(M)$. We put

$$M(\mathcal{N}) = \{x\in M\,|\,x\in \bigcap_{I-F} N_\beta \quad \text{for some finite subset } F \text{ of } I\}\ .$$

Then clearly $M(\mathcal{N})$ is a submodule of $M$, and the map from $M(\mathcal{N})$ to $\sum_I\oplus(M/N_\alpha)$

given by $x\to\sum_I(x+N_\alpha)$ is well defined. We denote this by $\eta_{\mathcal{N}}$ and the induced monomorphism: $M(\mathcal{N})/ker(\eta_{\mathcal{N}})\to\sum_I\oplus(M/N_\alpha)$ by $\bar{\eta}_{\mathcal{N}}$. We note that $M=M(\mathcal{N})$ if $\mathcal{N}$ is a finite family.

For our purpose we consider the following condition:

(*) For any finite subset $F$ of $I$,

$$M=(\bigcap_F N_\alpha)+(\bigcap_{I-F}N_\beta)\,.$$

As is easily seen, (*) is satisfied if and only if $\eta_{\mathcal{N}}$ is an epimorphism.

Here, we introduce the notion of a co-independent family which is dual to that of an independent family.

DEFINITION. Let $\mathcal{N}=\{N_\alpha\}_I$ be a subfamily of $\mathcal{L}(M)$. We say that $\mathcal{N}$ is co-independent if $N_\alpha\neq M$ for all $\alpha\in I$, $M=M(\mathcal{N})$ and $\eta_{\mathcal{N}}$ is an epimorphism.

**Proposition 4.1.** *Let $\mathcal{N}=\{N_\alpha\}_I$ be a co-independent subfamily of $\mathcal{L}(M)$ and put $T_\alpha=\bigcap_{I-\{\alpha\}}N_\beta$ for all $\alpha\in I$ and $X=\bigcap_I N_\alpha$. Then*

1) *$\bar{\eta}_{\mathcal{N}}(T_\alpha/X)=M/N_\alpha$ for all $\alpha\in I$; so $M=\sum_I T_\alpha$ and $M/X=\sum_I\oplus(T_\alpha/X)$.*
2) *$M=\sum_I T_\alpha$ is irredundant.*
3) *$N_\alpha=\sum_{I-\{\alpha\}}T_\beta$.*

Proof. 1) Since $M=N_\alpha+T_\alpha$ and $T_\alpha\subseteq N_\beta$ for $\alpha\neq\beta$, $\bar{\eta}_{\mathcal{N}}(T_\alpha/X)=M/N_\alpha$ for all $\alpha\in I$.

2) Since $0\neq T_\alpha/X\simeq M/N_\alpha$ for all $\alpha\in I$, $M=\sum_I T_\alpha$ is irredundant.

3) $\bar{\eta}_{\mathcal{N}}(T_\beta/X)=M/N_\beta$ shows that $T_\beta\subseteq N_\alpha$ for $\beta\neq\alpha$; whence $\sum_{I-\{\alpha\}}T_\beta\subseteq N_\alpha$. Since $\eta_{\mathcal{N}}(T_\alpha)\subseteq\sum_{I-\{\alpha\}}\oplus(M/N_\beta)$ we see that $N_\alpha/X\subseteq\sum_{I-\{\alpha\}}\oplus(T_\beta/X)$ and therefore $N_\alpha\subseteq\sum_{I-\{\alpha\}}T_\beta$. As a result $N_\alpha=\sum_{I-\{\alpha\}}T_\beta$.

**Proposition 4.2.** *Let $X$ be a submodule of $M$ and $M/X=\sum_I\oplus(T_\alpha/X)$ a decomposition with $T_\alpha\supsetneq X$ for all $\alpha\in I$. Put $N_\alpha=\sum_{I-\{\alpha\}}T_\beta$ for all $\alpha\in I$. Then*

1) *$M=\sum_I T_\alpha$ is irredundant.*
2) *$\mathcal{N}=\{N_\alpha\}_I$ is co-independent.*
3) *$X=\bigcap_I N_\alpha$ and $T_\alpha=\bigcap_{I-\{\alpha\}}N_\beta$ for all $\alpha\in I$.*

Proof. 1) is clear since $T_\alpha\supsetneq X$ for all $\alpha\in I$.

2) Since $M=\sum_I T_\alpha$ is irredundant, $N_\alpha\neq M$ for all $\alpha\in I$. Noting $\eta_{\mathcal{N}}(T_\alpha)=M/N_\alpha$ we see $M=M(\mathcal{N})$. Now if $F$ is a finite subset of $I$ then $\bigcap_F N_\alpha\supseteq\sum_{I-F}T_\beta$ and $\bigcap_{J-F}N_\beta\supseteq\sum_F T_\alpha$; whence $M=(\bigcap_F N_\alpha)+(\bigcap_{I-P}N_\beta)$. Thus $\mathcal{N}$ is co-independent.

3) Since $X\subseteq T_\alpha$ for all $\alpha\in I$ it is clear that $X\subseteq N_\alpha$ for all $\alpha\in I$. As a result, $X\subseteq \bigcap_I N_\alpha$. Let $a\in \bigcap_I N_\alpha$ and express it in $M=\sum_I T_\alpha$ as $a=a_{\alpha_1}+\cdots+a_{\alpha_n}$. Then it follows from $a_{\alpha_i}\in N_{\alpha_i}$ that $a_{\alpha_i}\in X$ for all $i$. Hence $a\in X$ and we get $X=\bigcap_I N_\alpha$.

REMARK. By Propositions 4.1 and 4.2 there exists a one to one map between the family of all co-independent subfamilies of $\mathcal{L}(M)$ and the family of all decompositions of all homomorphic images of $M$.

**Proposition 4.3** *Let $M=\sum_I T_\alpha$ be an irredundant sum and put $N_\alpha=\sum_{I-\{\alpha\}} T_\beta$ for all $\alpha\in I$ and $X=\bigcap_I N_\alpha$. Then*

1) *$\mathcal{N}=\{N_\alpha\}_I$ is co-independent.*
2) *$T_\alpha+X=\bigcap_{I-\{\alpha\}} N_\beta$ for all $\alpha\in I$.*
3) *$M/X=\sum_{\alpha\in I}\oplus((T_\alpha+X)/X)$.*

Proof. The proofs of 1) and 2) are done as that of Proposition 4.2. 3) is evident.

**Proposition 4.4.** *Let $\mathcal{N}=\{N_\alpha\}_I$ be a co-independent subfamily of $\mathcal{L}(M)$ and let $\{N_\alpha^*\}_I$ be a subfamily of $\mathcal{L}(M)$ with $N_\alpha^*\subseteq_c N_\alpha$ in $M$ for all $\alpha\in I$. Then for any finite subset $F$ of $I$, $\{N_\alpha^*\}_F\cup\{N_\beta\}_{I-F}$ is co-independent.*

Proof. It is enough to show that $\{N_\alpha^*\}\cup\{N_\beta\}_{I-\{\alpha\}}$ is co-independent for any $\alpha\in I$. So, let $\alpha\in I$. Inasmuch as $N_\alpha^*\subseteq_c N_\alpha$ in $M$ and $M=N_\alpha+(\bigcap_{I-\{\alpha\}} N_\beta)$ we infer that $M=N_\alpha^*+(\bigcap_{I-\{\alpha\}} N_\beta)$. As a result $M/Y\simeq M/N_\alpha^*\oplus M/(\bigcap_{I-\{\alpha\}} N_\beta)$, canonically, where $Y=N_\alpha^*\cap(\bigcap_{I-\{\alpha\}} N_\beta)$. On the other hand $M/(\bigcap_{I-\{\alpha\}} N_\beta)\simeq \sum_{I-\{\alpha\}}\oplus(M/N_\beta)$ since $\mathcal{N}-\{N_\alpha\}$ is co-independent. Accordingly we see

$$M/Y\simeq M/N_\alpha^*\oplus\sum_{I-\{\alpha\}}\oplus(M/N_\beta)$$

canonically; whence $\{N_\alpha^*\}\cup\{N_\beta\}_{I-\{\alpha\}}$ is co-independent.

DEFINITION. Let $X$ be a submodule of $M$ and $M/X=\sum_I\oplus(T_\alpha/X)$ a decomposition of $M/X$ with $X\subsetneqq T_\alpha$ for all $\alpha\in I$. We say that $M/X=\sum_I\oplus T_\alpha/X$ is *co-essentially lifted to a decomposition of $M$* if there exists a decomposition $M=X^*\oplus\sum_I\oplus T_\alpha^*$ such that $X^*\subseteq X$, $T_\alpha=X+T_\alpha^*$ and $0\subseteq_c(T_\alpha^*\cap X)$ in $T_\alpha^*$ for all $\alpha\in I$.

**Proposition 4.5.** *Let $\mathcal{N}=\{N_\alpha\}_I$ be a co-independent subfamily of $\mathcal{L}(M)$,*

*and let* $M/X=\sum_I \oplus(T_\alpha/X)$ *be its corresponding decomposition,* i.e., $X=\bigcap_I N_\alpha$ *and* $T_\alpha=\bigcap_{I-\{\alpha\}} N_\beta$ *for all* $\alpha\in I$. *Then* $M/X=\sum_I \oplus(T_\alpha/X)$ *is co-essentially lifted to a decomposition of* $M$ *if and only if there exists a subfamily* $\{N_\alpha^*\}_I$ *such that* $N_\alpha^*\subseteq_c N_\alpha$ *in* $M$ *for all* $\alpha\in I$ *and* $\bigcap_I N_\alpha^*<\oplus M$.

Proof. This is clear by Proposition 4.1.

DEFINITION. Let $\mathcal{A}$ be a subfamily of $\mathcal{L}(M)$. We say that $M$ has the lifting property of direct sums for $\mathcal{A}$, provided that, for any co-independent subfamily $\mathcal{N}$ of $\mathcal{A}$, the corresponding decomposition to $\mathcal{N}$ is co-essentially lifted to a decomposition of $M$. If this condition holds whenever $\mathcal{N}$ is a finite family we say that $M$ has the lifting property of finite direct sums for $\mathcal{A}$.

REMARK. In [14], the notion of the extending property of direct sums for $\mathcal{A}$ has been introduced. In view of Proposition 4.5, we see that its dual is just the above notion of the lifting property of direct sums for $\mathcal{A}$.

Now, our main purpose of this section is to show that a quasi-semiperfect $R$-module $M$ has the lifting property of direct sums for $\mathcal{L}(M)$. Before proving this, we further observe some properties of quasi-semiperfect modules.

**Theorem 4.6.** *Let* $M$ *be a quasi-semiperfect* $R$*-module,* $X$ *a small submodule of* $M$ *and* $M/X=\sum_I \oplus(T_\alpha/X)$ *a decomposition of* $M/X$ *with* $X\subsetneq T_\alpha$ *for all* $\alpha\in I$. *If* $\{T_\alpha^*\}_I$ *is a subfamily of* $\mathcal{L}(M)$ *such that* $T_\alpha^*<\oplus M$ *and* $T_\alpha=T_\alpha^*+X$ *for all* $\alpha\in I$ *then* $M=\sum_I \oplus T_\alpha^*$.

Proof. Since $X$ is small in $M$, it follows from $M=(\sum_I T_\alpha^*)+X$ that $M=\sum_I T_\alpha^*$. Now, let $F$ be a finite subset of $I$ and assume that $\{T_\alpha^*\}_F$ is independent and $\sum_F \oplus T_\alpha^*<\oplus M$. Put $T=\sum_F \oplus T_\alpha^*$. Now, to show the lemma, it suffices to show that $T+T_\beta^*=T\oplus T_\beta^*<\oplus M$ for every $\beta\in I-F$ by Proposition 3.3. By the condition $(C_1)$ for $\mathcal{L}(M)$, we can take a direct summand $Q$ of $M$ with $Q\subseteq_c \sum_{I-F} T_\beta^*$ in $M$. Then $M=T\oplus Q$ since $M=T+\sum_{I-F} T_\beta^*$. Hence, we see from $T\cap Q\subseteq X$ that $M=T\oplus Q$. Here, let $\beta\in I-F$. By Theorem 3.5 and Corollary 3.11 we get

$$M = T_\beta^*\oplus T'\oplus Q'$$

for some submodules $T'\subseteq T$ and $Q'\subseteq Q$. Since $T_\beta^*\oplus Q'\subseteq \sum_{I-F} T_\gamma$, we see that $T+X=T'+X$. Inasmuch as $X$ is small in $M$, it follows that $T=T'$. Thus $(\sum_F \oplus T_\alpha^*)\oplus T_\beta^*<\oplus M$.

**Corollary 4.7.** *Let* $M$ *be a quasi-semiperfect* $R$*-module, and* $M=\sum_I T_\alpha$

*an irredundant sum with* $T_\alpha < \oplus M$ *for all* $\alpha \in I$. *If* $\bigcap_{\alpha\in I}(\sum_{I-\{\alpha\}} T_\beta)$ *is small in* $M$ *then* $M=\sum_I \oplus T_\alpha$.

Proof. Put $X=\bigcap_{\alpha\in I}(\sum_{I-\{\alpha\}} T_\beta)$ and $T'_\alpha = T_\alpha + X$ for all $\alpha\in I$. Then, by Proposition 4.3, we get

$$M/X = \sum_I \oplus (T'_\alpha/X)$$

with $X \subsetneqq T'_\alpha$ for all $\alpha\in I$. Hence, by Theorem 4.6, $M=\sum_I \oplus T_\alpha$.

**Corollary 4.8.** *Let* $M$ *be a quasi-semiperfect* $R$*-module such that* $\mathrm{J}(M)$ *is small in* $M$. *If* $\{T_\alpha\}_I$ *is a family of direct summands of* $M$ *which is independent modulo* $\mathrm{J}(M)$ *then* $\{T_\alpha\}_I$ *is independent and* $\sum_I \oplus T_\alpha < \oplus M$.

Proof. By Proposition 3.1 we can assume that $M=\sum_I T_\alpha$. Since $\{T_\alpha\}_I$ is independent modulo $\mathrm{J}(M)$ and $T_\alpha \subsetneqq \mathrm{J}(M)$ for all $\alpha\in I$, clearly, $M=\sum_I T_\alpha$ is irredundant. Thus $M=\sum_I \oplus T_\alpha$ by Corollary 4.7.

Now, we are in a position to show our main theorem in this section.

**Theorem 4.9.** *For a given* $R$*-module* $M$, *the following conditions are equivalent*:

1) $M$ *is quasi-semiperfect.*
2) $M$ *has the lifting property of finite direct sums for* $\mathcal{L}(M)$.
3) $M$ *has the lifting property of direct sums for* $\mathcal{L}(M)$.

Proof. 3)⇒2)⇒1) is evident. So we show the implication 1)⇒3). Let $\{N_\alpha\}_I$ be a co-independent subfamily of $\mathcal{L}(M)$, and let $M/X=\sum_I \oplus (T_\alpha/X)$ be its corresponding decomposition ($X=\bigcap_I N_\alpha$ and $T_\alpha=\bigcap_{I-\{\alpha\}} N_\beta$ for all $\alpha\in I$). By the condition $(C_1)$ for $\mathcal{L}(M)$ we have $M=X^*\oplus X^{**}$ with $X^*\subseteq X$ and $X\cap X^{**}$ is small in $X$. Then $\{N_\alpha \cap X^{**}\}_I$ is a co-independent subfamily of $\mathcal{L}(M)$ and $\bigcap_I (N_\alpha \cap X^{**}) = X\cap X^{**}$. As is easily seen, $\sum \oplus (T_\alpha/X)$ is co-essentially lifted to a decomposition of $M$ if and only if $X^{**}/(X\cap X^{**})=\sum_J \oplus (X^{**}/(X\cap N_\alpha))$ is co-essentially lifted to a decomposition of $X^{**}$. Thus we can assume that $X$ is small in $M$.

Now, by the condition $(C_1)$ for $\mathcal{L}(M)$, there exists a direct summand $N^*_\alpha$ of $M$ with $N^*_\alpha \subseteq_c N_\alpha$ in $M$ for all $\alpha\in I$, and a direct summand $T^*_\alpha$ of $M$ with $T^*_\alpha \subseteq_c T_\alpha = \bigcap_{I-\{\alpha\}} N_\beta$ in $M$ for all $\alpha\in I$. Then, it follow from $M=N_\alpha+T_\alpha$ that $M=N^*_\alpha+T^*_\alpha$ for all $\alpha\in I$. We claim that $T_\alpha = T^*_\alpha + X$ for all $\alpha\in I$. In fact, it is clear that $T^*_\alpha + X \subseteq T_\alpha$. Conversely, let $a\in T_\alpha$ and express it in

$M=N_\alpha^*+T_\alpha^*$ as $a=n+t$, where $n\in N_\alpha^*$ and $t\in T_\alpha^*$. Since $n\in N_\alpha^*\subseteq N_\alpha$ and $n=a-t\in T_\alpha=\bigcap_{I-\{\alpha\}} N_\beta$ we see that $n\in\bigcap_I N_\alpha=X$; so $a\in T_\alpha^*+X$ and hence $T_\alpha\subseteq T_\beta^*+X$. Thus $T_\alpha=T_\alpha^*+X$ for all $\alpha\in I$.

Inasmuch as $T_\alpha=T_\alpha^*+X$, $T_\alpha^*<\oplus M$ for all $\alpha\in I$ and $X$ is small in $M$, we infer from Theorem 4.6 that $M=\sum_I\oplus T_\alpha^*$. The proof is completed.

Combining Theorem 4.9 to Theorem 2.1 we have

**Corollary 4.10.** *A quasi-projective R-module M over a right perfect ring R has the lifting property of direct sums for $\mathcal{L}(M)$.*

**Theorem 4.11.** *Let M be* a quasi-semiperfect $R$-module, and let $\{A_\alpha\}_I$ a family of indecomposable direct summands of $M$ with $M=\sum_I A_\alpha$. If the sum $M=\sum_I A_\alpha$ is irredundant then $M=\sum_I\oplus A_\alpha$.

Proof. For a subset $K$ of $I$, we put $M(K)=\sum_K A_\beta$. Consider the family $\mathcal{S}$ consisting of all subsets $J$ of $I$ such that $\{A_\beta\}_J$ is independent and $\sum_J\oplus A_\beta$ is a locally direct summand of $M$. Then $\mathcal{S}$ becomes a partially ordered set by inclusion and has a maximal member by Zorn's lemma. Let $J_0$ be one of maximal members in $\mathcal{S}$. We wish to show that $I=J_0$. So, assume $I\neq J_0$.

By Proposition 3.2, $M(J_0)$ is a direct summand of $M$; so $M=M(J_0)\oplus T$ for some submodule $T$. Take $\alpha_0\in I-J_0$. Then, by the maximality of $J_0$ and Corollary 3.11, there must exist $\beta_0\in J_0$ such that $M=A_{\alpha_0}\oplus M(J_0-\{\beta_0\})\oplus T$; whence $A_{\alpha_0}\simeq A_{\beta_0}$. Let $\pi_T$ be the projection: $M=M(J_0)\oplus T\to T$. If $\pi_T(A_{\alpha_0})$ is small then we have $M=M(I-\{\alpha_0\})$, a contradiction. Hence $\pi_T(A_{\alpha_0})$ is not small in $T$. So, by the condition $(C_1)$ for $\mathcal{L}(T)$, there exists a decomposition $T=T^*\oplus T^{**}$ with $\pi_T(A_{\alpha_0})_c\supseteq T^*$ in $T$. Let $\pi_{T^*}$ be the projection: $M=M(J_0)\oplus T^*\oplus T^{**}\to T^*$. Then, $\pi_{T^*}(A_{\alpha_0})=T^*$; whence $T^*$ is indecomposable and $\pi_{T^*}|A_{\alpha_0}$ is an isomorphism by Lemma 3.9 and the fact: $A_{\alpha_0}\simeq A_{\beta_0}$. Consequently, $T^*$ can be exchanged by $A_{\alpha_0}$, i.e., $M=M(J_0)\oplus A_{\alpha_0}\oplus T^{**}$. This contracts the maximality of $J_0$. Thus we must have $I=J_0$.

## 5. Quasi-semiperfect modules over Dedekind domains

The purpose of this section is determine all types of quasi-semiperfect modules over Dedekind domains. Therefore, from now on, we assume that $R$ is a Dedekind domain and $Q$ denotes its quotient field.

For a prime ideal $P$ of $R$, we denote by $E(R/P)$ the injective hull of $R/P$ as an $R$-module. It is well known that the submodules of $E(R/P)$ are totally ordered by inclusion, more precisely, there exists a countable subset $\{x_1, x_2, \cdots\}$

of $E(R/P)$ with the property that $\{x_iR \mid i=1, 2, \cdots\}$ of the set is all submodules of $E(R/P)$, $x_1R \subsetneqq x_2/R \subsetneqq \cdots$, $E(R/P)=\bigcup_{i=1}^{\infty} x_iR$ and $x_{n+k}R/x_nR \simeq R/P^k$ for any $k>n$. We use later the fact that there exists an endomorphism of $E(R/P)$ which is an epimorphism but not an isomorphism. For a non-zero element $r$ in $P$ the endo morphism $\phi$ of $E(R/P)$ given by $\phi(x)=xr$ is such an endomorphism (see [27, Proposition 2.26 Corollary]). It is well known that for distinct prime ideals $P_1$ and $P_2$, $\mathrm{Hom}_R(E(R/P_1), E(R/P_2/))=0$. The following result is due to Kaplansky ([19]): 1) *Every R-module which is not torsion-free contains a direct summand which is either of type* $R/P^n$ *or* $E(R/P)$ *for some prime ideal P.* 2) *Every torsion-free R-module of finite rank is a direct sum of modules of rank one.* 3) *In the case when R is a complete discrete valuation ring, every torsion-free R-module with countable rank is a direct sum of modules of rank one.*

We now attend to the following result which is due to Harada [7] and Rangaswamy [26]: *An R-module H is hollow if and only if it is one of the following*: i) $R/P^n$, ii) $E(R/P)$ *where P a prime ideal and* iii) *R or Q when R is a discrete valuation ring. So, all hollow modules are completely indecomposable.*

By this result and Theorem 3.5 we see that a quasi-semiperfect $R$-module $M$ is expressed as $M=\sum_I \oplus M_\alpha$ where each $M_\alpha$ is isomorphic to one of i)$\sim$iii) above. Thus our work is to observe all types of modules $M$ expressed as $M=\sum_I \oplus M_\alpha$ with each $M_\alpha$ one of i)$\sim$iii) above, and is to check which types of these are quasi-semiperfect.

**Lemma 5.1.** *Let P be a prime ideal of R and k an integer* $>1$. *Then the type* $M=\sum_{i=1}^{\infty} \oplus M_i$ *with each* $M_i \simeq R/P^k$ *is not quasi-semiperfect.*

Proof. There exists an epimorphism $f$ from $R/P^k$ to $P^{k-1}/P^k$. Then note $f^2=0$. For each $i$, $f_i$ denotes the corresponding map: $M_i \to M_{i+1}$ to $f$. We put $M_i'=\{x+f_i(x) \mid x \in M_i\}$, $i=1, 2, \cdots$. Then $M=(\sum_{i=1}^{\infty} \oplus M_i')+(\sum_{i=1}^{\infty} \oplus f_i(M_i))$. Since $f_if_{i+1}=0$ for each $i$, we see that $M \neq \sum_{i=1}^{\infty} \oplus M_i'$, so $\sum_{i=1}^{\infty} \oplus f_i(M_i)$ is not small in $M$. As a result, $M$ is not quasi-semiperfect by Proposition 3.7.

**Lemma 5.2.** *Free R-modules with infinite rank are not quasi-semiperfect.*

Proof. To show this statement we can assume that $R$ is a discrete valuation ring by Harada-Rangaswamy's theorem and Theorem 3.5. Further we may show that $M=\sum_{i=1}^{\infty} \oplus R_i$ with each $R_i \simeq R$ is not quasi-semiperfect. We can take a monomorphism $f_i: R_i \to R_{i+1}$ which is not an epimorphism, $i=1, 2, \cdots$.

Put $R_i'=\{x+f_i(x)\,|\,R_i\}$ for each $i$. Then $M=(\sum_{i=1}^{\infty}\oplus R_i')+(\sum_{i=1}^{\infty}f_i(R_i))$ but $M\neq\sum_{i=1}^{\infty}\oplus R_i'$; so $M$ is not quasi-semiperfect by a similar reason as in the proof of Lemma 5.1.

**Lemma 5.3.** *If $R$ is not a complete discrete valuation ring then the type $Q\oplus Q$ is not quasi-semiperfect.*

Proof. If $Q\oplus Q$ is quasi-semiperfect then $Q$ is quasi-projective by Corollary 3.13. Hence it follows from [25, Lemma 5.1] that $R$ is a complete discrete valuation ring, a contradiction. Thus $Q\oplus Q$ is not quasi-semiperfect.

**Lemma 5.4.** *If $R$ is a complete discrete valuation ring, then $M=\sum_{i=1}^{\infty}\oplus Q_i$ with each $Q_i\simeq Q$ is not quasi-semiperfect.*

Proof. If $M$ is quasi-semiperfect, then it is quasi-projective by Corollary 3.13. So, we see from [25, Theorem 5.8] that $R$ is not complete. Thus $M$ must be not quasi-semiperfect.

**Lemma 5.5.** *The following types are not quasi-semiperfect*: 1) $\mathrm{E}(R/P)\oplus\mathrm{E}(R/P)$, 2) $R\oplus R/P^k$ $(K\geq 1)$, 3) $R/P^k\oplus R/P^j$ $(k>j\geq 1)$ *and* 4) $R/P^k\oplus\mathrm{E}(R/P)$ $(k\geq 1)$, *where $P$ is a prime ideal.*

Proof. There exists an epimorphism from $\mathrm{E}(R/P)$ to $\mathrm{E}(R/P)$ which is not an isomorphism; so $\mathrm{E}(R/P)\oplus\mathrm{E}(R/P)$ is not quasi-semiperfect by Lemma 3.9. Similarly we can show that types 2) and 3) are not quasi-semiperfect. To check 4) we take a submodule $A$ of $\mathrm{E}(R/P)$ whose composition length is $k+1$, and denote the canonical map: $\mathrm{E}(R/P)\to\mathrm{E}(R/P)/A$ by $\eta$. Then $\eta f=0$ for any homomorphism $f$ from $R/P^k$ to $\mathrm{E}(R/P)$. But there exists a non-zero homomorphism from $R/P^k$ to $\mathrm{E}(R/P)$. As a result, the type 4) is not quasi-semiperfect by Theorem 3.16.

**Lemma 5.6.** *For a prime ideal $P$ of $R$ and positive integer $k$, $M=R/P^k\oplus R/P^k\oplus Q\oplus Q$ is not quasi-semiperfect.*

Proof. If $M$ is a quasi-semiperfect then it is quasi-projective by Corollary 3.13. But this contracts the result ([25, Theorem 5.12]): Every quasi-projective module over a Dedekind domain is either torsion or torsion-free. Thus $M$ is not quasi-semiperfect.

NOTATION. For a prime ideal $P$ of $R$ and positive integer $n$, $k$, we denote the type of the form $R/P^k\oplus\cdots\oplus R/P^k$($n$-copies) by $\mathrm{M}(P, k, n)$.

**Lemma 5.7.** $\mathrm{M}(P, k, n)$ *is semiperfect.*

Proof. By [25, Theorem 5.10] $M(P, k, n)$ is quasi-projective. Further we can see that every homomorphic image of it has a projective cover. As a result $M(P, k, n)$ is semiperfect by Theorem 2.1.

**Lemma 5.8.** *For distinct prime ideals $\{P_\alpha\}_I$, the module $M$ of the form $M=\sum_I \oplus E(R/P_\alpha)$ is semiperfect.*

Proof. This is easily seen by noting the fact that if $A$ is a submodule of $M$ then $A=\sum_I \oplus(E(R/P_\alpha)\cap A)$.

**Lemma 5.9.** *Let $M=\sum_I \oplus M_\alpha \oplus \sum_J M_\beta$ be the type such that $M_\alpha$ is of the form $M(P_\alpha, k_\alpha, n_\alpha)$ for each $\alpha\in I$, $M_\beta$ of the form $E(R/P_\beta)$ for each $\beta\in J$ and $\{P_\alpha\}_I \cup \{P_\beta\}_J$ is a set of distinct prime ideals. Then $M$ is quasi-semiperfect.*

Proof. This is also shown by noting the fact that if $A$ is a submodule of $M$ then $A=\sum_I \oplus M_\alpha\cap A)\oplus \sum_J \oplus(E(R/P_\beta)\cap A)$.

**Lemma 5.10.** *Assume that $R$ is a complete discrete valuation ring. Then every torsion-free $R$-module of finite rank is semiperfect.*

Proof. Note that $R$ and $Q$ are complete indecomposable since $R$ is a discrete valuation ring. Let $M$ be a torsion-free $R$-module of finite rank. Inasmuch as $R$ is a complete discrete valuation ring, [25, Theorem 5.8] says that $M$ is quasi-projective; whence it follows from Kaplansky's result that $M$ is expresses as $M=R_1\oplus\cdots\oplus R_n\oplus Q_1\oplus\cdots\oplus Q_m$ with each $R_i\simeq R$ and each $Q_j\simeq Q$. By Theorem 3.18 we may show that $M$ satisfies the condition $(C_6)$ for $\mathcal{L}(M)$. Let $A$ be a submodule of $M$. By again Kaplansky's result $A$ is written as $A=R'_1\oplus\cdots\oplus R'_l\oplus Q'_1\oplus\cdots\oplus Q'_k$ with each $R'_i\simeq R$ and $Q'_i\simeq Q$. Here we can assume $Q'_i=Q_i$, $i=1, 2, \cdots, l$ by the Krull-Remak-Schmidt-Azumaya's theorem ([1]). By $\pi_i$ we denote the projection: $M=R_1\oplus\cdots\oplus R_n\oplus Q_1\oplus\cdots\oplus Q_m\to R_i$, $i=1, 2, \cdots, n$. We may assume that $R'_1, \cdots, R'_{j-1}$ are small in $M$ and $R'_j, \cdots, R'_l$ are not small in $M$. Clearly $\pi_i(R'_j)=R_i$ for some $i$. We can say $i=1$. In this case, we get $M=R'_j\oplus R_2\oplus\cdots\oplus R_n\oplus Q_1\oplus\cdots\oplus Q_m$. Since $R'_j\oplus R'_{j+1}<\oplus M$ we also see that $\pi_i(R'_{j+1})=R_i$ for some $i\geq 2$. We can say $i=2$. Then $M=R'_j\oplus R'_{j+1}\oplus R_3\oplus\cdots\oplus R_n\oplus Q_1\oplus\cdots\oplus Q_m$. Continuing this argument, we obtain $M=R'_j\oplus\cdots\oplus R'_l\oplus R_{l-j+1}\oplus\cdots\oplus R_n\oplus Q_1\oplus\cdots\oplus Q_m$. Put $N=R_{l-j+1}\oplus\cdots\oplus R_n\oplus Q_{k+1}\oplus\cdots\oplus Q_m$. Then $M=A+N$ and $A\cap N\subseteq R'_1\oplus\cdots\oplus R'_{j-1}$. Hence $M$ surely satisfies the condition $(C_6)$ for $\mathcal{L}(M)$.

**Lemma 5.11.** *Assume that $R$ is a discrete valuation ring. Then the type $M=R_1\oplus\cdots\oplus R_n\oplus Q$ with each $R_i\simeq R$ is semiperfect.*

Proof. $\pi_i$ denotes the projection: $M=R_1\oplus\cdots\oplus R_n\oplus Q\to R_i$, $i=1, \cdots, n$

and $\pi$ the projection $M=R_1\oplus\cdots\oplus R_n\oplus Q\to Q$. Now let $A$ be a submodule of $M$. If there exists $i$ such that $\pi_i(A)=R_i$ then, as is easily seen, there exists a direct summand $R'$ of $M$ such that $A\supseteq R'\simeq R$. If $\pi(A)=Q$ we see $Q\subseteq A$ by Kaplansky's result. Noting these facts, we see that $A$ is written as $A=R_1'\oplus\cdots\oplus R_l'\oplus A'$ or $A=R_1'\oplus\cdots\oplus R_l'\oplus Q\oplus A'$, where $R_1'\oplus\cdots\oplus R_l'<\oplus M$, each $R_i'\simeq R$ and $A'$ is small in $M$. Consequently by the Krull-Remak-Schmidt-Azumaya's theorem ([1]) we can assume $R_i'=R_i$ for $i=1, 2, \cdots, l$; so if $A\not\supseteq Q$ then $A'$ is replaced by a submodule of $R_{l+1}\oplus\cdots\oplus R_n\oplus Q$, and if $A\supseteq Q$ it is replaced by one of $R_{l+1}\oplus\cdots\oplus R_n$. Thus it follows that $M$ satisfies the condition $(C_1)$ for $\mathcal{L}(M)$.

Next, let $A_1$ and $A_2$ be direct summands of $M$ with $M=A_1+A_2$ and $A_1\cap A_2$ is small in $M$. Then one of $A_i$ contains $Q$, say $A_1\supseteq Q$ and $A_1$ and $A_2$ are expressed as $A_1=R_1''\oplus\cdots\oplus R_l''\oplus Q$ $(l\leq n)$ and $A_2=R_1'\oplus\cdots\oplus R_k'$, where each $R_i''\simeq R$ and each $R_i'\simeq R$ (cf. Kaplansky's result). We can assume that $R_i''=R_i$, $i=1, 2, \cdots, l$. If $\pi_{l+1}(A_2)\neq R_{l+1}$ then $A_1+A_2$ is not equal to $M$. Therefore $\pi_{l+1}(A_2)=R_{l+1}$; so $\pi_{l+1}(R_i')=R_{l+1}$ for some $i$. We can assume $i=1$. Then $R_{l+1}$ can be exchanged by $R_1'$; $M=R_1\oplus\cdots\oplus R_l\oplus R_1'\oplus R_{l+2}\oplus\cdots\oplus R_n\oplus Q$. If $k=1$ then $l+1=n$ and $M=A_1\oplus A_2$. If $k\neq 1$ then we see from $R_1'\oplus R_2'<\oplus M$ and $0\subseteq_c(A_1\cap A_2)$ in $M$ that $l+1\neq n$. So in this case $R_{l+2}$ exists and, by similar argument to the above, $R_{l+2}$ is exchanged by some $R_i'\in\{R_2', \cdots, R_l'\}$. We can say $i=2$. Then $M=A\oplus R_1'\oplus R_2'\oplus R_{l+3}\oplus\cdots\oplus R_n$. Continuing this procedure, we obtain $M=A_1\oplus R_1'\oplus\cdots\oplus R_k'=A_1\oplus A_2$. As a result, $M$ satisfies the condition $(C_3)$ for $\mathcal{L}(M)$. Thus $M$ is quasi-semiperfect. Since $R$ and $Q$ satisfy the condition (E–I), $M$ is indeed semiperfect by Theorem 3.17.

**Lemma 5.12.** *Assume that $R$ is a discrete valuation ring. Then the form $M=\mathrm{M}(P, k, n)\oplus Q$ is quasi-semiperfect.*

Proof. Note that $Q$ is hollow since $R$ is a discrete valuation ring. By this note and Kaplansky's result we can see that if $A$ is a submodule of $M$ then it is expressed as either $A=M_1'\oplus\cdots\oplus M_l'\oplus X$ or $A=Q\oplus M_1'\oplus\cdots\oplus M_l'\oplus X$, where $X$ is small in $M$ and each $M_i'$ is of the form $R/Pk_i$. We put $M=M_1\oplus\cdots\oplus M_n\oplus Q$, where each $M_i\simeq R/P^k$. If $k_i=k$ then we can assume $M_i=M_i'$ (that is $M_i$ can be exchanged by $M_i'$) and if $A<\oplus M$ then each $l_i=k$ and $X=0$. By these observations it is easily seen that $M$ satisfies the condition $(C_1)$ and $(C_3)$ for $\mathcal{L}(M)$. So $M$ is quasi-semiperfect.

**Lemma 5.13.** *Assume that $R$ is a complete discrete valuation ring, and $P$ its prime ideal. Then the modules $M$ expressed as $M=R/P^k\oplus Q_1\oplus\cdots\oplus Q_n$ with each $Q_i\simeq Q$ are quasi-semiperfect.*

Proof. Again by the Kaplansky's result if $A$ is a submodule of $M$ then, it is contained in $R/P^k\oplus B$ where $B\subseteq A$ and $B$ is expressed as $B=M_1\oplus\cdots$

$\oplus M_l \oplus Q_1^1 \oplus \cdots \oplus Q_m'$ with each $M_i \simeq R$ and each $Q_i' \simeq Q$. Then we see that $M_1 \oplus \cdots \oplus M_l$ is small in $M$ and we can assume $Q_i' = Q_i$, $i=1, \cdots, m$. By these facts and Lemma 5.10 it is not difficult to verify that $M$ satisfies the conditions $(C_1)$ and $(C_3)$ for $\mathcal{L}(M)$. Hence $M$ is quasi-semiperfect.

By Theorem 3.5, Harada-Rangaswamy's theorem and Lemmas 5.1~5.13 we now obtain the following theorems.

**Theorem 5.14.** *Let $R$ be a Dedekind domain which is not a discrete valuation ring. Then an $R$-module is quasi-semiperfect if and only if it is isomorphic to a direct summand of the form: $M = \sum_I \oplus \mathrm{M}(P_\alpha, k_\alpha, n_\alpha) \oplus \sum_J \oplus \mathrm{E}(R/P_\alpha)$ where $\{P_\alpha\}_I \cup \{P_\alpha\}_J$ is a set of distinct prime ideals.*

**Theorem 5.15.** *Let $R$ be a discrete valuation ring but not complete. Then an $R$-module is quasi-semiperfect if and only if it is one of the following types:*

1) $\mathrm{M}(P, k, n)$, 2) $\mathrm{E}(R/P)$, 3) $\mathrm{M}(P, k, n) \oplus \mathrm{E}(R/P)$, 4) *torsion-free $R$-modules of finite rank,* 5) *$F \oplus Q$ where $F$ a type of* 4), 6) $\mathrm{M}(P, k, n) \oplus Q$, *where $P$ is the prime ideal of $R$.*

**Theorem 5.16.** *Let $R$ be a complete discrete valuation ring with the maximal ideal $P$. Then an $R$-module is quasi-semiperfect if and only if it is one of the following:*

1) $\mathrm{M}(P, k, n)$, 2) $\mathrm{E}(R/P)$, 3) $\mathrm{M}(P, k, n) \oplus \mathrm{E}(R/P)$, 4) *Torsion-free $R$-modules of finite rank,* 5) *$F \oplus Q$, where $F$ a type* 4), 6) $\mathrm{M}(P, k, n) \oplus Q$, 7) *$R/P^k \oplus Q_1 \oplus \cdots \oplus Q_n$ with each $Q_i \simeq Q$.*

Added in Proof. In S. Mohamed and B.J. Müller [Decomposition of dual-continuous modules; Module theory, Lecture Notes in Mathematics. No. 700, Springer-Verlag, 1979], dual continuous modules are introduced. This concept just coincides with that of our semi-perfect modules. In [ibid. p. 227], S. Mohamed asked 'what is the structure of a dual-continuous module $M$ with $\mathrm{J}(M) = M$?'. Our results in sections 3 and 4 give a complete solution for this problem.

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Department of Mathematics
Yamaguchi University
Yamaguchi 753, Japan