Harada, M.
Osaka J. Math.
19 (1982), 189-201

ON MODULES WITH LIFTING PROPERTIES

MANABU HARADA

(Received April 21, 1980)

We have studied the lifiting property on a direct sum of completely indecomposable and cyclic hollow modules over a ring R in [8].

In this note, we shall define the lifting property of decompositions with finite direct summands in §1 and give characterizations of this in terms of endomorphism rings of R-modules in §2. We shall study, in §3, R-modules with lifting properties and show that they are very closed to R-modules with direct decomposition of completely indecomposable and cyclic hollow modules when R is right noetherian.

We shall give the dual results for the extending property and its applications in the forthcoming papers.

1. Definitions

Throughout this paper we assume that a ring R contains an identity and every R-module M is a unitary right R-module. We recall here definitions in [8].

If $\mathrm{End}_R(M)$ is a local ring, we call M a *completely indecomposable module*. We denote the Jacobson radical and an injective envelope of M by $J(M)$ and $E(M)$, respectively. By $\bar{M}$ we denote $M/J(M)$. If N is a submodule of M and $N/J(N)$ is canonically monomorphic into $M/J(M)$, then we mean $\bar{N}$ both $N/J(N)$ and the image of $N/J(N)$ into $M/J(M)$.

If $J(M)$ is a unique maximal and small submodule in M, we call M a *cyclic hollow module* (actually M is cyclic). If, for each simple submodule A of $\bar{M}$, there exists a completely indecomposable and cyclic hollow direct summand M_1 of M such that $\bar{M}_1=A$, then we say M has *the lifting property of simple modules* (*modulo radical*). More generally, if for any direct summand B of $\bar{M}$, there exists a direct summand M' of M such that $\bar{M}'=B$, we say M has *the lifting property of direct summands* (*modulo radical*). Finally if, for any finite decomposition of $\bar{M}$; $\bar{M}=C_1\oplus C_2\oplus\cdots\oplus C_n$, there exists a decomposition of M; $M=M_1\oplus M_2\oplus\cdots\oplus M_n$ such that $\bar{M}_i=C_i$, we say M has *the lifting property of decompositions with finite direct summands* (*modulo radical*). If the above property is satisfied for any direct decompositions, we say M has *the lifting property of decompositions* (*modulo radical*).

We recall the definition in [8].

(E-I) *Every epimorphism of M onto itself is an isomorphism.*

Let $\{M_\alpha\}_I$ be a set of completely indecomposable modules. We assume each M_α satisfies (E-I). We shall define a relation $\leqslant$ in $\{M_\alpha\}_I$. If $M_\alpha \approx M_\beta$, we put $M_\alpha \equiv M_\beta$. If there exists an epimorphism of M_α to M_β, we put $M_\alpha \geqslant M_\beta$. Then $\leqslant$ defines a partial order in $\{M_\alpha\}_I$. We refer the reader for other definitions to [8]. We call briefly lifting property modulo racical lifting property.

2. Lifting property on direct sums

Let $\{M_\alpha\}_I$ be a set of completely indecomposable modules and $M=\sum_I \oplus M_\alpha$. In this section we shall study the lifting property of M when each M_α is a cyclic hollow module.

First we shall quote a well known property on semi-perfect module [11].

Proposition 1. *Let P be a projective module. We assume that $P/J(P)$ is semi-simple and $J(P)$ is small in P. Then the following conditions are equivalent:*

1) *P is semi-perfect.*
2) *P has the lifting property of simple modules.*
3) *P has the lifting property of decompositions.*

Proof. It is clear from [11], Theorem 4.3 or [5], Lemma 6 and [6], Proposition 1.

We shall give an example which shows that the assumption on $J(P)$ is necessary in the above proposition.

Let R be the ring of infinite lower-triagular and column finite matrices over a field K and let $\{e_{ij}\}$ be the set of matrix units. Then $P=\sum_I^{\infty} \oplus e_{ii}R$ is a projective module with $P/J(P)$ semisimple. Since $\overline{e_{ii}R} \not\approx \overline{e_{jj}R}$ for $i \neq j$, P has trivially the lifting property of decompositions. However, $J(P)$ is not small in P by [7] and so P is not semi-perfect.

In the above proposition, the lifting property of simple modules implies that P has a decomposition $P=\sum_I \oplus P_\alpha$ and the P_α are cyclic hollow and completely indecomposable modules. However this fact is not true in general for any module N with $N/J(N)$ semi-simple. We shall study this problem in the next section.

Thus, we assume from now on that $M=\sum_I \oplus M_\alpha$ and the M_α are cyclic hollow and completely indecomposables. We give one remark on any R-module M'.

Proposition 2. *Let M' be an R-module. Let N_1 and N_2 be completely indecomposable direct summands of M'. If $\bar{N}_1=\bar{N}_2$ in $\bar{M}'$, N_1 is isomorphic to N_2.*

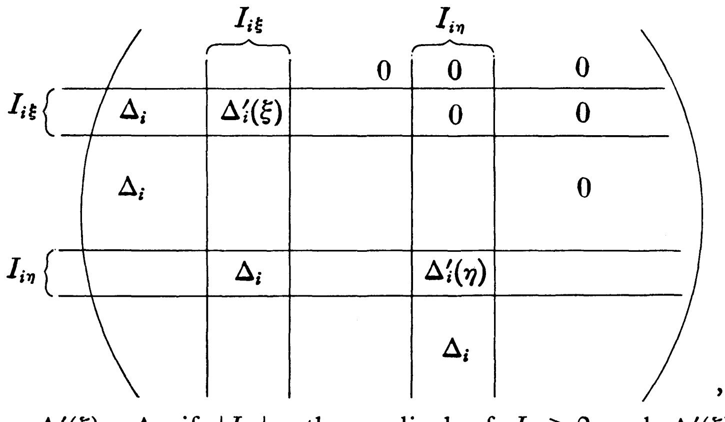
Proof. Let $M'=N_1 \oplus N_1'=N_2 \oplus N_2'$. Since N_1 has the exchange property by [12], Proposition 1, $M'=N_1\oplus N_2\oplus N_2''$ ($N_2''\subseteq N_2'$) or $M'=N_1\oplus N_2'$. Since $\bar{N}_1=\bar{N}_2$, we have the latter decomposition and so $N_1\approx N_2$.

First we consider a special case. Let $\{M_i\}_K$ be the subset of $\{M_\alpha\}_I$ such that $\bar{M}_i\not\approx\bar{M}_\alpha$ for any $\alpha\neq i$. Then $\bar{M}=\sum_K\oplus\bar{M}_i\oplus\sum_{I-K}\oplus\bar{M}_\alpha$ is a (special) decomposition of $\bar{M}$. Since $\bar{M}_i\not\approx\bar{M}_j$ for $i\neq j$ in K, $\sum_K\oplus M_i$ has the lifting property and so M has the lifting property if so does $\sum_{I-K}\oplus M_\beta$. Hence, when we study the lifting property, we may assume, without loss of generality, the following.

(♯) *For each α in I, there exists $\beta(\alpha)\neq\alpha$ in I such that $\bar{M}_{\beta(\alpha)}\approx\bar{M}_\alpha$.*

Let M be as above. Put $S_R=\mathrm{End}_R(M)$, $\bar{S}=\mathrm{End}_R(\bar{M})$ and $J_0(S)=\mathrm{Hom}_R(M, J(M))$. Then every element in S (resp. $\bar{S}$) is expressed as an infinite column finite matrix with entries $f_{\alpha\beta}\in\mathrm{Hom}_R(M_\beta, M_\alpha)$ (resp. $\mathrm{Hom}_R(\bar{M}_\beta, \bar{M}_\alpha)$). It is clear $\bar{S}\supseteq S/J_0(S)$. We shall take a partition of I; $I=\cup I_i$ such that for any α, β in I_i $\bar{M}_\alpha\approx\bar{M}_\beta$ and for any $\alpha'\in I_i$ and $\beta'\in I_j(i\neq j)$, $\bar{M}_{\alpha'}\not\approx\bar{M}_{\beta'}$. We assume here that each M_α satisfies (E-I) and M has the lifting property of simple modules. Then the subset $\{M_\gamma\}_{I_i}$ is a linearly ordered set with respect to the relation $\leqslant$ by [8], Theorem 2 for each i. Further, we obtain a partition of $I_i=\bigcup_{j\in K_i}I_{ij}$ such that for α, β in I_{ij} $M_\alpha\approx M_\alpha$ and for $\gamma\in I_{ij}$, $\delta\in I_{ik}$ $(j<k)$ $M_\gamma>M_\delta$. For the sake of simplicity we assume $K_i=(\cdots,\xi,\cdots,\eta,\cdots)$ and $M_\delta>M_\varepsilon$ for $\delta\in I_{i\xi}$ and $\varepsilon\in I_{i\eta}$.

Now in general $\bar{S}\approx\prod_i\mathrm{End}_R(\sum_{I_i}\oplus\bar{M}_\gamma)$ and $\mathrm{End}_R(\sum_{I_i}\oplus\bar{M}_\gamma)$ is the ring of column finite matrices over the division ring $\Delta_i=\mathrm{End}_R(\bar{M}_\gamma)$. For the partition $I_i=\cup I_{ij}$, by $T(\cdots, I_{i\xi},\cdots, I_{i\eta},\cdots)$ we denote a subring of $\mathrm{End}_R(\sum_{I_i}\oplus\bar{M}_\gamma)$ which consists of all lower tri-angular matrices



[2]

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where $\Delta_i'(\xi)=\Delta_i$ if $|I_{i\xi}|=$ the cardinal of $I_{i\xi}\geqslant 2$ and $\Delta_i'(\xi)\approx\mathrm{End}_R(M_\gamma)/\mathrm{Hom}_R(M_\gamma, J(M_\gamma))\subseteq\Delta_i$ if $|I_{i\xi}|=1$.

We can restate [8], Theorem 2 as follows.

Proposition 3. *Let $\{M_\alpha\}_I$ and M be as above. We assume that each M_α*

satisfies (E-I). *Then* M *has the lifting property of simple modules if and only if* $S/J_0(S)$ *is a subring* $\prod_i T(\cdots, I_{i\xi}, \cdots, I_{i\xi}, \cdots)$ *in* $\bar{S}$ *via the natural monomorphism* $S/J_0(S)\to\bar{S}$.

REMARK. Since there exists a cyclic hollow module Q such that $\mathrm{End}_R(Q)\to \mathrm{End}(\bar{Q}_R, \bar{Q})\to 0$ is not exact, we need the restriction $\Delta_i'(\xi)$.

We shall consider the case $\bar{S}=S/J_0(S)$ after the following theorem.

Theorem 1. *Let* $\{M_\alpha\}_I$ *be a set of completely indecomposable and cyclic hollow modules. We put* $M=\sum_I\oplus M_\alpha$ *and assume* $\{M_\alpha\}_I$ *is a semi-T-nilpotent set* [3]. *Then the following conditions are equivalent*:

1) *M has the lifting property of simple modules.*
2) *M has the lifting property of direct summands.*
3) *Every direct summand of M has the above property.*

Proof. 2)→3). (see [8], the proof of Proposition 2). Let $M=T_1\oplus T_2$ and $\bar{T}_1=A\oplus B$. We put $C=A\oplus\bar{T}_2$. Then from 2) we obtain direct summands K_i such that $M=K_1\oplus K_2$ and $\bar{K}_1=C$. Since $\{M_\alpha\}_I$ is semi-T-nilpotent, K_1 has the exchange property in M by [10], Theorem. Hence, $M=K_1\oplus T_1'\oplus T_2'$; $T_i'\subseteq T_i$. Since $\bar{T}_2'\subseteq\bar{T}_2\subseteq C=\bar{K}_1$, $\bar{T}_2'=0$ and so $T_2'=0$ by [10] and the assumption. Accordingly, $M=K_1\oplus T_1'$ and $T_1=T_1'\oplus(T_1\cap K_1)$. On the other hand, since $\bar{M}=\bar{K}_1\oplus\bar{T}_1'=C\oplus\bar{T}_1'$, $A\cap\bar{T}_1'=0$ *and* $\overline{T_1\cup K_1}\subseteq\bar{T}_1\cap\bar{K}_1=\bar{T}_1\cap(A\oplus\bar{T}_2)=A$. Hence, $A=\overline{T_1\cap K_1}\oplus(A\cap\bar{T}_1')=\overline{T_1\cap K_1}$ and $T_1\cap K_1$ is a direct summand of T_1.
3)→1). It is clear.
1)→2). Since $\bar{M}$ is semi-simple, we may assume $\bar{M}=A\oplus B$ and $A=\sum_X\oplus A_\alpha$ with A_α simple. We assume X is a well ordered set. We show by induction that for each $\alpha>\beta\geqslant\kappa$ there exists a summand N_κ of M such that $\sum_{\kappa\leqslant\beta}\oplus N_\kappa$ is a direct summand of M, $\{N_\kappa\}_{\kappa\leqslant\beta}\subseteq\{N_{\kappa'}\}_{\kappa'\leqslant\beta'}$ if $\beta\leqslant\beta'$ and $\overline{\sum_{\kappa\leqslant\beta}\oplus N_\kappa}=\sum_{\kappa\leqslant\beta}\oplus A_\kappa$. If $\alpha=1$, we have $N_1=M_1$ by 1). Since $\sum_{\beta<\alpha}\oplus N_\beta$ is locally direct summand of M, by [9], Lemma 3 and [10], Theorem, it is a direct summand of M; $M=\sum_{\beta<\alpha}\oplus N_\beta\oplus N$ and $\overline{\sum_{\beta<\alpha}\oplus N_\beta}=\sum_{\beta<\alpha}\oplus A_\beta$. Put $A'=\sum_{\beta<\alpha}\oplus A_\beta\oplus A_\alpha$. Then $A'=\sum_{\beta<\alpha}\oplus A_\beta\oplus(A'\cap\bar{N})$ and $A'\cap\bar{N}\approx A_\alpha$ is simple. Hence, there exists a direct summand N_α of N such that $\bar{N}_\alpha=A'\cap\bar{N}$ by [8], Proposition 2. Therefore, $\sum_{\beta<\alpha}\oplus N_\beta\oplus N_\alpha$ is a direct summand of M and $\overline{\sum_{\beta<\alpha}\oplus N_\beta}\oplus\bar{N}_\alpha=\sum_{\beta<\alpha}\oplus A_\beta\oplus(A'\cap\bar{N})=A'$.

In the following theorem we consider an R-module M which has less assumptions.

Theorem 2. *Let* $\{M_\alpha\}_I$ *be a set of R-modules with* $M_\alpha/J(M_\alpha)$ *simple* (not necessarily either hollow or completely indecomposable). *We put* $M=\sum_I \oplus M_\alpha$. *Then the following conditions are equivalent*:

1) *M has the lifting property of decompositions with finite direct summands.*
2) *M has the lifting property of decompositions with two direct summands.*
3) $\mathrm{Hom}_R(\bar{M}_\alpha, \bar{M}_\beta)$ *is lifted to* $\mathrm{Hom}_R(M_\alpha, M_\beta)$ *for* $\alpha \neq \beta$ *in I.*
4) $\bar{S}=S/J_0(S)$ (*under the assumption* (♯)), *where* $S=\mathrm{End}_R(M)$, $\bar{S}=\mathrm{End}_R(\bar{M})$ *and* $J_0(S)=\mathrm{Hom}_R(M, J(M))$.

Proof. 1)→2). It is clear.
2)→3). Let f be in $\mathrm{Hom}_R(\bar{M}_1, \bar{M}_2)$ and $\bar{M}_1(f)=\{\bar{x}+f(\bar{x})\,|\,x\in M_1\}$. Then $\bar{M}=\bar{M}_1(f)\oplus(\bar{M}_2\oplus\bar{M}_3\oplus\cdots)$. By 2) we obtain a decomposition of M; $M=N_1\oplus N_2$ such that $\bar{N}_1=\bar{M}_1(f)$ and $\bar{N}_2=(\bar{M}_2\oplus\bar{M}_3\oplus\cdots)$. Let $\pi_i\colon M\to N_i$ and $\rho_i\colon M\to M_i$ projections with respect to the decompositions $M=\sum_{i=1}^{2}\oplus N_i$ and $M=\sum_I\oplus M_\alpha$, respectively. Put $g=\rho_2\pi_1\,|\,M_1\in\mathrm{Hom}_R(M_1, M_2)$. Let m be in M_1. Then $m=\sum_{i=1}^{2}\pi_i(m)$ and $\pi_1(m)=\sum_I\rho_\alpha(\pi_1(m))$. Hence, since $\bar{m}=\sum\overline{\pi_i(m)}$ and $\overline{\pi_1(m)}=\bar{x}+f(\bar{x})$ for some $x\in M_1$, $\bar{m}=\bar{x}$. Accordingly, $\bar{g}(\bar{m})=\overline{g(m)}=\overline{\rho_2\pi_1(m)}=\rho_2(\bar{x}+f(\bar{x}))=f(\bar{x})=f(\bar{m})$. Hence, f is lifted to g.
3)→4). We assume (♯). Let $\varphi\colon\bar{M}_\alpha\approx\bar{M}_{\rho(\alpha)}$ and $\alpha\neq\rho(\alpha)$. For any $g\in\mathrm{End}_R(\bar{M}_\alpha)$, $g=\varphi^{-1}(\varphi g)$. Then φ^{-1}, φg are lifted to $H\in\mathrm{Hom}_R(M_{\rho(\alpha)}, M_\alpha)$ and $H\in\mathrm{Hom}_R(M_\alpha, M_{\rho(\alpha)})$ by 3), respectively. Hence, g is lifted to FH. Therefore, $\bar{S}=S/J_0(S)$ by 3).
4)→1). First we shall show 4)→2). Let $\bar{M}=A\oplus B$ and $\bar{M}=A\oplus\sum_P\oplus\bar{M}_\delta$. Now, $M=\sum_{I-P}\oplus M_\gamma\oplus\sum_P\oplus M_\delta$. Let π be the projection of $\bar{M}$ onto $\sum_P\oplus\bar{M}_\delta$ with respect to the decomposition $\bar{M}=A\oplus\sum_P\oplus\bar{M}_\delta$. Since $\bar{S}=S/J_0(S)$, we can choose, from the representation of column finite matrices, a homomorphism $f\colon M\to\sum_P\oplus M_\delta$ such that $\bar{f}=-\pi$. We put $M_1(f)=\{x+f(x)\,|\,x\in\sum_{I-P}\oplus M\}\subseteq M$. Then $M=M_1(f)\oplus\sum_P\oplus M_\delta$ and $\overline{M_1(f)}=A$. Next we take the projection π' of $\bar{M}$ onto A with respect to the decomposition $\bar{M}=A\oplus B$. Since $A=\overline{M_1(f)}$, there exists $g\colon M\to M_1(f)$ such that $\bar{g}=-\pi'$ (note $M_1(f)\approx\sum_{I-P}\oplus M_\gamma$ and $M=M_1(f)\oplus\sum_P\oplus M_\delta$). Put $M_2(g)=\{y+g(y)\,|\,y\in\sum_P\oplus M_\delta\}$. Then $M=M_1(f)\oplus M_2(g)$ and $\overline{M_2(g)}=B$. Next we shall show 4)→1). Since $M_1(f)\approx\sum_{I-P}\oplus M_\gamma$ and $\overline{M_2(g)}\approx\sum_P\oplus M_\delta$, we can prove it by induction on number of summand (note that 4) is satisfied for the direct summand $M_2(g)$).

REMARK. We know, from the proof and the remark before (♯), that 1)∼3) are equivalent without (♯).

Corollary 1. *Let $\{M_\alpha\}_I$ and M be as above. We assume further each M_α is completely indecomposable and $\{M_\alpha\}_I$ is semi-T-nilpotent. Then the above conditions* 1)$\sim$4) *are equivalent to*

5) *M has the lifting property of direct decmpositions.*

Proof. 5)$\to$1). It is clear.
1)$\to$5). We shall use the same method in the proof 1)$\to$2) of Theorem 1. Let $\bar{M}=\sum_L \oplus B_\gamma$. We shall prove it by transfinite induction on L. We assume L is a well ordered set. We assume that for each $\rho\leqslant\beta<\alpha$ there exist direct summands N_ρ such that $M=\sum_{\rho<\beta}\oplus N_\rho\oplus M'$ and $\bar{N}_\rho=B_\rho$. We note that every direct summand of M is a direct sum of completely indecomposable modules by [10], Theorem. Since $T=\sum_{\rho<\alpha}\oplus N_\rho$ is a locally direct summand, T is a direct summand of M by [9], say $M=T\oplus M'$. Now $\bar{M}=\sum_{\rho<\alpha}\oplus B_\rho\oplus\sum_{\alpha\leqslant\gamma}\oplus B_\gamma$, $M=T\oplus M'$ and $\bar{T}=\sum_{\rho<\alpha}\oplus\bar{N}_\gamma=\sum_{\rho<\alpha}\oplus B_\rho$. Then we know from the proof 4)$\to$1) of the theorem that there exists a decomposition $M=T\oplus M'(g)$ and $\overline{M'(g)}=\sum_{\alpha\leqslant\gamma}\oplus B_\gamma$. $M'(g)$ has also the lifting property of decompositions with two direct summands by the theorem. Hence, we obtain a direct summand N_α of M such that $\bar{N}_\alpha=B_\alpha$ and $M=\sum_{\beta\leqslant\alpha}\oplus N_\beta\oplus M''$.

If M_α is a cyclic hollow module with (*E-I*), then $\mathrm{Hom}_R(M_\alpha, J(M_\alpha))=J(\mathrm{End}_R(M_\alpha))$. Hence, under the condition in Theorem 2, $J_0(S)\subseteq J(S)$ if and only if $\{M_\alpha\}_I$ is a semi-T-inlpotent set by [5], Proposition 2. If $J(M)$ is small in M, $\{M_\alpha\}_I$ is semi-T-nilpotent by [7], Corollary 1 to Proposition 1.

Corollary 2. *Let $\{M_\alpha\}_I$ and M be as above. We assume each M_α is a cyclic hollow module with (E-I). We assume one of the following.*

i) $J_0(S)\subseteq J(S)$.

ii) *$\{M_\alpha\}_I$ is a semi-T-nilpotent set.*

iii) *$J(M)$ is small in M* (e.g. R is right perfect).

Then the conditions in Theorem 2 *are equivalent to* 5).

Proof. Let $\bar{M}=\sum_I\oplus A_\alpha$ with A_α simple and let N_α be a direct summand of M such that $\bar{N}_\alpha=A_\alpha$. Then the inclusion map $i\colon\sum_I N_\alpha\to M$ modulo $J(M)$ gives an isomorphism to $\bar{M}$. Hence, i is an isomorphism if $J_0(S)\subseteq J(S)$ (cf. [3]).

Corollary 3. *We further assume in Theorem* 2 *that $M_\alpha\approx M_1$ for all α in I. Then the conditions in Theorem* 2 *are equivalent to*

6) *M has the lifting property of simple modules.*

3. Modules with lifting property

In this section, we shall study R-modules M with lifting property and $M/J(M)$ semi-simple.

Theorem 3. *Let M be an R-module with $M/J(M)$ semi simple. We assume that M has the lifting property of simple module and satisfies one of the following conditions.*

1) *M has the lifting property of decomposition with two direct summands.*

2) *$\bar{S}=S/J_0(S)$, where $S=\mathrm{End}_R(M)$, $\bar{S}=\mathrm{End}_R(\bar{M})$ and $J_0(S)=\mathrm{Hom}_R(M, J(M))$.*

3) *For any cyclic hollow and completely indecomposable direct summand N of M and an element f in $\mathrm{Hom}_R(N, M)$, if $\bar{f}\colon \bar{N}\to\bar{M}$ is a monomorphism, so is f.*

4) *For any two direct summands M_1, M_2 as in 3) and $f\in\mathrm{Hom}_R(M_1, M_2)$, if $\bar{f}$ is an epimorphism, then f is an isomorphism.*

Then M contains a submodule M' satisfying the following.

a) *$M'=\sum_I\oplus M_\alpha$; the M_α are cyclic hollow and completely indecomposables.*

b) *M' has the lifting property of simple modules and decompositions with finite direct summands.*

c) *$\sum_I\oplus M_\alpha$ is a locally direct summand of M.*

d) *$M=M'+J(M)$.*

Conversely, if M contains a submodule M' above, M has the lifting property of simple modules. Furthermore, if each M_α satisfies (E-I), *then M satisfies* 3) *and* 4).

Proof. We assume M has the lifting property of simple modules. Let $\boldsymbol{N}$ be the set of submodules N of M such that $N=\sum_J\oplus N_\gamma$; the N_γ are cyclic hollow and completely indecomposable and N is a locally direct summand of M (with respect to $\sum_J\oplus N_\gamma$). Since M has the lifting property of simple modules, $\boldsymbol{N}$ is not empty. We can define a partial order in $\boldsymbol{N}$ by the members of direct summands of N. Then we can find a maximal element, say $M'=\sum_I\oplus N_\alpha$, in $\boldsymbol{N}$ by Zorn's lemma. Since N is a locally direct summand, $\bar{M}'\subseteq\bar{M}$.

Case 1). We put $\bar{M}=\bar{M}'\oplus K$ and show $K=0$. If $K\neq 0$, we have a simple submodule A such that $\bar{M}=\bar{M}'\oplus K'\oplus A$. Since M has the lifting property of decompositions with two direct summands, $M=L\oplus N$ and $\bar{L}=\bar{M}'\oplus K'$, $\bar{N}=A$. Let J be any finite subset of I. Then $M=\sum_J\oplus N_\gamma\oplus P$. Since $\sum_J\oplus N_\gamma$ has the exchange property by [1], Lemma 3.10 and [12], Proposition 1, $M=\sum_J\oplus N_\gamma\oplus L'\oplus N'$, where $L'=L'\oplus L''$ and $N=N'\oplus N''$. Since $\bar{N}=\bar{N}'\oplus\bar{N}''=A$, $N'=0$ or $\bar{N}''=0$. If $\bar{N}'=0$, $\bar{M}=\sum_J\oplus\bar{N}_\gamma\oplus\bar{L}'\subseteq\bar{L}$. Hence $\bar{N}'\neq 0$, and so $\bar{N}''=0$. On the other hand, N'' is isomorphic to a direct ummand of $\sum_J\oplus N_\gamma$.

Hence $N''=0$ and so $N=N'$. Therefore, $\sum_{J}\oplus N_{\gamma}\oplus N$ is a direct summand of M, which contradicts the maximality of M' in N. Therefore, $M=M'+J(M)$. M' has the lifting property of simple modules by [8], Theorem 2 and Proposition 2. Next we shall show that M' has the lifting property of decompositions with finite direct summands. Let N_1, N_2 be in $\{N_{\alpha}\}_I$, then $N_1\oplus N_2$ is a direct summand of M by c), say $M=N_1\oplus N_2\oplus M^*$. We can apply the argument in the proof of Theorem 2 to the decomposition $M=N_1\oplus N_2\oplus M^*$, instead of $M=\sum_{I}\oplus M_{\alpha}$. Hence, $\mathrm{Hom}_R(\bar{N}_1, \bar{N}_2)$ is lifted to $\mathrm{Hom}_R(N_1, N_2)$. Therefore, M' has the lifting property of decompositions with finite direct summands by Theorem 2 and has the lifting property of simple modules by [8], Theorem 2 and Proposition 2.

Case 2). We also show $K=0$. Put $N_0=\sum_{J}\oplus N_i$ for a finite subset J of I. Then $M=N_0\oplus P$ and $\bar{M}=\bar{N}_0\oplus\bar{P}=\bar{N}_0\oplus\sum_{I-J}\oplus\bar{N}_{\gamma}\oplus K$. Let π' be the projection of $\bar{M}$ onto $\bar{N}_0$ with respect to the latter decomposition and $\pi=\pi'|\bar{P}$. Since $\bar{S}=S/J_0(S)$ and

$$S=\begin{pmatrix}\mathrm{Hom}_R(N_0, N_0) & \mathrm{Hom}_R(P, N_0)\\ \mathrm{Hom}_R(N_0, P) & \mathrm{Hom}_R(P, P)\end{pmatrix},$$

there exists $f\in\mathrm{Hom}_R(P, N_0)$ with $\bar{f}=-\pi$. Then $M=P(f)\oplus N_0$ and $\overline{P(f)}=\sum\oplus\bar{N}_{\delta}\oplus K$ (cf. the proof of Theorem 2). We assume $K\neq 0$ and K_{α} is a simple component of K. Since M has the lifting property of simple modules, there exists a direct summand M_{α} of M such that $\bar{M}_{\alpha}=K_{\alpha}$. M_{α} has the exchange property and so $M=M_{\alpha}\oplus P(f)'\oplus N_0$ and $P(f)'\subseteq P(f)$, since $\bar{M}_{\alpha}\subseteq\overline{P(f)}$. Hence, $M_{\alpha}\oplus N_0$ is a direct summand of M for any finite subset J of I. Hence, $M_{\alpha}\oplus N_0$ is a locally direct summand of M, which is a contradiction. Therefore, $K=0$ and $\bar{M}=\bar{M}'$. It is clear that every direct summand of M satisfies the condition 2). Hence, M' has the lifting property of simple modules and decompositions with finite direct summands by Theorem 2 (note the remark before (#)).

Cases 3) or 4). Let M_{α} be a cyclic hollow and completely indecomposable direct summand of M with $\bar{M}_{\alpha}=A_{\alpha}$. Put $M'=\sum_{I}M_{\alpha}$. Then $\bar{M}'=\bar{M}$. We shall show $\sum_{I}M_{\alpha}$ is a direct sum (cf. [8], Theorem 1). Let $\sum_{i=1}^{n}M_i=M(n)$ be any finite sum in $\sum_{I}M_{\alpha}$. We show $M(n)=\sum_{i=1}^{n}\oplus M_i$ and $M(n)$ is a direct summand of M by induction on n. If $n=1$, it is clear. We assume $M=M(n-1)\oplus T$. Let π be the projection of M onto T and $f=\pi|M_n$. Since $f(M_n)\not\subset J(T)$, $\overline{f(M_n)}$ is a simple component of $\bar{T}$ and T has the lifting property by [8], Proposition 2 and so there exists a cyclic hollow and completely indecomposable direct summand T_1 of T such that $\bar{T}_1=\overline{f(M_n)}$. Let $T=T_1\oplus T_2$ and π_1 the projection of T onto T_1. Put $g=\pi_1 f$. Then $\overline{g(M_n)}=\bar{T}_1$. Hence, $\bar{g}$ is a monomorphism and so g is a monomorphism by 3) or 4). Accordingly, $M=M_n\oplus\ker\pi_1\pi=M_n\oplus T_2\oplus M(n-1)=M(n)\oplus T_2$ and $M(n)=\sum_{i=1}^{n}\oplus M_i$. Cases 3) and 4) imply that $\{M_\alpha\}_I$ satisfies (E-I). Hence, M' has the lifting property of decompositions with finite direct summands by Theorems 1 and 2. Conversely, we assume M has the submodule M'. Since $M'=\sum_I\oplus M_\alpha$ is a locally direct summand of M, $\bar{M}=\sum_I\oplus\bar{M}_\alpha=\bar{M}'$. Let A be a simple submodule of $\bar{M}$. Then there exists a finite subset J of I such that $A\subseteq\sum_J\oplus\bar{M}_\gamma$. Since M' has the lifting property of simple module, $\sum_J\oplus M_\gamma$ contains a direct summand N with $\bar{N}=A$ by [8], Proposition 2 and so N is also a direct summand of M. Finally we assume that each M_α satisfies (E-I). Let N_α be any direct summand of M as in 3). Since $\bar{M}=\bar{M}'$, there exists a direct summand M_α of M' such that $\bar{M}_\alpha=\bar{N}_\alpha$. Hence, $M_\alpha\approx N_\alpha$ by Proposition 2. Accordingly, 4) is satisfied by Theorem 2. Let f be as in 3). Since $\bar{M}=\bar{M}'$ and M' has the lifting property of simple modules, there exists a direct summand T_α with $\bar{T}_\alpha=\overline{f(N)}$. T_α is cyclic and so T_α is contained in a direct summand $\sum_J\oplus M_\alpha$ of M by c). Hence, T_α is a direct summand of M. Let π be the projection of M onto T_α. Then $\pi f\colon N_\alpha\to T_\alpha$ is an isomorphism by the above and Theorem 2. Hence, f is a monomorphism.

Corollary. *Let R be a right artinian ring and M an R-module. Then M has the lifting property of decompositions if and only if M has the lifting property of decompositions with two direct summands.*

Proof. If R is right artinian, every R-module N with $N/J(N)$ simple is a cyclic hollow and completely indecomposable module. Hence, if M has the lifting property of decompositions with two direct summands, then M has the lifting property of simple modules and so $M=\sum_I\oplus M_\alpha$ by the theorem, where the M_α are cyclic hollow modules. Hence, M has the lifting property of decompositions by Corollary 2 to Theorem 2.

REMARKS 1. We note that if $J(M)$ is small in M, $M=M'$ in Theorem 3. If further each M_α satisfies (E-I), then all conditions 1)$\sim$4) in Theorem 3 are equivalent when M has the lifting property of simple modules.

2. We assume $M=M'\oplus K$ with $K=J(K)$. Then M satisfies 1) and 2) if M' satisfies a) and b). We do not know whether this fact is true or not without assumptions.

3. Let Z be the ring of integers and p a prime. Put $M=\sum_i\oplus Z/p^i\oplus E(Z/p)$ and $N=(Z/p^i)^{(I)}\oplus E(Z/p)$, where $(Z/p^i)^{(I)}$ is the direct sum of $|I|$-copies of Z/p^i. Then M has the lifting property of simple modules but not of decompositions and N has the lifting property of decompositions.

Next we shall study R-modules satisfying the lifting property of simple modules.

Proposition 4. *Let M be an R-module. We assume that M is countably generated, $M/J(M)$ is semi-simple and $J(M)$ is small in M. Then if M has the lifting property of simple modules, $M=\sum_{i=1}^{\infty}\oplus M_i$ with M_i indecomposable.*

Proof. Let $\{m_1, m_2, \cdots, m_n, \cdots\}$ be a set of generators. Since $J(M)$ is small, we may assume $m_i \notin J(M)$ for all i. Further we may assume $\overline{m_i R}$ is simple. We assume there exists a set of indecomposable direct summands M_i such that $M(n)=\sum_{i=1}^{t_n}\oplus M_i$ is a direct summand of M and $\sum_{i=1}^{n}\overline{m_i R}\subseteq\overline{M(n)}$. Let $M=M(n)\oplus T$ and $m_{n+1}=x+t$; $x\in M(n)$, $t\in T$. If $t\notin J(T)$, there exists an indecomposable direct summand $M_{t_{n+1}}$ of T such that $\bar{M}_{t_{n+1}}=\overline{tR}$ by [8], Proposition 2. Hence, there exists a direct summand $M(n+1)$ of M such that $\overline{M(n+1)}\supseteq\sum_{i=1}^{n+1}\overline{m_i R}$. Accordingly, $\bigcup_n M(n)=\sum_1^{\infty}\oplus M_{t_i}=M$, since $J(M)$ is small in M.

Lemma 1. *Let M be an R-module and let $\sum_{I_1}\oplus N_{\alpha_1}$, $\sum_{I_2}\oplus N_{\alpha_2}, \cdots$ be submodules in M. We assume that the N_{α_i} are completely indecomposable and $\sum_{I_i}\oplus N_{\alpha_i}$ is a locally direct summand of M for all i. If $N_{\alpha_i}\not\approx N_{\alpha_j}$ for any $\alpha_i\in I_i$ and any $\alpha_j\in I_j$, $\sum_i\sum_{I_i}\oplus N_{\alpha_i}$ is a locally direct summand of M.*

Proof. Let J_i be any finite subset of I_i. Then $M=\sum_{J_i}\oplus N_{\alpha'}\oplus M(i)$. Since $\sum_{J_1}\oplus N_{\alpha_1'}$ has the exchange property, $M=\sum_{J_1}\oplus N_{\alpha_1'}\oplus\sum_{J_2}\oplus N_{\alpha_2'}\oplus M^*(2)$ where $M^*(2)\subseteq M(2)$, since $N_{\alpha_1'}\not\approx N_{\alpha_2'}$. Repeating this argument, we know $\sum_i\sum_{I_i}\oplus N_{\alpha_i}$ is a locally direct summand of M.

Corollary. *Let M be an R-module such that $M/J(M)=\sum_I\oplus A_\alpha$ with A_α simple and $A_\alpha\not\approx A_\beta$ for $\alpha\neq\beta\in I$. Then M has the lifting property of simple modules if and only if M contains a submodule M' wihch has the lifting property of simples modules and satisfies a), c) and d).*

Proof. If M has the lifting property of simple modules, we obtain a cyclic hollow and completely indecomposable module M_α with $\bar{M}_\alpha=A_\alpha$. Then $M_\alpha\not\approx M_\beta$ if $\alpha\neq\beta$. Hence, $\sum_I M_\alpha=M'$ satisfies a), c) and d) by Lemma 1. The remaining part is clear.

Theorem 4. *Let M be an R-module with $M/J(M)$ semi-simple. We assume every cyclic direct summand of M is noetherian* (e.g. R is right noetherian). *Then M has the lifting property of simple modules if and only if M contains a submodule M' such that a) $M'=\sum_I\oplus M_\alpha$; the M_α are cyclic hollow and completely indecomposa-*

ble, b) M' has the lifting property of simple modules, c) $\sum_I \oplus M_\alpha$ is a locally direct summand of M and d) $M = M' + J(M)$.

Proof. "If" part is clear from Theorem 3. We assume M has the lifting property of simple modules. Let M_1, M_2 be completely indecomposable and cyclic hollow direct summands of M, say $M = M_1 \oplus N_1 = M_2 \oplus N_2$. Then $M_1 \approx M_2$ or $M = M_1 \oplus M_2 \oplus N_2'$ by Lemma 1. In the latter case, since $M_1 \oplus M_2$ has the lifting property of simple modules by [8], Proposition 2, $M_1 \leqslant M_2$ or $M_1 \geqslant M_2$ if $\bar{M}_1 \approx \bar{M}_2$ by [8], Theorem 2. Therefore, we may assume in any cases $M_1 \geqslant M_2$ or $M_1 > M_2$ whenever $\bar{M}_1 \approx \bar{M}_2$. We note that every cyclic direct summand satisfies (E-I) by the assumption. Now let $\{M_\gamma\}_L$ be a representative set of cyclic hollow and completely indecomposable direct summands of M with $\bar{M}_\gamma \approx \bar{M}_{\gamma_0}$ for all $\gamma \in L$ (γ_0 is fixed). Then $\{M_\gamma\}_L$ is a linearly ordered set with respect to $\geqslant$. Since M_γ is noetherian, we have the minimal member M_1 and a finite chain $M_\gamma > M_{\gamma-1} > \cdots > M_1$. Hence, L is well ordered and so we may assume $\{M_\gamma\}_L = \{M_n\}$ with $M_n > M_{n-1}$. We define the set $\boldsymbol{N}_1$ of submodules of M: $\boldsymbol{N}_1 = \{T = \sum_{\alpha \in I_1'} \oplus M_{1\alpha} \mid T$ is a locally direct summano of M and $M_{1\alpha} \approx M_1$ for all $\alpha \in I_1'\}$. Let $T_1 = \sum_{I_1} \oplus M_{1\alpha}$ be a maximal member in $\boldsymbol{N}_1$. Then $\bar{M} = \bar{T} \oplus K_1$ and $\bar{T} = \sum_{I_1} \oplus \bar{M}_{1\alpha}$. Let A_1 be a simple submodule of $\bar{M}$ not contained in $\bar{T}_1$. Since M has the lifting property of simple modules, there exists a completely indecomposable and cyclic hollow direct summand N_1 of M such that $\bar{N}_1 = A_1$. We shall show $N_1 \not\approx M_1$. Let J_1 be any finite subset of I_1. Then $M = \sum_{J_1} \oplus M_{1\alpha} \oplus M'$. Let $\pi_{M'}\colon M \to M'$ be the projection. Since $\bar{N}_1 \not\subset \overline{\sum_{J_1} \oplus M_1}$, $\bar{N} \approx \overline{\pi_{M'}(N_1)}$ and so M' contains a cyclic hollow direct summand N_1' with $\bar{N}_1' = \overline{\pi_{M'}(N_1)}$ say $M' = N_1' \oplus M''$ by [8], Proposition 2. Then $M = \sum_{J_1} \oplus M_{1\alpha} \oplus M'' \oplus N_1'$. Let $\pi\colon M \to N_1'$ and $\pi'\colon M' \to N_1'$ be the projections, respectively. Then $\overline{\pi(N_1)} = \overline{\pi'(\pi_{M'}(N_1))} = \pi'(\bar{N}_1') = \bar{N}_1'$. Hence, $\pi \mid N_1 \to N_1'$ is an epimorphism. If $N_1 \approx M_1$, $N_1' \approx M_1$ since M_1 is minimal and so $\pi \mid N_1$ is an isomorphism by (E-I). Therefore, $M = N_1 \oplus \ker \pi = N_1 \oplus \sum_{J_1} \oplus M_{1\alpha} \oplus M''$. Since J_1 is any finite subset of I_1, $N_1 \oplus T_1$ is a locally direct summand of M and $N_1 \approx M_1$, which contradicts the maximality of T_1 in $\boldsymbol{N}_1$. Hence, $N_1 \not\approx M_1$. Now $\bar{M} = \bar{T}_1 \oplus K_1$ and $M_2 \oplus T_1$ is a locally direct summand of M by Lemma 1. Let $\boldsymbol{N}_2 = \{T = \sum_{I_1} \oplus M_{1\alpha} \oplus \sum_{I_2'} \oplus M_{2\beta}$ is a locally direct summand of M and $M_{2\beta} \approx M_2\}$. Let T_2 be a maximal member in $\boldsymbol{N}_2$. Then $\bar{M} = \bar{T} \oplus \sum_{I_2} \oplus \bar{M}_{2\beta} \oplus \bar{K}_2$. Let A_2 be a simple submodule not contained in $\bar{T}_2$ and N_2 a cyclic hollow direct summand of M with $\bar{N}_2 = A_2$. Since $A_2 \not\subseteq \bar{T}_1$, $N_2 \not\approx M_1$ by the above. Let J_1 and J_2 be any finite subsets of I_1 and I_2, respectively, and $M=\sum_{J_1}\oplus M_{1\alpha}\oplus\sum_{J_2}\oplus M_{2\beta}\oplus M^*$. Let π_{M^*}: $M\to M^*$ be the projection. Since $\bar{N}_2=A_2\not\subseteq\bar{T}_2$ and $\overline{\sum_{J_1}\oplus M_{1\alpha}\oplus\sum_{J_2}\oplus M_{2\beta}}\subseteq\bar{T}_2$, $\overline{\pi_{M^*}(N_2)}\not\subseteq\bar{T}_2$. Let N_2' be a cyclic hollow direct summand of M^* with $\bar{N}_2'=\pi_{M^*}(N_2)$. Since $\overline{\pi_{M^*}(N_2)}\not\subset\bar{T}_1$, $N_2'\not\approx M_1$ from the above. We obtain the epimorphism $\pi|N_2\to N_2'$ similarly to the above. If, since $N_2'\not\approx M_1$, $N_2\approx M_2$, we have a contradiction. Hence, $N_2\not\approx M_i$ $(i=1,2)$. Since L is well ordered, using inductively the above, we obtain a locally direct summand $T(\gamma_0)=\sum_{I_1}\oplus M_{1\alpha_1}\oplus\sum_{I_2}\oplus M_{2\alpha_2}\oplus\cdots\sum_{I_n}\oplus M_{n\alpha_n}\oplus\cdots$, which has the lifting property of simple modules and each $M_{i\alpha_i}$ is a cyclic hollow and completely indecomposable direct summand of M. Let $\{S_\gamma\}_P$ be the representative set of simple modules in $\bar{M}$. For each S_γ we can obtain $T(\gamma)$ as above. Then $\sum_P T(\gamma)$ is a locally direct summand of M by Lemma 1 and $\bar{M}=\sum_P\oplus\overline{T(\gamma)}$ from the above argument. Thus $M'=\sum\oplus T(\gamma)$ is the desired submodule of M by [8], Theorem 2.

Corollary. *Let R be a right artinian ring and M an R-module. Then M has the lifting property of simple modules if and only if $M=\sum_I\oplus M_\alpha$ with M_α indecomposable hollow and $\{M_\alpha\}$ satisfies the conditions in* [8], *Theorem* 2.

Proof. If R is right artinian, $J(M)$ is small in M and $M/J(M)$ is semi-simple.

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Department of Mathematics
Osaka City University
Sumiyoshi-Ku, Osaka 558
Japan