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# ON PERFECT RINGS AND THE EXCHANGE PROPERTY

Dedicated to Professor Kiiti Morita on his 60th birthday

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Let $R$ be a ring with unit element. We always consider unitary right $R$-modules. Let $T$ be an $R$-module and $\eta$ a cardinal number. If for any module $K$ containing $T$ as a direct summand and for any decomposition of $K$ with $\eta$ components: $K=\sum_{\alpha\in I}\oplus A_\alpha$, there exist submodules $A_\alpha'$ of $A_\alpha$ for all $\alpha$ such that $K=T\oplus\sum_{\alpha\in I}\oplus A_\alpha'$, then we say $T$ has the *$\eta$-exchange property* [2]. If $T$ has the $\eta$-exchange property for any $\eta$, we say $T$ has the *exchange property.*

In this short note we shall show that $R$ is a right perfect ring if and only if for every projective module P, P has the exchange property and $\mathrm{End}_R(P)/J(\mathrm{End}_R(P))$ is a regular ring in the sense of Von Neumann. This is a refinement of Theorem 7 in [4] and we shall give its proof as an application of [6].

After submitting this paper to the journal, the authors have received a manuscript of Yamagata [13] and found that one of main theorems in this paper overlaped with one in [13]. The authors would like to express their thanks to Dr. Yamagata for his kindness.

## 1. Preliminaries

First we shall recall some definitions given in [3], [4] and [6]. Let $T$ be an $R$-module. If $\mathrm{End}_R(T)$ is a local ring, $T$ is called *completely indecomposable.* We take a set $\{M_\alpha\}_I$ of completely indecomposable modules and define the full additive subcategory $\mathfrak{A}$ of all right $R$-modules which is induced from $\{M_\alpha\}_I$, namely the objects in $\mathfrak{A}$ consist of all modules which are isomorphic to direct-sums of completely indecomposable modules in $\{M_\alpha\}_I$. We define an ideal $\mathfrak{J}'$ in $\mathfrak{A}$ as follows: let $A=\sum_{\alpha\in K}\oplus A_\alpha$, $B=\sum_{\beta\in L}\oplus B_\beta$ be in $\mathfrak{A}$, where $A_\alpha$, $B_\beta$ are isomorphic to some in $\{M_\alpha\}_I$, then $\mathfrak{J}'\cap[A, B]=\{f\mid\in\mathrm{Hom}_R(A, B), p_\beta f i_\alpha$ are not isomorphic for all $\alpha\in K, \beta\in L\}$, where $i_\alpha: A_\alpha\to A, p_\beta: B\to B_\beta$ are the inclusion and the projection, respectively. By $\bar{\mathfrak{A}}$ we denote the factor category of $\mathfrak{A}$ with respect to $\mathfrak{J}'$ [3]. For any object $A$ and morphism $f$ in $\mathfrak{A}$, by $\bar{A}$ and $\bar{f}$ we denote the residue classes of $A$ and $f$ in $\bar{\mathfrak{A}}$, respectively.

We take a countable subset $\{M_{\alpha_i}\}_1^\infty$ of $\{M_\alpha\}_I$ (resp. we can take same modules $M_{\alpha_i}$ in $\{M_{\alpha_i}\}_1^\infty$ as many as we want) and a set of homomorphisms $f_i \in \mathfrak{J}' \Pi [M_{\alpha_i}, M_{\alpha_{i+1}}]$. If for any element $m$ in $M_{\alpha_1}$ there exists $n$, which depends on $m$, such that $f_n f_{n-1} \cdots f_1(m)=0$, $\{f_i\}$ is called *locally semi-T-nilpotent* (resp. *T-nilpotent*). If every $\{f_i\}$ is locally semi-$T$-nilpotent (resp. $T$-nilpotent) for every subset $\{M_{\alpha_i}\}$, we say $\{M_\alpha\}_I$ is a *locally semi-T-nilpotent* (resp. *T-nilpotent*) *system* [4]. Finally, let $M \supset N$ be modules and $N = \sum_{\gamma \in J} \oplus N_\gamma$. If for any finite subset $J'$ of $J$ $\sum_{\gamma \in J'} \oplus N_\gamma$ is a direct summand of M, $N$ is called a *locally direct summand* of M (with respect to the decomposition $\sum_{\gamma \in J} \oplus N_\gamma$) [6].

## 2. Perfect rings

Let $\{M_\alpha\}$ be a set of completely indecomposable modules and $M = \sum_{\alpha \in I} \oplus M_\alpha$. We understand $p_\alpha$ means the projection of $M$ to $M_\alpha$ in the decomposition if there are no confusions. Let $N$ be a submodole of M, which is isomorphic to one in $\{M_\alpha\}_I$. We shall consider a strong condition:

*each N above is a direct summand of M* $\cdots$(*)

**Lemma 1.** *Let M and* $\{M_\alpha\}_I$ *be as above. We assume* $\{M_\alpha\}_I$ *is a locally semi-T-nilpotent system and M satisfies* (*). *Let A be a submodule of M. Then we have* $A = A_1 \oplus A_2$, *where* $A_1$ *is a direct summand of M (and hence* $A_1 \in \mathfrak{A}$) *and* $A_2$ *does not contain any submodules which are isomorphic to some in* $\{M_\alpha\}_I$.

Proof. Let $\mathfrak{S}$ be the set of submodules $A'$ in $A$ as follows: $A$ is in $\mathfrak{A}$, say $A' = \sum_{\alpha \in J} \oplus A_\alpha$; $A_\alpha$ are isomorphic to some in $\{M_\alpha\}_I$ and $A'$ is a locally direct summand of $M$ with respect to this decomposition. We can define a partial order in $\mathfrak{S}$ by members of direct components (cf. [6]). Then we obtain a maximal one in $\mathfrak{S}$ by Zorn's lemma, say $A_1$. Since $\{M_\alpha\}_I$ is locally semi-$T$-nilpotent, $A_1$ is a direct summand of $M$: $M = A_1 \oplus M_1$ by Theorem 9 in [3], Theorem in [7] and Lemma 3 and Corollary 2 to Lemma 2 in [6]. Hence, $A = A_1 \oplus (A \cap M_1)$ and $A \cap M_1$ does not contain any submodules in $\mathfrak{A}$ from the assumption and the maximality of $A_1$.

The following lemma is a modification of one part of Theorem 2.6 in [12].

**Lemma** 2. *Let* $\{M_\alpha\}_I$ *and M be as above. We assume M satisfies* (*). *Then M has the exchange property if and only if* $\{M_\alpha\}_I$ *is a locally semi-T-nilpotent system.*

Proof. If M has the exchange property, then $\{M_\alpha\}_I$ is a locally semi-$\Gamma$-nilpotent system by [4'], Corollary 2 to Proposition 1. Conversely, we assume that $\{M_\alpha\}_I$ is semi-$\Gamma$-nilpotent. Let $A = M \oplus N = \sum_{\alpha \in J} \oplus A_\alpha$ We may assume from [2], Theorem 8.2 that all $A_\alpha$ are isomorphic to submodules in M, in order to show that $M$ has the exchange property. Then from the assumption and Lemma 1, $A_\alpha = A_\alpha' \oplus A_\alpha''$, where $A_\alpha' \in \mathfrak{A}$ and $A_\alpha''$ does not contain any submodules, isomorphic to some in $\{M_\alpha\}_I$. Put $A' = \sum_{\alpha \in J} \oplus A_\alpha'$ and $A'' = \sum_{\alpha \in J} \oplus A_\alpha''$, then $A = A' \oplus A''$. Let $\varphi: A \to A/A''$ be the natural epimorphism. We shall show that M is a locally direct summand of $A/A''$ through $\varphi$. Let $I'$ be a finite subset of $I$ and $M' = \sum_{\alpha \in I'} \oplus M_\alpha$. Since $M'$ has the exchange property by [11], Proposition 1 and [2], Lemma 3.10, $A = M' \oplus A_0' \oplus A_0''$, where $A_0' \subset A'$ and $A_0'' \subset A''$. Then $A'' = A_0'' \oplus K''$ and $K''$ is isomorphic to a direct summand of $M'$. If $K'' \neq 0$, $K''$ contains a completely indecomposable module $K_1$ (isomorphic to one in $\{M_\alpha\}_I$) as a direct summand by Krull-Remak-Schmidt theorem. Since $K_1$ has the exchange property, we know from the argument above that some $A_\alpha''$ contains a submodule isomorphic to $K_1$. Which is a contradiction. Hence, $A = M' \oplus A_0' \oplus A''$ and $\varphi(M) \approx M$ is a locally direct summand of $A/A''$. Since $A/A'' \approx A' \in \mathfrak{A}$ and $\{M_\alpha\}_I$ is locally semi-$T$-nilpotent, $\varphi(M)$ is a direct summand of $A/A''$ by [6], Lemma 3; $A/A'' = \varphi(M) \oplus \varphi(K)$ and $K \subset A'$. Furthermore, $\varphi(M)$ has the exchange property in $\mathfrak{A}$ by [4], Corollary 2 to Proposition 1 and hence $A/A'' = \varphi(M) \oplus \sum_{\alpha \in J} \oplus \varphi(A_\alpha''')$ where $A_\alpha''' \subset A_\alpha'$. Therefore, $A = M \oplus \sum_{\alpha \in J} \oplus (A_\alpha''' \oplus A_\alpha'')$.

Next, we shall consider some cases where $M$ satisfies (*).

**Lemma 3.** *Let $\{M_\alpha\}_I$, $M$ and $N$ be as in (*) and $i: N \to M$ the inclusion. Then $N$ is a direct summand of $M$ if and only if $p_\alpha i$ is isomorphic for some $\alpha$ in $I$.*

Proof. It is clear from the definition of $\mathfrak{F}'$.

**Lemma 4.** *Let $M_1$ be a completely indecomposable module. We assume $M_1$ is a locally $T$-nilpotent system itself. Then $M = \sum_{\alpha \in I} \oplus M_\alpha$; $M_\alpha \approx M_1$ has the exchange property for any set $I$.*

Proof. We shall show that $M$ satisfies (*). We may assume $N = M_1$. We put $f_\alpha = p_\alpha i$ and assume that $f_\alpha$ are not isomorphic for all $\alpha \in I$. Let $m \neq 0 \in N$ and $i(m) = \sum_{i=1}^{n} f_{\alpha_i}(m)$. Since $i$ is monomorphic, we may assume $f_{\alpha_1}(m) = m_2 \neq 0$. Let $i(m_2) = \sum_{i=1}^{n'} f_{\alpha_i'}(m_2)$. Repeating this argument, we obtain a sequence $\{f_{\beta_i}\}_1^\infty$ such that $f_{\beta_n} f_{\beta_{n-1}} \cdots f_{\beta_1}(m) \neq 0$ for any $n$, which contradicts the $T$-nilpotency of $\{M_1\}$. Therefore, $M$ satisfies (*) by Lemma 3.

Let $A$, $B$ be $R$-modules and $f \in \mathrm{Hom}_R(A, B)$. If $\mathrm{Im}\, f$ is small in $B$, $f$ is called a *small homomorphism*. We note that if $A = B$ are $R$-projective, then the Jacobson radical $J(\mathrm{End}_R(A))$ of $\mathrm{End}_R(A)$ consists of all small homomorphisms by [10], Lemma 1.

**Lemma** 5. *Let* $\{P_\alpha\}_I$ *be a set of R-modules and* $P=\sum_{\alpha\in I}\oplus P_\alpha$ *If P has the* $\aleph_0$*-exchange property, then any sequence of small homomorphisms* $\{n_{\alpha_i}: P_{\alpha_i}\to P_{\alpha_{i+1}}\}$ *is locally semi- T-nilpotent for any countable subset* $\{P_{\alpha_i}\}_1^\infty$ *of* $\{P_\alpha\}_I$.

Proof. We make use of the same argument in [3], Lemma 9. Since $P^*=\sum_{i=1}^\infty\oplus P_{\alpha_i}$ has the $\aleph_0$-exchange property by [2], Lemma 3.10, we may assume $I=\{\alpha_i\}^\infty$. Let $\{n_i\}$ be the given small homomorphisms. Put $P_i'=\{p_i+n_ip_i \mid p_i\in P_i\}\subset P_i\oplus P_{i+1}$. Then $P=P_1\oplus P_2'\oplus P_3\oplus P_4'\oplus\cdots=P_1'\oplus P_2\oplus P_3'\oplus P_4\oplus\cdots$. Since $P_i'\approx P_i$, $\sum_{n=0}^\infty\oplus P_{2n+1}'$ has the $\aleph_0$-exchange property. Hence, $P=\sum_{n=0}^\infty\oplus P_{2n+1}'\oplus P_1^{(1)}\oplus P_2'^{(1)}\oplus P_3^{(1)}\oplus P_4'^{(1)}\oplus\cdots$, where $P_{2n+1}^{(1)}$ and $P_{2n+2}'^{(1)}$ are direct summands of $P_{2n+1}$ and $P_{2n+2}'$, respectively. Since $P_{2n+2}'\approx P_{2n+2}$, $P_{2n+2}'^{(1)}=P_{2n+2}^{(1)\prime}$, where $P_{2n+2}=P_{2n+2}^{(1)}\oplus P_{2n+2}^{(2)}$. Let $p_{2n}$ be the projection of P to $P_{2n}$ with respect to the decomposition $P=\sum_{i=1}^\infty\oplus P_i$ Then $P_{2n}=p_{2n}(P)=n_{2n-1}(P_{2n-1})+P_{2n}^{(1)}$ from the latest decomposition. On the other hand, $n_{2n-1}(P_{2n-1})$ is small in $P_{2n}$ by the definition and hence $P_{2n}^{(1)}=P_{2n}$. We consider the two decompositions $P=(P_1'\oplus P_1^{(1)})\oplus\{P_2'\oplus(P_3'\oplus P_3^{(1)})\oplus P_4'\oplus\cdot\}=\sum_{i=1}^\infty\oplus P_i$. We shall show $P_1^{(1)}=(0)$. Let $x$ be in $P_1^{(1)}$. If $n_1x$ is contained in $\{P_2'\oplus(P_3'\oplus P_3^{(1)})\oplus P_4'\oplus\ \}$, then $x=x+n_1x+(-n_1x)\in(P_1'\oplus\{P_2'\oplus(P_3'\oplus P_3^{(1)})\oplus P_4'\oplus\ \})\cap P_1^{(1)}=(0)$ from the former decomposition and so $x=0$, which implies that $n_1\,_{P_1^{(1)}}$ is monomorphic. Let $y$ be any element in $P_2$ in the latter decomposition. Consider the expression of $y$ in the former, then $y=x_1+x_1'+n_1x_1'+x_2'+n_2x_2'+y'$, where $x_1\in P_1^{(1)}$, $x_1'\in P_1$, $x_2'\in P_2$ and $y\in(P_3'\oplus P_3^{(1)}\oplus P_4'\oplus\ )$. We consider this expression in the latter, then $x_1=-x_1'$, $n_2x_2'=-y'$ and $y=n_1x_1'+x_2'$. We define a submodule $N$ in $P_2$ as follows: $N=\{z\mid\in P_2, n_2z\in(P_3'\oplus P_3^{(1)})\oplus P_4'\oplus\ \}$. Then we obtain $P_2=n_1(P_1^{(1)})\oplus N$ from the above arguments. On the other hand, $n_1(P_1^{(1)})$ is small in $P_2$ and hence, $n_1(P_1^{(1)})=0$. We have already known that $n_1\,_{P_1^{(1)}}$ is monomorphic. Therefore, $P_1^{(1)}=(0)$ and $P=P_1'\oplus P_2'\oplus(P_3'\oplus P_3^{(1)})\oplus P_4'\oplus\cdots$. Consider an expression of element $x_3$ in $P_3$ in the above decomposition, then $x_3=x_3^{(1)}+x_3'+n_3x_3'+y$, $x_3^{(1)}\in P_3^{(1)}$, $x_3'\in P_3$ and $y\in\{P_4'\oplus\ \cdot\}$. Hence if we repeat the same argument on the direct summand $P_3\oplus P_4$ instead of the direct summand $P_1\oplus P_2$, we know $P_3^{(1)}=(0)$. Similarly, we obtain $P_{2n+1}^{(1)}=(0)$ for all $n$. Thus, we have $P=\sum_{i=1}^\infty\oplus P_i'$. It is easy from this fact to prove the lemma (cf. [3], Lemma 9).

**Theorem 1.** *Let R be a ring. Then the following statements are equivalent.*

1) *R is a right perfect ring* (see [1]).

2) *For every projective module $P$,*
   i *$P$ has the exchange property,*
   ii $\mathrm{End}_R(P)/J(\mathrm{End}_R(P))$ *is a regular ring in the sense of Von Neumann.*

3) *Put $P_0=\sum_1^{\infty}\oplus R$.*
   i *$P_0$ has the exchange property,*
   ii $\mathrm{End}_R(P_0)/J(\mathrm{End}_R(P_0))$ *is a regular ring.*

4) i *$P_0$ has the exchange property,*
   ii *$R/J(R)$ is artinian.*

Proof. 1)→2) Let $R$ be perfect and $R=\sum_{i=1}^{n}\oplus e_iR$, where $\{e_i\}$ is a complete set of mutually orthogonal primitive idempotents (see [1]). Let P be a projective $R$-module. Then $P\approx\sum_i(\sum_{J_i}\oplus e_iR)$ and $\{e_iR\}$ is a $T$-nilpotent system of completely indecomposable modules by [1]. Hence, $\sum_{J_i}\oplus e_iR$ has the exchange property by Lemma 4 and so does $P$ from [2], Lemma 3.10. ii is obtained by [8].

2)→3) It is clear.

3)→4) Since $P_0$ has the exchange property, $J(R)$ is right T-nilpotent from Lemma 5 and [10], Lemma 1. It is well known that $\mathrm{End}_R(P_0)$ is isomorphic to the ring of column finite matrices over $R$ with degree $\aleph_0$. Since $J(R)$ is right T-nilpotent, $J(\mathrm{End}_R(P_0))$ is isomorphic to the subring of column finite matrices over $J(R)$ by [9] or [5], Corollary 1 to Proposition 1. Hence, $\mathrm{End}_R(P_0)/J(\mathrm{End}_R(P_0))$ is isomorphic to the ring of column finite matrices over $R/J(R)$. Therefore, $R/J(R)$ is artinian by [5], Corollary to Lemma 2.

4)→1) It is clear from Lemma 5 and [1].

**Proposition 1.** *Let $R$ be a semi-perfect ring (see* [1]) *and $P$ a projective $R$-module. Then $P$ is semi-perfect (see* [8]) *if and only if $P$ has the exchange property.*

Proof. Since $R$ is semi-perfect, P is isomorphic to a module $\sum_{i=1}^{n}\sum_{J_i}\oplus e_iR$ by [11]. If $P$ is semi-perfect, $\{e_iR\}_{J_i}$ is semi-$T$-nilpotent by [8] or [4], Theorem 7. Hence, P has the exchange property from Lemma 4. The converse is clear from [4], Theorem 7.

Finally, we shall add here some remarks concerned with (*).

**Lemma 6.** *Let $\{M_\alpha\}_I$, $M$ and $N$ be as in* (*). *If $N$ is uniform, $p_\alpha i$ is monomorphic for some $\alpha$.*

Proof. Let $x\neq 0$ be in $N$ and $n=\sum_{i=1}^{n}m_{\alpha_i}$. Put $M_0=\sum_{I-\{\alpha_i\}}\oplus M_\alpha$ and let $p_0$: $M\to M_0$ be its projection. Then $0=(\cap\mathrm{Ker}(p_{\alpha_j}i))\cap\mathrm{Ker}(p_0i)$. Since $N$ is uniform and $\mathrm{Ker}(p_0i)\neq 0$, $\mathrm{Ker}(p_{\alpha_j}i)=0$ for some $\alpha_j$.

**Corollary 1** (cf. [12]) *Let $\{M_\alpha\}_I$ and $M$ be as above. We assume that all $M_\alpha$ are uniform and each $M_\alpha$ is not isomorphic to a proper submodule in $M_\beta$ for all $\alpha$, $\beta$ (e.g. all $M_\alpha$ are injective). Then $M$ has the exchange property if and only if $\{M_\alpha\}_I$ is a locally semi-T-nilpotent system.*

Proof. It is clear from Lemma 6 and [4].

For an $R$-module $L$ we denote its composition length by $||L||$.

**Corollary** 2. *We assume all $M_\alpha$ are uniform and of $|M_\alpha|| \leqslant n < \infty$ for all a. Then $M = \sum_{\alpha \in I} \oplus M_\alpha$ has the exchange property.*

Proof. Put $M(i) = \sum_{J_i} \oplus M_\gamma$, where $||M_\gamma|| = i$. Then $M(i)$ satisfies (*) by Lemma 6. On the other hand, $\{M_\alpha\}_I$ is $T$-nilpotent by [3], Corollary to Lemma 12. Hence, $M$ has the exchange property by Lemma 2 and [2], Lemma 3.10.

## 3. $\aleph_0$-exchange property

Let $\{M_\alpha\}_I$ be a set of completely indecomposable modules and $\mathfrak{A}$ the induced category from $\{M_\alpha\}_I$. We have shown in [4] that every object in $\mathfrak{A}$ has the exchange property in $\mathfrak{A}$ if and only if $\{M_\alpha\}_I$ is a locally $T$-nilpotent system.

In this section we shall study a similar theorem to the above. We rearrange $\{M_\alpha\}_I$ as follows: $\{M_{\alpha\beta}\}_{\alpha \in \mathrm{K}, \beta \in J_\alpha}$ such that $M_{\alpha\beta} \approx M_{\alpha\beta'}$ and $M_{\alpha\beta} \not\approx M_{\alpha'\beta'}$ if $\alpha \neq \alpha'$. Put $\mathrm{K}^{(1)} = \{\alpha | \in \mathrm{K}, |J_\alpha| < \aleph_0\}$, $\mathrm{K}^{(2)} = \{\alpha | \in \mathrm{K}, |J_\alpha| \geqslant \aleph_0\}$ and $M^{(i)} = \sum_{\alpha \in \mathrm{K}^{(i)}} \sum_{\beta \in J_\alpha} \oplus M_{\alpha\beta}$, where $|\mathrm{K}|$ means the cardinal of K. Then $M = M^{(1)} \oplus M^{(2)}$. In Section 2, we have mainly studied a case $M = M^{(2)}$. We shall consider here a case $M = M^{(1)}$.

**Lemma** 7. *Let $M = M^{(1)} \oplus M^{(2)}$ be as above. We assume $\{M_\alpha\}_I$ is a locally T-nilpotent system. If either $M^{(2)} = 0$ or $M^{(1)} = 0$ and $|\mathrm{K}^{(2)}| = 1$, then every monomorphism $f$ in $Hom_R(M, M)$ has a left inverse, namely $Im f$ is a direct summand of $M$, (cf.* [13], Proposition 6).

Proof. We first note that there exist indices $a$, $\beta \in I$ such that $p_\beta f|_{M_\alpha}$ is isomorphic from the proof of Lemma 4. Let $\mathfrak{S}$ be the set of direct summands $\sum_{I'} \oplus M_\delta$ of $M$ such that $\sum_{I'} \oplus f(M_\delta)$ are locally direct summands of $M$. Then $\mathfrak{S}$ contains a maximal element $K = \sum_L \oplus M_\delta$ with respect to the inclusion. Since $\{f(M_\delta)\}_L$ is locally $T$-nilpotent, $f(K)$ is a direct summand of $M$ and $M = f(K) \oplus \sum_{I'} \oplus M_\varepsilon'$ by Lemma 10 in [3] and Lemma 3 in [6], where $M_\varepsilon'$ are isomorphic to some in $\{M_\alpha\}_I$. We assume $K \neq M$. Then $f(M) = f(K) \oplus f(M) \cap (\sum_{I'} \oplus M_\varepsilon')$. Let $p$ be the projection of $M$ to $\sum_{I'} \oplus M_\varepsilon'$, then $pf|_{\sum_{I-L} \oplus M_\alpha}$ is monomorphic. On the other hand, if $M^{(2)}=0$, $\sum_{I-L}\oplus M_{\alpha}\approx\sum_{I'}\oplus M_{\varepsilon}'$ and we may assume $pf|_{\sum_{I-L}\oplus M_{\alpha}}$ is a monomorphism in $\mathrm{Hom}_R(\sum_{I'}\oplus M_{\varepsilon}', \sum_{I'}\oplus M_{\varepsilon}')$. Hence, in either case $M^{(1)}=0$ or $M^{(2)}=0$, there exists $\alpha'$ in $I-L$ such that $pf(M_{\alpha'})$is a direct summand of $\sum_{I'}\oplus M_{\varepsilon}'$ from the first argument and Lemma 3. Therefore, $f(K\oplus M_{\alpha'})$ is a direct summand of M, which contradicts the maximality of $K$. Thus, we have proved the lemma.

REMARK. Lemma 7 is not true for the different cases from the assumption.

The following lemma is substantially due to [2], Lemma 3.11.

**Lemma 8.** *We assume that an R-module T has the finite exchange property and $T\oplus T'=\sum_{i=1}^{\infty}\oplus A_i=A$. Then each $A_i$ contains a direct summand $A_i'$ such that $T\cap(\sum_{i=1}^{\infty}\oplus A_i')=0$ and $T$® $\sum_{i=1}^{n}\oplus A_i'$ is a direct summand of A for any n.*

Proof. Put $K_n=\sum_{i\geqslant n}\oplus A_i$. We assume $A=T\oplus A_1'\oplus A_2'\oplus\cdots\oplus A_n'\oplus K_{n+1}'$, where $A$ and $K_{n+1}'$are direct summands of $A_i$ and $K_{n+1}$, respectively, say $A_i=A_i'\oplus A_i''$, $K_{n+1}=K_{n+1}'\oplus K_{n+1}''$. Since $T\approx\sum_{i=1}^{n}\oplus A_i''\oplus K_{n+1}''$,$K_{n+1}''$has the finite exchange property by [2], Lemma 3.10. We may assume $K_{n+1}''=(T\oplus A_1'\oplus A_2'\oplus\cdots\oplus A_n')\cap K_{n+1}$ and hence $T\oplus A_1'\oplus A_2'\oplus\cdots\oplus A_n'$ contains $K_{n+1}''$ as a direct summand; $T\oplus A_1'\oplus A_2'\oplus\cdots\oplus A_n'=K_{n+1}''\oplus P$. *Put* $K_{n+1}=A_{n+1}\oplus K_{n+2}$. Then $K_{n+1}=K_{n+1}''\oplus A_{n+1}'\oplus K_{n+2}'$, since$K_{n+1}''$ has the finite exchange property. Thus, $A=T\oplus A_1'\oplus A_2'\oplus\cdots\oplus A_n'\oplus K_{n+1}'=K_{r+1}''\oplus P\oplus K_{n+1}'=K_{n+1}\oplus P=K_{n+1}''\oplus P\oplus A_{n+1}'\oplus K_{n+2}'=T\oplus A_1'\oplus A_2'\oplus\cdots\oplus A_n'\oplus A_{n+1}'\oplus K_{n+2}'$.

In Lemma 8 we have obtained $A_i=A_i'\oplus A_i''$and $A=(\sum_{i=1}^{\infty}\oplus A_i')\oplus(\sum_{i=1}^{\infty}\oplus A_i'')$. Let $p_n$ be the projection of $A$ to $\sum_{i=1}^{n}\oplus A_i''$in the decomposition above.

**Lemma 9.** $p_n(T)=\sum_{i=1}^{n}\oplus A_i$*/for any n.*

Proof. Let $p$ be the projection of $A$ to $\sum_{i=1}^{\infty}\oplus A_i''$ Since $A=\sum_{i=1}^{n}\oplus A_i'\oplus T\oplus K_{n+1}'$ and $K_{n+1}=\sum_{i>n}\Theta A_i'\oplus\sum_{i>n}\oplus A_i''$, $p(A)=p(T\oplus K_{n+1}')\subseteq p(T)+p(K_{n+1})=p(T)+\sum_{i>n}\oplus A_i''\subseteq p(A)$. Hence, $p_n(T)=\sum_{i=1}^{n}\oplus A_i''$.

**Lemma 10.** *If $\{M_{\alpha}\}_I$ ιsfo^α/fy T-nilpotentand $M^{(2)}=0$, then M has the $\aleph_0$-exchange property.*

Proof. Let $M=M^{(1)}=\sum_{\alpha\in K}\sum_{\beta\in J_{\alpha}}\oplus M_{\alpha\beta}$ as above and $M\oplus N=\sum_{i=1}^{\infty}\oplus A_i=A$. Then $M$ has the finite exchange property by [12], Proposition 1.7. Hence, we

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obtain direct summands $A$ of $A_i$ such that $M\cap(\sum_{i=1}\oplus A_i')=(0)$ from Lemma 8. Since

$$\sum_{i=1}^{n}\oplus A_i''\oplus K_{n+1}''\approx M\cdots(**)\,,$$

$A_i''\approx\sum_{\alpha\in K^{(i)}}\sum_{\beta\in J_\alpha^{(i)}}\oplus M_{\alpha\beta}$ and $J_\alpha^{(i)}\subseteqq J_\alpha$, $K^{(i)}\subseteqq K$ by [3], Theorem 9. Since $|J_\alpha|<\aleph_0$, $\sum_1^\infty|J_\alpha^{(i)}|$ is finite and $|J_\alpha|\geqslant\sum|J_\alpha^{(i)}|$ from (**). Now, $\sum_1^\infty\oplus A_i=\sum_1^\infty\oplus A_i'\oplus\sum_1^\infty\oplus A_i''$ and let $p$ be the projection to $\sum_1^\infty\oplus A_i''$ Since $M\cap(\sum_1\oplus A_i')=(0)$, M is isomorphic to $p(M)\subseteqq\sum_1^\infty\oplus A_i''$. From the above argument we may assume that $p\backslash_M$ is a monomorphism in $\mathrm{Hom}_R(M, \mathrm{M})$. Then $p(M)$ is a direct summand of $\sum_1^\infty\oplus A_i''$ by Lemma 7. Hence, $\sum_1^\infty\oplus A_i''=p(M)\oplus\sum_1^\infty\oplus A_i'''$; $A_i'''\subseteqq A_i''$, by [4′], Corollary 2 to Proposition 1. Thus, $\sum_1^\infty\Theta A_i=M\oplus\sum_1^\infty\oplus(A_i'\oplus A_i''')$.

**Lemma 11.** *// $\{M_\alpha\}_I$ is locally semi-T-nilpotentand $M=M^{(1)}$ (i.e. $M^{(2)}=0$) is R-projective, then M has the* tf ${}_0$*-exchangeproperty.*

Proof. Let $M=M^{(1)}=\sum_{\alpha\in K^{(1)}}\sum_{\beta\in\overline{J}_\alpha^{(1)}}\oplus M_{\alpha\beta}$. We shall use the same notations as in the proof of Lemma 10. We have obtained the monomorphism $p$ of $M$ to $\sum_1^\infty\oplus A_i''$and $A_i''$ are in $\mathfrak{A}$ from (**) and [4], Theorem 4. Put $A_i''=\sum_{\alpha\in\overline{K}^{(i)}}\sum_{\beta\in J^{(i)}}\oplus M_{\alpha\beta}'$. Since $|J_\alpha^{(1)}|<\aleph_0$, $\sum_1^\infty|J_\alpha^{(i)}\leqslant J_\alpha|$ from (**). We consider all $M_{\alpha\beta}'$ and $p$ in $\overline{\mathfrak{A}}$. Since M is projective, so is $A$ $'$ from (**). Furthermore, $p_np$ $_M$ is epimorphic to $\sum_1^n\oplus A_i''$ by Lemma 9 and so $p_np\backslash_M$splits. Therefore, $\overline{p_np}\backslash_{\overline{M}}$ is epimorphic in $\overline{\mathfrak{A}}$. Now, $\sum_1^\infty\oplus A_i''=\sum_{\overline{\cup K^{(i)}}}\sum_{\overline{\sum_i J_\alpha^{(i)}}}\oplus M_{\alpha\beta}'$. Since $\sum_i|J_\alpha^{(i)}|<\aleph_0$, $\sum_{\Sigma\overline{J}_\alpha^{(i)}}\oplus\bar{M}_{\alpha\beta}'$ is a direct summand of some $\sum_1^t\oplus\bar{A}_i''$. Let $q$ be the projection of $\sum_1^\infty\oplus A_i''$ to $\sum_{\Sigma J_\alpha^{(i)}}\oplus M_{\alpha\beta}'$. Then $\overline{qp}|_{\overline{M}}$ is epimorphic from the above. On the other hand, $\overline{qp}(\sum_{\alpha'\neq\alpha}\oplus\bar{M}_{\alpha'\beta})=(0)$, since $\bar{M}_{\alpha'\beta}$are minimal and $\bar{M}_{\alpha'\beta}\not\approx\bar{M}_{\alpha\beta}$. Hence, $\bar{p}(\sum_{\beta\in J_\alpha}\oplus\bar{M}_{\alpha\beta})=\sum_{\sum_i J_\alpha^{(i)}}\oplus\bar{M}_{\alpha\beta}'$, which implies $\bar{p}|_{\overline{M}}$ is epimorphic. Since $\overline{\mathfrak{A}}$ is a regular abelian category from [3], Theorem 7, there exists $t\colon\sum_1^\infty\oplus A_i''\to M$such that $pt=1_{\Sigma\oplus\bar{A}_i''}$ Therefore, $p$ is epimorphic as $R$-modules by [6] and [7]. Thus, $A=\sum_1^\infty\oplus A_i'\oplus M$.

Let $\mathfrak{A}(f)$ be the subadditive category of $\mathfrak{A}$, whose objects consist of all $A$

such that $A=A^{(1)}\oplus A^{(2)}$ and $|K^{(2)}|<\aleph_0$.

Summarizing the above we have

**Proposition 2.** *Let $\{M_\alpha\}_I$ be a set of completely indecomposable modules and $\mathfrak{A}(f)$ as above. Then $\{M_\alpha\}_I$ is a locally T-nilpotent system if and only if every module in $\mathfrak{A}(f)$ has the $\aleph_0$-exchange property.*

**Proposition 3.** *Let P be a projective R-module in $\mathfrak{A}(f)$. Then P is semi-perfect if and only if P has the $\aleph_0$-exchange property.*

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