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ON LOCALLY DIRECT SUMMANDS OF MODULES

Dedicated to Professor Kiiti Morita on his 60th birthday

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Throughout R will represent a ring with unit element 1, and all modules will be unitary R-modules. We call a module M a *completely indecomposable module* if the endomorphism ring of M is a local ring. Let $\mathfrak{M}=\{M_\alpha\}_I$ be a set of completely indecomposable right R-modules, and $\mathfrak{A}$ the full subadditive category of the category of all right R-modules, whose objects consist of all R-modules which are isomorphic to direct sums of M_α's in $\mathfrak{M}$. We define the subclass $\mathfrak{J}'$ of the morphisms in $\mathfrak{A}$ as follows: for any objects $M=\sum_{\alpha\in K}\oplus M_\alpha'$, $N=\sum_{\beta\in L}\oplus N_\beta$ in $\mathfrak{A}$, $\mathfrak{J}'\cap \mathrm{Hom}_R(M, N)=\{f\in \mathrm{Hom}_R(M,N) \setminus p_\beta f i_\alpha$ is not isomorphic, for all $\alpha\in K, \beta\in L$, where $i_\alpha: M_\alpha'\to M$ is the inclusion and $p_\beta: N\to N_\beta$ is the projection$\}$. Then, $\mathfrak{J}'$ does not depend on the decompositions of M and N (see Corollary to Lemma 5 in [5]).

M. Harada and Y. Sai [4], [5] gave several equivalent conditions for $S_M\cap\mathfrak{J}'$ to be equal to the Jacobson radical $J(S_M)$ of S_M, where $M\in\mathfrak{A}$ and $S_M=\mathrm{Hom}_R$ (M, N). Among those conditions, they made great use of structures of the factor cagegory $\mathfrak{A}/\mathfrak{J}'$ in order to show the following fact: if $J(S_M)=S_M\cap\mathfrak{J}'$, then for any two decompositions $M=\sum_{\alpha\in K}\oplus M_\alpha=\sum_{\beta\in L}\oplus N_\beta$ and any subset K' of K, there exists a one-to-one mapping φ of K' into L such that $M_\alpha\approx N_{\varphi(\alpha)}$ for all $\alpha\in K'$ and $M=\sum_{\alpha\in K'}\oplus N_{\varphi(\alpha)}\oplus\sum_{\alpha'\in K-K'}\oplus M_{\alpha'}$.

The purpose of this note is to give a ring-theoretical proof of the above fact by using a few structure of $\mathfrak{A}/\mathfrak{J}'$. We shall define a concept of locally direct summands of M in $\mathfrak{A}$ for this purpose. Let $N=\sum_{\gamma\in L}\oplus N_\gamma$ be a submodule of M in $\mathfrak{A}$. If $\sum_{\gamma'\in L'}\oplus N_{\gamma'}$ is a direct summand of M for every finite subset L' of L, we call a *locally direct summand* of M (with respect to the decomposition $N=\sum_{\gamma\in L}\oplus N_\gamma$) We shall give a relation between some locally direct summands of M and dense submodules of M defined in [4], and using this relation we shall give a proof of the statement above.

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We begin with preliminary definitions and results on $\mathfrak{J}'$ and S_M. From now, we understand that a module M is in $\mathfrak{A}$ and that M_α's are completely indecomposable, if there are no confusions.

Let M, N be in $\mathfrak{A}$, and $f \in \mathrm{Hom}_R(M, N)$. f is said to be *left regular* modulo $\mathfrak{J}'$ if, for any homomorphism g of any L in $\mathfrak{A}$ to M, fg in $\mathfrak{J}'$ implies g in $\mathfrak{J}'$. The right regularity of / modulo $\mathfrak{J}'$ is defined similarly. / is said to be an *isomorphism* modulo $\mathfrak{J}'$ if there exists some g: $N \to M$ such that $gf = 1_M \text{mod.} \mathfrak{J}'$ and $fg = 1_N$ mod. $\mathfrak{J}'$.

REMARK 1. Let $M = \sum_{\beta \in K} \oplus M_\beta$, $N = 2_{\beta' \in K'} \oplus M_{\beta'}$ be in $\mathfrak{A}$ where K' is a subset of K, i the inclusion of N to M and p the projection of M onto N. Then, i is left regular mod. $\mathfrak{J}'$ and p is right regular mod. $\mathfrak{J}'$.

Lemma 1. *For any morphismf in* SI *and any g in* $\mathfrak{J}'$, *fg and gf are in* $\mathfrak{J}'$. See Lemma 5 in [5].

Lemma *2. Let* $M = \sum_{\alpha \in K} \oplus M_\alpha$ *be in* $\mathfrak{A}$, *and* S_M *the endomorphismring of M. Then,*

(1) $S_M / S_M \Pi \mathfrak{J}'$ *is a regular ring* (*in the sense of von Neumann*), *moreover*

(2) for any fin S_M *with* $f = f^2$ *modulo* $\mathfrak{J}' \cap S_M$, *there exist some elements a and e in* S_M *such that a is regular in* $S_M / S_M \cap \mathfrak{J}'$, *e is a projection of M to* $\sum_{\alpha' \in K'} \oplus M_{\alpha'}$ *for some subset* K' *of K and* $f = aea'$ *modulo* $\mathfrak{J}'$, *where* $aa' = a'a = 1$ *modulo* $\mathfrak{J}'$ *and* $a' \in S_M$.

See [1], Lemma 6 and Theorem 7 in [5] and [6].

Corollary 1. *Let M, N be in* $\mathfrak{A}$, *and* $f: M \to N$. *Then,*

(1) *f is left (resp. right) regular mod.* $\mathfrak{J}'$ *if and only if there exists some* g: $N \to M$ *such that* $gf = 1_M$ (*resp.* $fg = 1_N$) *mod.* $\mathfrak{J}'$, *and*

(2) f is an isomorphism mod. $\mathfrak{J}'$ *if and only iff is left and right regular mod.* $\mathfrak{J}'$.

Proof. (1) "If" part is trivial. Conversely, we assume that f is left regular mod. $\mathfrak{J}'$ Since $S_M / S_M \cap \mathfrak{J}'$ is a regular ring by the lemma, there exists some g: $N \to M$ such that $fgf = f$ mod. $\mathfrak{J}'$. The left regularity of / mod. $\mathfrak{J}'$ implies that $gf = 1_M \text{mod.} \mathfrak{J}'$. The right regularity is similar. *(2)* is clear.

Corollary *2. If f: $M \to N$ is left regular mod. $\mathfrak{J}'$ for M, N in $\mathfrak{A}$ and $S_M \Pi \mathfrak{J}'$ is equal to the Jacobson radical $J(S_M)$ of S_M, then f is an R-monomorphism and M is R-isomorphic to a direct summand of N.*

Proof. By Corollary 1(1), there exists some g: $N \to M$ such that $gf = 1_M$ mod. $\mathfrak{J}'$, since f is left regular mod. $\mathfrak{J}'$. Hence, $1_M - gf \in S_M \cap \mathfrak{J}' = J(S_M)$ and so gf is an R-isomorphism. Therefore, / is an R-monomorphism and M is R-isomorphic to a direct summand of N.

Let U, V be right R-modules, /: $U\to V$, and $U=\sum_{\gamma\in K}\theta U_\gamma$ a direct sum of right R-submodules of U. Then, we consider the following condition:

/ is an R-monomorphism and ··(*)

for any finite subset K' of K, $f(\sum_{\gamma'\in K'}\Phi U_{\gamma'})$ is a direct summand of V.

If f satisfies the above (*)-condition, we call f a (*)-monomorphism (with respect to this decomposition of U).

For example, let f, U and V be as above. If f is an R-monomorphism and each U_γ is injective, then / is a (*)-monomorphism (with respect to the decomposition $U=\sum_{\gamma\in K}\oplus U_\gamma$).

From now on, (*)-monomorphisms will be considered in $\mathfrak{A}$.

The following lemma on (*)-monomorphisms is essential in this note.

Lemma 3. *Let $M=\sum_{\alpha\in K}\oplus M_\alpha$, N be in $\mathfrak{A}$ and f: $M\to N$. Then, f is left regular mod. $\mathfrak{J}'$ if and only if f is a (*)-monomorphism (w.r.t. the decomposition $M=\sum_{\alpha\in K}\oplus M_\alpha$).*

Proof. First, we assume that f is left regular mod. $\mathfrak{J}'$. Put $M_0=\sum_{\alpha'\in K'}\oplus M_{\alpha'}$ for any finite subset K' of K. Let i be the inclusion of M_0 to M. Then, fi is left regular mod. $\mathfrak{J}'$ and $S_{M_0}\cap\mathfrak{J}'=(S_{M_0})$ by Lemma 8 in [5], because K' is a finite set. Hence, fi is an R-monomorphism and $fi(M_0)$ is a direct summand of N by Corollary 2 to Lemma 2. Therefore, / is an R-monomorphism and $f(M_0)$ is a direct summand of N, i.e. / is a (*)-monomorphism (w.r.t. the decomposition $M=\sum_{\alpha\in K}\oplus M_\alpha$). Conversely, let $g\in\mathrm{Hom}_R(T, M)$ for any module $T=\sum_{\gamma\in L}\theta T_\gamma$ in $\mathfrak{A}$ and assume that fg in $\mathfrak{J}'$. Put $g_\gamma=gi_\gamma$, where i_γ is the inclusion of T_γ to T for all $\gamma\in L$. Then, we can express g_γ as a column-summable matrix for all $\gamma\in L$. Hence, g_γ is a column-matrix whose finite components are isomorphic and the others are all non-isomorphisms. We can rearrange g_γ as follows: the first n components are isomorphisms. Put $M_0=\sum_{i=1}^{n}\oplus M_i$ Let i be the inclusion of M_0 to M, and p the projection of M onto M_0. Then, $fipg_y=fg_y=fgi_{f_y}=Q$ mod. $\mathfrak{J}'$. Since fi is left regular mod. $\mathfrak{J}'$, pg_γ is in $\mathfrak{J}'$ Hence, g_γ and so g are in $\mathfrak{J}'$, because o: $pg_\gamma+(1-p)g_\gamma=g_\gamma$ mod. $\mathfrak{J}'$ Therefore, / is left regular mod. $\mathfrak{J}'$.

We note that a (*)-monomorphism does not depend on the decomposition of M from Lemma 3.

Corollary 1 (cf. Lemma 3.2.3 in [3]) (1) *// /: $M\to N$ is left regular mod. $\mathfrak{J}'$, then f is an R-monomorphism. (2) For any f in $S_M\cap\mathfrak{J}'$, 1_M-f is an R-monomorphism.*

Proof. (1) is clear by the lemma. (2) Since / is in $S_M\cap\mathfrak{J}'$, 1_M-f is left regular mod. $\mathfrak{J}'$ and hence an R-monomorphism by (1).

Corollary 2. *Let M, N be in* $\mathfrak{A}$, *and f*: $M\to N$ *an isomorphism mod.* $\mathfrak{J}'$. *Then f is an R-isomorphism provided either* $S_M\cap\mathfrak{J}'=J(S_M)$ *or* $S_N\cap\mathfrak{J}'=J(S_N)$. *Especially, if M is a finite direct mm of* M_α*'s in* $\mathfrak{M}$, *then an isomorphism mod.* $\mathfrak{J}'$ *means an R-isomorphism.*

Proof. Since / is isomorphic mod. $\mathfrak{J}'$, there exists some g: $N\to M$ such that $gf=1_M$ mod. $\mathfrak{J}'$ and $fg=1_N$ mod. $\mathfrak{J}'$. Hence, / and g are left regular mod. $\mathfrak{J}'$, that is, both are R-monomorphisms by Corollary 1. In case $S_N\cap\mathfrak{J}'=J(S_N)$, $1_N-fg\in J(S_N)$ Hence, fg is an R-isomorphism and so is /. On the other hand, if $S_M\cap\mathfrak{J}'$is equal to $J(S_M)$, then $1_M-gf\in J(S_M)$and hence gf is an R-isomorphism. Therefore, / is an R-isomorphism. The latter assertion is clear by Lemma 8 in [5].

We define here an important concept as follows (see [3]): let M, N be in $\mathfrak{A}$, and $N=\sum_{\beta\in K}\oplus N_\beta$ a submodule of M. Then, N is said to be a *locally direct summand* of M (with respect to the decomposition $N=2_{\beta\in K}\oplus N_\beta$) if the inclusion i: $N\to M$ is a (*)-monomorphism (with respect to this decomposition of N).

In the following lemma, we consider the existence of locally direct summands of a module M in SI.

We remark that in the above definition, the concept of locally direct summands of M in $\mathfrak{A}$ does not depend on the decomposition of M, since (*)-mononorphisms do not depend on the decomposition of M.

Lemma 4. *Let M, N be in* $\mathfrak{A}$, *andf*: $M\to N$. *Then, there exist a locally direct summand* N' *of N in* $\mathfrak{A}$ *via the inclusion* i: $N'\to N$ *and some* f': $M\to N'$ *such that* $f=if'$ *mod.* $\mathfrak{J}'$, *i is left regular mod.* $\mathfrak{J}'$ *andf is right regular mod.* $\mathfrak{J}'$.

Proof. We begin with the case $M=N$ and $f=f^2$ mod. $\mathfrak{J}'$. There exist a projection e of $M=\sum_{\alpha\in K}\oplus M_\alpha$ onto $\sum_{\alpha'\in K'}\oplus M_{\alpha'}$ for some subset K' of K and elements a, a' in S_M such that $f=aea'$ mod. $\mathfrak{J}'$ and $aa'=a'a=1$ mod. $\mathfrak{J}'$, by Lemma 2(2). Put $N'=aeM$, $N''=eM$, and consider the inclusions i: $N'\to M$, i': $N''\to M$. Then, by Lemma 3, N' is a locally direct summand of M and i is left regular mod. $\mathfrak{J}'$, since N' is isomorphic to N'' under ai' that is left regular mod. $\mathfrak{J}'$ Moreover, ea' is right regular mod. $\mathfrak{J}'$, and hence so is $f'=aea'$: $M\to N'$. Thus, our lemma holds. In the general case, for /: $M\to$, there exist some homomorphisms g: $N\to N$ and k: $N\to M$ such that $g=g^2$ mod. $\mathfrak{J}'$, $f=gf$ mod. $\mathfrak{J}'$ and $g=fk$ mod. $\mathfrak{J}'$, by Lemma *2*. For g: $N\to N$ with $g=g^2$ mod. $\mathfrak{J}'$, there exist a locally direct summand N' of N in SI, some g': $N\to N'$ and the inclusion i: $N'\to N$ such that $g=ig'$ mod. $\mathfrak{J}'$, g' and i are right and left regular mod. $\mathfrak{J}'$, respectively, by the above argument. We can easily show that $g'f$ is right regular mod.$\mathfrak{J}'$ since $g'i=1_{N'}$ mod.$\mathfrak{J}'$, and $f=ig'f$mod.$\mathfrak{J}'$.

Lemma 5. *For M and N in $\mathfrak{A}$, a homomorphism f: $M \to N$ is right regular mod. $\mathfrak{J}'$ if and only if there exist a locally direct summand M' of M in $\mathfrak{A}$ and some g: $N \to M'$ such that $fig= 1_N$ mod.$\mathfrak{J}'$ and g is an isomorphism mod.$\mathfrak{J}'$, where i is the inclusion of M' to M.*

Proof. "If" part is trivial. Conversely, suppose that / is right regular mod.$\mathfrak{J}'$. Then there exists some g': $N \to M$ such that $fg'=1_N$ mod.$\mathfrak{J}'$, by Corollary 1 to Lemma 2. Since g' is left regular mod.$\mathfrak{J}'$, there exists a locally direct summand $M'$of M in $\mathfrak{A}$ such that $g'=ig$ mod.$\mathfrak{J}'$, where g: $N \to M'$is right regular mod.$\mathfrak{J}'$ and the inclusion i:$M' \to M$ is left regular mod.$\mathfrak{J}'$, by Lemma 4. Therefore, $fig=1_N$ mod.$\mathfrak{J}'$ and g is an isomorphism mod.$\mathfrak{J}'$.

Lemma 6. *Let M, N be in $\mathfrak{A}$, e an idempotent element in S_M where N is contained in eM, and let the inclusion i: $N \to M$ be left regular mod.$\mathfrak{J}'$. Then, there exists a locally direct summand N' of M in $\mathfrak{A}$ such that $e=ip+i'p'$mod.$\mathfrak{J}'$, $pi=1_N$ mod.$\mathfrak{J}'$, $p'i'=1_{N'}$mod.$\mathfrak{J}'$ and $pi'=p'i=0$ mod.$\mathfrak{J}'$, where i' is the inclusion of N' to M and p, p' are homomorphisms of M to N, N' respectively. Furthermore, the formal direct sum $N \oplus N'$ is R-isomorphic to a locally direct summand of eM.*

Proof. For i: $N \to M$, there exists some p_0: $M \to N$such that $ip_0i=i$mod. $\mathfrak{J}'$. Since $ei=i$, $ip_0ei=i$mod.$\mathfrak{J}'$. Put $p=p_0e$, that is, p: $M \to N$and $ipi=i$mod.$\mathfrak{J}'$. Since i is left regular mod.$\mathfrak{J}'$, $pi=1_N$ mod.$\mathfrak{J}'$. Now, we put $f=ip$and $g=e-f$. Then, $ef=fe=f$and $eg=ge=g$. For g: $M \to M$ there exist a locally direct summand N' of M in $\mathfrak{A}$ and some p': $M \to N'$ such that $g=i'p'$ mod.$\mathfrak{J}'$, the inclusion i': $N' \to M$ is left regular mod.$\mathfrak{J}'$ and p' is right regular mod.$\mathfrak{J}'$, by Lemma 4. Therefore, $e=f+g=ip+i'p$mod.$\mathfrak{J}'$. Since $gf=fg=0, ipi'p'=i'p'ip$ $=0$ mod.$\mathfrak{J}'$ implies $pi'=p'i=0$ mod.$\mathfrak{J}'$, because i and i' are left regular mod.$\mathfrak{J}'$ and p, p' are right regular mod.$\mathfrak{J}'$. Moreover, $g= g^2$ mod.$\mathfrak{J}'$ implies $p'i'=1_{N'}$ mod.$\mathfrak{J}'$. Finally, we show that the formal direct sum $N \oplus N'$ is R-isomorphic to a locally direct summand of eM. Let $I=(i, i')$:$N \oplus N' \to M$and $t=\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$: $T \to N \oplus N'$for any Tin $\mathfrak{A}$. Suppose that $It=it_1+i't_2$is in $\mathfrak{J}'$. Then, $pit_1+pi't_2$ is in $\mathfrak{J}'$. Since pi' is in $\mathfrak{J}'$, t_1 is in $\mathfrak{J}'$ and so is t_2. Hence, t is in $\mathfrak{J}'$. It follows that / is a (*)-monomorphism. Therefore, $N \oplus N'$is a locally direct summand of M in $\mathfrak{A}$. On the other hand, $g=eg$ and $g=i'p'$mod.$\mathfrak{J}'$ imply $ei'=i'$mod.$\mathfrak{J}'$, and so we may assume $ei'=i'$ in the above. Since i' is a (*)-monomorphism, Im(i') is contained in eM. Hence, Im(I) is contained in eM, whence $N \oplus N'$is R-isomorphic to a locally direct summand of eM.

Corollary. *Let $N \subset M$ be in $\mathfrak{A}$. If the inclusion i: $N \to M$ is left regular mod. $\mathfrak{J}'$, then there exist a locally direct summand N' of M in $\mathfrak{A}$, the inclusion i':*

$N' \to M$ and some p, p' of M to N, N' respectively such that $1_M = ip + i'p'$ mod. $\mathfrak{J}'$, $pi = 1_N$ mod. $\mathfrak{J}'$, $p'i' = 1_{N'}$ mod. $\mathfrak{J}'$ and $p'i = pi' = 0$ mod. $\mathfrak{J}'$.

Proof. Put $e = 1_M$ in the lemma.

Let N be an R-module, and $\{N_j\} = \sum_{i \in I_j} \oplus N_i^{(j)}\}_{j \in J}$ the set of submodules of N in $\mathfrak{A}$ which are locally direct summands of N. We define an order $>$ in the set $\{N_j\}$ as follows:

for each locally direct summand N_j of N,

$N_j > N_k$ if and only if $\{N_i^{(j)}\}_{I_j} \supset \{N_i^{(k)}\}_{I_k}$, for any $\neq k$ in J.

Then, there exists a maximal submodule of N among the set $\{N_j\}$ with respect to this order $>$, by Zorn's lemma. We call it a *maximal* locally direct summand of N.

Proposition 7. *Assume that all M_α in $\mathfrak{M}$ are injective. Let $N \subset M$ be in $\mathfrak{A}$. Then, N is essential in M if and only if N is a maximal locally direct summand of M.*

Proof. "Only if" part is trivial. Conversely, if N is not essential, there exists a cyclic submodule N' of M with $N \cap N' = (0)$. Then, the injective hull $E(N')$ in M is a direct summand of M. On the other hand, $N \cap E(N') = (0)$. Since $E(N')$ contains an injective submodule M_β for some β, this contradicts the maximality of N. Hence, N is an essential submodule of M.

Next, we show that a dense submodule of a module in $\mathfrak{A}$ defined in [4] is equal to a maximal locally direct summand of the module.

Lemma 8. *Let $N \subset M$ be in $\mathfrak{A}$. Then, N is a maximal locally direct summand of M if and only if the inclusion i of N to M is an isomorphism mod. $\mathfrak{J}'$.*

Proof. First, we assume that N is a maximal locally direct summand of M. If the inclusion i: $N \to M$ is not isomorphic modulo $\mathfrak{J}'$, there exists a locally direct summand N' of M in $\mathfrak{A}$ such that $1_M = ip + i'p'$ mod. $\mathfrak{J}'$, where i' is the inclusion of N' to M, p: $M \to N$ and p': $M \to N'$, by Corollary to Lemma 6. Then, $I = (i, i')$: $N \oplus N' \to M$ is a (*)-monomorphism. Hence, the image of / is equal to a locally direct summand $N \oplus \mathrm{Im}(i')$ of M in $\mathfrak{A}$ which contains N; this contradicts the maximality of N. Hence, $N' = 0$. Therefore, $1_M = ip$ mod. $\mathfrak{J}'$ and so i is an isomorphism mod. $\mathfrak{J}'$. Conversely, suppose that i is an isomorphism mod. $\mathfrak{J}'$ Then, there exists some p: $M \to N$ such that $pi = 1_N$ mod. $\mathfrak{J}'$ and $ip = 1_M$, mod. $\mathfrak{J}'$, and also N is a locally direct summand of M. If N is not maximal in M, there exists a locally direct summand N' of M in $\mathfrak{A}$ such that $N \oplus N'$ is a locally direct summand of M in $\mathfrak{A}$. Hence, the inclusion $I = (i, i')$: $N \oplus N' \to M$ is left regular mod. $\mathfrak{J}'$, where i' is the inclusion *of* N' to M. Therefore, there exists some g: $M \to N \oplus N'$ such that $gI = 1_{N \oplus N'}$ mod. $\mathfrak{J}'$, by Corollary 1(1) to Lemma 2. Let p_1 be the projection of $N\oplus N'$ onto N. Then, $p_1gi=1_N=pi$ mod. $\mathfrak{J}'$ and so $pig=p$ rn.od.9f', which implies that $pi'=0$ mod. $\mathfrak{J}'$. Hence, $N'=0$; a contradiction. It follows that N is a maximal locally direct summand of M.

REMARK 3. The submodule N in the lemma is called a *dense* submodule of M, in [4]. We note that $N\oplus N'$ in Corollary to Lemma 6 is a dense submodule of M.

Corollary 1. *Let M,N be in $\mathfrak{A}$, and f: $M\to N$. Then, there exist locally direct summands M' and N' of M and N in $\mathfrak{A}$, respectively, such that the restriction $f|_{M'}$ to N' is an R-isomorphism. Especially, f is isomorphic mod. $\mathfrak{J}'$ if and only if M' and N' are dense in M and N, respectively.*

Proof. For f: $M\to N$, there exist a locally direct summand N'' of N in $\mathfrak{A}$, the inclusion i': $N''\to N$ and some f': $M\to N''$ such that $f=i'f'$ mod. $\mathfrak{J}'$, i' is left regular mod. $\mathfrak{J}'$ and f' is right regular mod. $\mathfrak{J}'$, by Lemma 4. Since f' is right regular mod. $\mathfrak{J}'$, there exist a locally direct summand M' of M in $\mathfrak{A}$ and some g: $N''\to M'$ such that $f'ig=1_{N'}$mod. $\mathfrak{J}'$ and g is isomorphic mod. $\mathfrak{J}'$, where i is the inclusion of M' to M, by Lemma 5. Since fig is left regular mod. $\mathfrak{J}'$ and g is isomorphic mod. $\mathfrak{J}'$, fi is left regular mod. $\mathfrak{J}'$ and so R-monomorphic. Let N' be the image of fi in N. Then, fi: $M'\to N'$ is an R-isomorphism, whence it follows that N' is a locally direct summand of N. Particularly, in case / is isomorphic mod. $\mathfrak{J}'$, i' and f' are isomorphic mod. $\mathfrak{J}'$ by Corollary 1(2) to Lemma 2 *and fig:* $N''\to N'$ is isomorphic mod. $\mathfrak{J}'$, so that N' and M' are dense in N and M, respectively, by the lemma. Conversely, if N' and M' are dense in N and M' respectively, i and i' are isomorphic mod. $\mathfrak{J}'$, and hence / is isomorphic mod. $\mathfrak{J}'$ by the lemma.

Corollary 2. *// $S_M\cap\mathfrak{J}'=J(S_M)$ for a module M in $\mathfrak{A}$, then M is the only one dense submodule in M.*

Proof. Let N be a dense submodule of M. Then, the inclusion i: $N\to M$ is isomorphic mod. $\mathfrak{J}'$ by the lemma. Hence, i is an R-isomorphism by Corollary 2 to Lemma 3 and so $N=M$.

Lemma *9. Let e be an idempotent element in S_M fora module $M=\sum_{\alpha\in K}\oplus M_\alpha$ in* SI. *Then, there exist a submodule N of eM in $\mathfrak{A}$ and p: $M\to N$ such that $e=ip$ mod. $\mathfrak{J}'$ and $pi=1_N$ mod. $\mathfrak{J}'$, where i: $N\to M$ is the inclusion.*

Proof. Since eM is a direct summand of M, eM contains some M_α by [2]. Hence, there exists a maximal locally direct summand of eM in $\mathfrak{A}$. Let N be the maximal one, and i the inclusion of N to M. Since i is left regular mod.$\mathfrak{J}'$, there exists a locally direct summand N' of M in $\mathfrak{A}$ such that $e=ip+i'p'$ mod.$\mathfrak{J}'$, $pi=1_{N'}$ mod.$\mathfrak{J}'$ and $N\oplus N'$ is R-isomorphic to a locally direct summand of eM, where p: $M\to N, p'$: $M\to N'$ and i' is the inclusion of N' to M, by Lemma 6. Since N is maximal in eM, $N'=0$ and hence $e=ip$ mod.$\mathfrak{J}'$

Corollary 1 (cf. Theorem 1 in [4]). *Let $P=\sum_{\alpha\in L}\oplus P_\alpha$ in $\mathfrak{A}$ (not necessarily each P_α is in $\mathfrak{M}$). Then, there exists a submodule N_α of P_α in $\mathfrak{A}$ such that $e_\alpha=i_\alpha p_\alpha$ mod. $\mathfrak{J}'$, where p_α: $P\to N_\alpha$, i_α: $N_\alpha\to P$ is the inclusion and e_α: $P\to P_\alpha$ is the projection, for each $\alpha\in L$. Moreover, $\sum_{\alpha\in L}\oplus N_\alpha$ is a maximal locally direct summand of P in $\mathfrak{A}$. (Such N_α is called a dense submodule of P_α,* in [4].)

Proof. We can find a maximal locally direct summand N_α of $e_\alpha P=P_\alpha$ such that $e_\alpha=i_\alpha p_\alpha$mod.$\mathfrak{J}'$, where p_α: $P\to N_\alpha$, i_α: $N_\alpha\to P$ is the inclusion and e_α: $P\to P_\alpha$ is the projection, for every $\alpha\in L$, by the lemma. Since a finite direct sum $\sum_{i=1}^{n}\oplus N_{\alpha_i}$ is a direct summand of P, $\sum_{\alpha\in L}\oplus N_\alpha$ is a locally direct summand of P. Hence, the inclusion I: $\sum_{\alpha\in L}\oplus N_\alpha\to P$ is left regular mod.$\mathfrak{J}'$ In order to see that $\sum_{\alpha\in L}\oplus N_\alpha$ is dense in P, we have only to prove that I is right regular mod.$\mathfrak{J}'$. Let t be a homomorphism of P to any module T in $\mathfrak{A}$ and assume that tI is in $\mathfrak{J}'$. If t is not in $\mathfrak{J}'$, there exists some direct summand P_β in P such that the restriction $t\backslash_{P_\beta}$ is not in $\mathfrak{J}'$. $\sum_{\alpha\in L}e_\alpha i=i$ and $e_\alpha i$ is non-isomorphic for almost all $\alpha\in L$, where i is the inclusion of P_β to P. Hence, for some integer n, $\sum_{j=1}^{n}e_{\alpha_j}\cdot i=i$ mod.$\mathfrak{J}'$ and so $ti=te_Ni=tI_Np_Ni-tIp_Ni=0$ mod.$\mathfrak{J}'$, where I_N: $\sum_{j=1}^{n}\oplus N_{\alpha_j}\to P$, p_N: $P\to\sum_{j=1}^{n}\oplus N_{\alpha_j}$ and e_N: $P\to\sum_{j=1}^{n}\oplus P_{\alpha_j}$. Therefore, ti is in $\mathfrak{J}'$, which is a contradiction. Thus, t is in $\mathfrak{J}'$ and so I is right regular mod.$\mathfrak{J}'$.

Corollary 2. *Let M be in $\mathfrak{A}$, and N a direct summand of M. If $S_M\cap\mathfrak{J}'$ is equal to $J(S_M)$, then N is in $\mathfrak{A}$.*

Proof. Since N is a direct summand of M, there exists a submodule N' of M such that $M=N\oplus N'$. Hence, there exist dense submodules N_0 and N_0' of N and N' in $\mathfrak{A}$, respectively such that $N_0\oplus N_0'$ is dense in M, by the above corollary. Hence, $N_0\oplus N_0'=M$ by Corollary 2 to Lemma 8, which implies that N is isomorphic to a direct sum of completely indecomposable modules M_α's in $\mathfrak{M}$.

Proposition 10. *Let M, N be in $\mathfrak{A}$, and f: $M\to N$. If either $S_M\cap\mathfrak{J}'=J(S_M)$ or $S_N\cap\mathfrak{J}'=J(S_N)$, then there exist submodules M_1 and M_2 of M in $\mathfrak{A}$ such that $M=M_1\oplus M_2$ and the restrictions of f to M_1 and M_2 are a zero homomorphism mod.$\mathfrak{J}'$ and an R-monomorphism, respectively.*

Proof. By Corollary 1(1) to Lemma 2 and Lemma 4, there exist a locally direct summand N' of N in $\mathfrak{A}$, f': $M\rightarrow N'$, g': $N'\rightarrow M$ and the inclusion i: $N'\rightarrow N$ such that $f=if'$ mod. $\mathfrak{J}'$, $f'g'=1_{N'}$ mod. $\mathfrak{J}'$, i is left regular rnod.^7 and f' is right regular mod. $\mathfrak{J}'$. In case $S_N\cap\mathfrak{J}'=J(S_N)$, $S_{N'}\Pi\,\mathfrak{J}'=J(S_{N'})$ and hence $f'g'$ is an R-isomorphism. Therefore, $M=\mathrm{Im}(g')\oplus\mathrm{Ker}(f')$. We put $M_1=\mathrm{Ker}(f')$ and $M_2=\mathrm{Im}(g')$. Then, the restriction $f|_{M_1}$is a zero homomorphism mod. $\mathfrak{J}'$ Since/I $_{M_2}=if'|_{M_2}$ mod. $\mathfrak{J}'$ and $f'|_{M_2}$is an isomorphismrn.od.3i', $f|_{M_2}$ is an R-monomorphism. On the other hand, if $S_M\cap\mathfrak{J}'=J(S_M)$, $S_{M'}\cap\mathfrak{J}'=J(S_{M'})$ where M' is a locally direct summand of M in $\mathfrak{A}$ such that some g: $N'\rightarrow M'$ is isomorphic mod. $\mathfrak{J}'$ (cf. Lemma 5). Since g is an R-isomorphism by Corollary 2 to Lemma 3, $S_{N'}\cap\mathfrak{J}'=J(S_{N'})$and so $M=\mathrm{Im}(g')\oplus\mathrm{Ker}(f')$ as above. We put $M_2=\mathrm{Im}(g')$. Then, M_1 and M_2 satisfy the proposition.

Now, we shall show ring-theoretically the main theorem in this note by $M_1=\mathrm{Ker}(f')$ and only using the concept "modulo $\mathfrak{J}'$".

Theorem 11. *Let* $M=\sum_{\alpha\in K}\oplus M_\alpha=\sum_{\beta\in J}\oplus N_\beta$ *be any two direct sum decompositions of a module M in* $\mathfrak{A}$ *into completely indecomposable modules* M_α*'s and* $N\beta$*'s, respectively and assume that* $S_M\Pi\,\mathfrak{J}'=J(S_M)$. *Then, for any subset* K' *of* K, *there exists a one-to-one mapping* φ *of* K' *into* J *such that* $M=\sum_{\alpha'\in K'}\oplus N_{\varphi(\alpha')}\oplus\sum_{\alpha''\in K-K'}\oplus M_{\alpha''}$ *and* $M_{\alpha'}\approx N_{\varphi(\alpha')}$ *for* $\alpha\in K'$.

Proof. For any subset K' of K, we put $M_0=\sum_{\alpha''\in K-K'}\oplus M_{\alpha''}$. Then, there exists a maximal member M* in the set $\{M_0\oplus\sum_{k\in J_i}\oplus N_{\gamma_k}\}_{i\in I}$ of locally direct summands of M with each subset J_i of J, by Zorn's lemma. Since M is the only one dense submodule of M by Corollary 2 to Lemma 8, M* is a direct summand of M, say, $M=M^*\oplus M'$ for some submodule M' of M. By Corollary 2 to Lemma 9, M' is in $\mathfrak{A}$ if $M'\neq 0$. And so by [2] there exists some N_β such that $M^*\oplus N_\beta$ is a direct summand of M. This contradiction shows that $M^*=M$. Since $\sum_{\alpha'\in K'}\oplus M_{\alpha'}\approx M/M_0\approx\sum_{\gamma'\in J'}\oplus N_{\gamma'}$ with some subset J' of J, by [2] we can find a one-to-one mapping 99 of K' onto J' such that $M_{\alpha'}\approx N_{\varphi(\alpha')}$ for $\alpha'\in K'$.

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