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Osaka J. Math.
10 (1973), 585–596

PERFECT CATEGORIES IV

(QUASI-FROBENIUS CATEGORIES)

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(Received November 8, 1972)

The author defined perfect Grothendieck categories and studied them [11]. In [12], [13] he developed [11] and determined hereditary perfect categories and hereditary perfect and QF-3 categories.

In this note, as a continuos work we define quasi-Frobenius categories (briefly QF) and generalize some properties of QF-rings.

Let $\mathfrak{A}$ be a Grothendieck category. We always assume $\mathfrak{A}$ contains a generating set $\{G_\alpha\}_I$ of small objects G_α, *e.g.* functor categories. If every projective objects in $\mathfrak{A}$ are injective, we call $\mathfrak{A}$ a QF-category. As we see in examples of QF-categories, some important properties of QF-rings are not inherited to QF-categories.

The object of this paper is to fill those gaps. We assume mainly that G_α's are projective, then QF-categories are perfect. It is clear that all of results in the category $\mathfrak{M}_R$ of modules over a ring R with identity are not valid in perfect categories $\mathfrak{A}$. However, modifying proofs in $\mathfrak{M}_R$, we sometimes succeed to extend some properties in $\mathfrak{M}_R$ to $\mathfrak{A}$. All of theorems in this note are well known in $\mathfrak{M}_R$ and so we shall give often only methods how to modify proofs in $\mathfrak{M}_R$.

In §1 we generalize the notion of Σ-injective [5] and obtain [5], Proposition 3 in $\mathfrak{A}$. We define a QF-category in §2 and generalize results in [4] and [14]. In §3 we deal with a problem whether a QF-category has the following property or not: every injetives are projective, (see [6]). In the final sction, we give some supplementary results of [10].

In this paper, rings S need not to have the identity, unless otherwise stated. We refer the readr to [11], [12] and [13] for notations and definitions.

1. Σ-injective

Let $\mathfrak{A}$ be a Grothendieck category. We always assume that $\mathfrak{A}$ has a generating set $\{G_\alpha\}_I$ of small objects G_α.

1) See [11] and [12] for the definitions.

Let M, N be objects in $\mathfrak{A}$ and $S=[N, N]$. Then $[M, N]$ is a left S-module. Let M_0 be a subobject of M. By $l_{[M,N]}(M_0)$ (bri.fly $l(M_0)$) we denote the left S-submodule of $[M, N]$ whose elements consist of all f such that $f|M_0=0$. By $l(M, N)$ we denote the set of such annihilator submodules of $[M, N]$. Conversely, for any left S-submodule K of $[M, N]$ we denote the subobject $\bigcap_{k\in K} \mathrm{Ker}\, k$ by $r_M(K)$ (briefly $r(K)$). Finally, by $r(M, N)$ we denote the set of such annihilator subobjects in M.

The following lemma is well known in the category $\mathfrak{M}_T$ of T-modules over a ring T with identity and we can prove it by modifying the proof of [7], Lemma 1 in p. 136.

Lemma 1 (Baer's condition). *An object Q in $\mathfrak{A}$ is injective if and only if any $f\in[G, Q]$ is extended to an element in $[G_\alpha, Q]$ for any subobject G of G_α, $\alpha\in I$.*

Following to Faith [5], we call an object Q *Σ-injective* if any coproducts of Q itself are injective.

The following results are some versions of [1] and [5] in $\mathfrak{A}$.

Lemma 2 ([5]). *Let M, N be objects in $\mathfrak{A}$. We assume that $r(M, N)$ is noetherian. Then for any subobject M_1 of M there exists a small subobject M_1' of M such that $l(M_1)=l(M_1')$.*

Proof. Since $r(M, N)$ is noetherian, $l(M, N)$ is artinian. From the assumption $M_1=\cup M_\alpha$, where M_α's are small objects. Then $l(M_1)=\bigcap_\alpha l(M_\alpha)=\bigcap_{i=1}^n l(M_{\alpha_i})$, since $l(M, N)$ is artinian. Hence, $l(M_1)=l(\bigcup_1^n M_{\alpha_i})$.

Theorem 1 ([1], [5]). *Let $\mathfrak{A}$ be a Grothendieck category with generating set $\{G_\alpha\}_I$ of small objects and let $Q, \{Q_\beta\}_J$ be a set of injective objects in $\mathfrak{A}$. Then*

1) *If Q is Σ-injective, $r(P, Q)$ is noetherian for any small object P. Conversely, if $r(G_\alpha, Q)$ is noetherian for all G_α, then Q is Σ-injective.*

2) *$\sum_J \oplus Q_\beta$ is injective if and only if for any $\alpha\in I$ and any chain $T_1\subsetneqq T_2\subsetneqq\cdots\subsetneqq T_n\subsetneqq\cdots$ of subobjects of G_α, there exist n_0 and a finite subset J_0 of J such that $[T_{n+1}/T_n, Q_\gamma]=0$ for all $n\geqslant n_0$ and $r\in J-J_0$.*

Proof. We assume that Q is Σ-injective and $r(P, Q)$ is not noetherian for a small object P. Let $P_1\subsetneqq P_2\subsetneqq\cdots\subsetneqq P_n\subsetneqq\cdots$ be a chain in $r(P, Q)$. Put $P_0=\cup P_i$ and let $f_i\in l(P_i)-l(P_{i+1})$. Then $f_i(P_j)=0$ for $j\leqslant i$ and $f_i(P_k)\neq 0$ for $k\geqslant i+1$. Put $f=\prod f_i\in[P_0, \prod Q]$. Since $f(P_i)\subset\sum\oplus Q$ and $P_0=\lim P_i$, $f(P_0)\subset \sum\oplus Q$. However, P is small and so $\mathrm{Im}\, f\subset \sum_1^m \oplus Q$, which contradicts to a fact $f|P_{m+1}=\sum_{i=1}^{m+1} f_i|P_{m+1}\not\subset \sum_1^m \oplus Q$. Hence, $r(P, Q)$ is noetherian. Conversely, we assume that $r(G_\alpha, Q)$ is noetherian for all $\alpha\in I$. We consider a diagram for a subobject P of G_α

$$\begin{array}{ccc} 0 \to & P & \to G_\alpha . \\ & \downarrow f & \\ \sum_J \oplus Q & & \end{array}$$

Let π_γ be the projection of $\sum_J \oplus Q$ to the γ-th compoment Q. From Lemma 2 we obtain a small subobject P' of G_α such that $l(P)=l(P')$. Since P' is small, $\pi_\gamma f\,|\,P'=0$ for almost all $\gamma\in J$. Hence, $\pi_\gamma f\,|\,P=0$ for almost all γ, which means that $\mathrm{Im} f\subset \sum_1^m \oplus Q$. Therefore, f is extended to an element in $[G_\alpha, \sum_J \oplus Q]$, since Q is injective. Hence, $\sum_J \oplus Q$ is injective by Lemma 1. We can prove 2) similarly to the case of modules.

Corollary 1. *Let Q be a Σ-injective and small object in $\mathfrak{A}$. Then $[Q, Q]$ is a semi-primary ring.*

Proof. It is clear from Theorem 1 and [10], Theorem 1.

Corollary 2 ([2]). *Let $\{Q_\alpha\}_J$ be a set of Σ-injectives. If $\sum_J \oplus Q_\alpha$ is injective, $\sum_J \oplus Q_\alpha$ is Σ-injective.*

Proof. It is clear from Theorem 1.

From Chase's method [3] and Theorem 1 we obtain

Corollary 3 ([3]). *Let $\mathfrak{A}$ be as above. Then every injectives are Σ-injective if and only if $\mathfrak{A}$ is locally noetherian.*

2. *QF*-categories

We have many characterizations of quasi-Frobenius rings R with identities. The categorical ones among them are

I *Every projective modules is injective* [5] and
II *Every injective module is projective* [6].

We shall define a quasi-Frobenius category by taking the property I. Let $\mathfrak{A}$ be a Grothendieck category with generating set $\{G_\alpha\}_I$ of small objects. $\mathfrak{A}$ is called QF if every projectives are injective.

First, we have

Proposition 1. *Let $\mathfrak{A}$ be as above and G_α projective for all $\alpha\in I$. Then $\mathfrak{A}$ is QF if and only if $\mathfrak{A}$ is perfect and $\sum_I \oplus G_\alpha$ is Σ-injective.*

Proof. We assume $\mathfrak{A}$ is QF. Then G_α is Σ-injective. Hence, G_α is a coproduct of completely indecomposable objects $\{P_\alpha^{(i)}\}$ by Corollary 1 to Theorem 1. Furthermore, since $\sum_K \oplus G_\alpha$ is injective, for any K, $\{P_\alpha^{(i)}\}_{\alpha,i}$ is a right T-niloptent system by [9], Corollary to Proposition 10. Hence, $\mathfrak{A}$ is perfect from [11], Corollary 1 to Theorem 4. Conversely, if $\mathfrak{A}$ is perfect, $\mathfrak{A}$ contains a generating set $\{P'_{\alpha'}\}_{I'}$ of small projectives and every projectives are coproduct of some family of $P'_{\alpha'}$ [11], §3. On the other hand $\sum_{I'} \oplus P'_{\alpha'}$ is Σ-injective if so is $\sum_I \oplus G_\alpha$. Therefore, $\mathfrak{A}$ is QF.

We know many interesting properties of a QF-ring and in this note we shall generalize some of them in $\mathfrak{A}$.

First, we shall give examples of QF-Grothendieck categories.

EXAMPLE 1. Let $\{K_i\}$ be a family of QF-rings. Then $\prod_i \mathfrak{M}_{R_i}$ is QF.

The following example is a slight modification of [18], p. 379.

EXAMPLE 2. Let K be a field and R be a vector space over K with basis $\{e_i, f_i\}$: $R=\sum_1^\infty \oplus(e_iK\oplus f_iK)$. We define a multiplication in R as follows:

$$e_ie_j = \delta_{i,j}e_i,\ e_if_j = \delta_{i,j}f_i,\ f_ie_j = \delta_{i,j-1}f_i \quad \text{and} \quad f_if_j = 0\,,$$

where $\delta_{i,j}$ is the Kronecker δ.
It is easily seen that R is an associative ring and $R=\sum \oplus e_iR=\sum \oplus Re_i$. Since $e_iR=e_iK\oplus f_iK$ is an artinian and noetherian R-module, $\mathfrak{M}_R^{+1)}$ is a locally artinian and noetherian perfect Grothendieck category from [11], §3. We shall show that e_iR is injective in $\mathfrak{M}_R^+$. Every e_iR has only one proper submodule f_iK. Let g be in $[f_jK, e_iR]$. It is clear that $g=0$ if $i\neq i$. It $i=j$, $g(f_i)=f_ik$ for some k in K. Hence, $e_ik\in[e_iR, e_iR]$ and $e_ik\,|\,f_iK=g$. Therefore, e_iR is injective by Lemma 1. Hence, $\mathfrak{M}_R^+$ is a perfect QF-category by Corollary 3 and Proposition 1.

Next, we consider ${}_R\mathfrak{M}^+$. Let g be an element in $[Kf_1, Re_1]$ such that $g(f_1)=e_1$. Then it is clear that g is not extended to an element in $[Re_2, Re_1]$. Hence, Re_1 is not injective in ${}_R\mathfrak{M}^+$. On the other hand, all of other Re_i are injective as above. Thus, ${}_R\mathfrak{M}^+$ is a QF-3 perfect category from [13], but not QF. Furthermore, R is a cogenerator in ${}_R\mathfrak{M}^+$, but not in $\mathfrak{M}_R^+$.

This example shows that a perfect QF-category does not inherit some properties of QF-rings, Furthermore, the example given in [9], p. 331 is a QF-Grothendieck category with generator and cogenerator object, however it is neither locally noetherian nor artinian (this category does not contain a generating set of small objects).

We do not know whether QF-categories with generating set of small objects are locally noetherian (or artinian).

Let $\mathfrak{A}$ be the category as above. Put $G=\sum_I \oplus G_\alpha$ and $S=[G, G]=\prod_\alpha [G_\alpha, G]$. Then S contains the ring $R=\sum_{\alpha,\beta} \oplus[G_\alpha, G_\beta]$. Let $\prod_\alpha f_\alpha$, $\prod_\alpha g_\alpha$ be elements in S. Since G_α is small, $\sum_\gamma f_\gamma g_\alpha$ is in $[G_\alpha, G]$. Hence, $(\prod f_\alpha)(\prod g_\alpha)=\prod_\alpha (\sum_\gamma f_\gamma g_\alpha)$. If $\prod g_\alpha$ is in R, $g_\alpha=0$ for almost all α. Hence, $(\prod f_\alpha)(\prod g_\alpha)\in R$ and $SR\subset R$. For any subobject Q of G we put $l_R(Q)=l_S(Q)\cap R$.

Lemma 3. *Let G and G_α be as above. For any subobject Q of G_α we have* $r(l_R(Q))=r(l_S(Q))$.

Proof. Put $G_\alpha=G_1$, $G_2=\sum_{\alpha\neq\beta} \oplus G_\beta$ and $S_{ij}=[G_j, G_i]$. Then $S=\sum_{i,j=1}^{2} \oplus S_{ij}$ and S_{i1} are in R. $l_S(Q)=\sum_{i=1}^{2} S_{i2}\oplus \sum_{i=1}^{2} (l_S(Q)\cap S_{i1})$. Then $r(l_S(Q))=r(\sum_i \oplus S_{i2})\cap r(T)=G_1\cap r(T)$, where $T=\sum_{i=1}^{2} (l_S(Q)\cap S_{i1})$. On the other hand, $l_R(Q)=(R\cap \sum_{i=1}^{2} S_{i2})\oplus T$ and $r(l_R(Q))=G_1\cap r(T)=r(l_S(Q))$.

Proposition 2. *Let $\mathfrak{A}$ be the Grothendieck category with G_α. If G_α is Σ-injective for all α and G is an injective cogenerator, then $\mathfrak{A}$ is locally noetherian.*

Proof. Let S and R be as above. Then $r(l_S(Q))=Q$ for any subobject Q of G by the assumption, (cf. [10], §2). Put $R=[G_\alpha, G]\oplus \sum_{\alpha\neq\beta} \oplus[G_\beta, G]$. Then for any subobject Q' of G_α, $l_R(Q')=l_R(Q')\cap [G_\alpha, G]\oplus \sum_{\alpha\neq\beta} \oplus[G_\beta, G]$. Hence, $Q'=r(l_S(Q'))=r(l(Q'))=r(l_R(Q')\cap [G_\alpha, G])\cap r(\sum_{\alpha\neq\beta}\oplus[G_\beta, G])=G_\alpha\cap r(l_{[G_\alpha,G]}(Q'))$ $=r_{G_\alpha}(l_{[G_\alpha,G]}(Q'))$. Since G is Σ-injective, G_α is noetherian from Theorem 1.

Corollary. *Let $R=\sum_I \oplus e_\alpha R=\sum_I \oplus Re_\alpha$ be the induced ring from a category.*[1] *We assume that $\mathfrak{M}_R^+$ is QF and R is a cogenerator in $\mathfrak{M}_R^+$. If for a given α, $e_\alpha Re_\beta=0$ for almost all $\beta\in I$, $e_\alpha R$ is artinian and noetherian.*

Proof. There exists an idempotent $E=e_\alpha+e_{\alpha_2}+\cdots+e_{\alpha_n}$ such that $e_\alpha R=e_\alpha RE\subset ERE$. ER is noetherian by Proposition 2. Hence, ERE is right noetherian and semi-primary by [10], Theorem 1. Therefore, ERE is right artinian and so $e_\alpha R$ is artinian as an R-module.

The following theorem is a version of [14] in $\mathfrak{A}$.

Theorem 2 ([14]). *Let $\mathfrak{A}$ be a locally noetherian category with generating set $\{P_\alpha\}_I$ of small projectives. We put $P=\sum_I \oplus P_\alpha$ and $S=[P, P]$. Then $\mathfrak{A}$ is QF if and only if*

1) *For any $\alpha\in I$ and any finitely generated S-module $\mathfrak{l}$ of $[P_\alpha, P]$ $l(r(\mathfrak{l}))=\mathfrak{l}$.*

2) *For any $\alpha\in I$ and any subobjects P_1, P_2 in P_α $l(P_1\cap P_2)=l(P_1)+l(P_1)$ in $[P_\alpha, P]$.*

Proof. Let Q be an injective object. Then 1) and 2) are valid if we replace P and S by Q and $[Q, Q]$ (cf. [10]). We assume 1) and 2) and show that P_α is injective. Let P_1 be a subobject of P_β such that $P_1=\operatorname{Im} x$; $x\in[P_\gamma, P_\beta]$, We may assume $x\in[P_\gamma, P]$. Let f be in $[P_1, P_\alpha]$. $x\colon P_\gamma \overset{x'}{\to} P_1 \overset{i}{\to} P_\beta$ and put $K=\operatorname{Ker} x'$. Then $r_{P_\gamma}(x)=K$. Since $l(K)=l(r(x))=Sx$ and $fx_1\in l(K)$, $fx'=sx$ for some $s\in S$. We may assume $s\in[P_\beta, P_\alpha]$. Then, $f=si$ and $s\,|\,P=f$. Since P_β is noetherian, every subobject of P_β is of form $\bigcup_1^n \operatorname{Im} x_i$; $x_i\in[P_{\gamma_i}, P]$. We can prove, analogously to the case of modules, from 2) that every element in $[P', P_\alpha]$ is extended to one in $[P_\beta, P_\alpha]$, (cf. [10]). Hence, P_α is injective by Lemma 1. Since $\mathfrak{A}$ is locally noetherian, $\mathfrak{A}$ is perfect from Corollary 3 to Theorem 1 and [9], Corollary to Proposition 10. Hence, $\mathfrak{A}$ is QF by Proposition 1.

Let T be a ring with identity. If T is right artinian and self injective as a right T-module, then T is QF and T is left artinian and self injective as a left T-module. However, as shown in Example 2, this fact is not true for $\mathfrak{A}$.

Theorem 3. *Let $R=\sum_I \oplus e_\alpha R=\sum_I \oplus Re_\alpha$ be the induced ring from the category $\mathfrak{A}$. Then the following are equivalent.*

1) *$\mathfrak{M}_R^+$ and ${}_R\mathfrak{M}^+$ are QF.*
2) *$\mathfrak{M}_R^+$ is locally noetherian and R is injective in $\mathfrak{M}_R^+$ and ${}_R\mathfrak{M}^+$.*
3) *$\mathfrak{M}_R^+$ is QF and R is injective in ${}_R\mathfrak{M}^+$.*
4) *$\mathfrak{M}_R^+$ is QF and locally artinian and R is a cogenerator in $\mathfrak{M}_R^+$*, (cf. [4]).

Proof. We first show the following fact. If $\mathfrak{M}_R^+$ is QF and R is injective in ${}_R\mathfrak{M}^+$, then R is a cogenerator in $\mathfrak{M}_R^+$. We may assume e_α's are primitive. From the remark before Lemma 3 and the first part of the proof of Theorem 2, we have $rl(\mathfrak{r}')=\mathfrak{r}'$ for any finitely generated right R-module $\mathfrak{r}'$ in $[Re_\alpha, R]=e_\alpha R$. Let $\mathfrak{r}$ be any right R-module in $e_\alpha R$. Then $l(\mathfrak{r})=\cap\, l(\mathfrak{r}')$, where $\mathfrak{r}'$ runs through all finitely generated R-modules. Since $\mathfrak{M}_R^+$ is perfect from the assumption and Proposition 1, ${}_R\mathfrak{M}^+$ is semi-artinian by [11], Theorem 5. Hence, Re_α contains a unique minimal submodule S_α. Since $l(\mathfrak{r}')\neq 0$, $l(\mathfrak{r})=\cap\, l(\mathfrak{r}')\supseteq S_\alpha\neq 0$. Therefore, $e_\alpha R/\mathfrak{r}$ is contained in R and hence, R is a cogenerator in $\mathfrak{M}_R^+$.

1)→2), 3) and 4). Since ${}_R\mathfrak{M}^+$ is perfect, $\mathfrak{M}_R^+$ is semi-artinian. On the other hand, R is a cogenerator in $\mathfrak{M}_R^+$ from the above. Hence, $\mathfrak{M}_R^+$ is locally neotherian and artinian by Proposition 2. Therefore, 1) implies 2), 3) and 4).

2)→3) and 4). Since $e_\alpha R$ is injective and noetherian, $e_\alpha Re_\alpha$ is semi-primary by [10], Theorem 1. Furthermore, we may assume that $e_\alpha R$'s are indecomposable. Then so are the Re_α's. Since $R=\sum_I \oplus Re_\alpha$ is injective, $\{Re_\alpha\}_I$ is a semi-T-niloptent system by [9], Corollary to Proposition 10. However, $e_\alpha Re_\alpha$ is semi-

primary and hence, $\{Re_\alpha\}_I$ is a T-nilpotent system. Therefore, ${}_R\mathfrak{M}^+$ is perfect. Similarly, we obtain from Corollary 3 to Theorem 1 that $\mathfrak{M}_R^+$ is QF. Hence 2) implies 3) and 4) from the first statement. 3)→2). R is a cogenerator in $\mathfrak{M}_R^+$ from the first remark. Hence, $\mathfrak{M}_R^+$ is locally noetherian.
4)→1). We may assume that $e_\alpha R$ is perfect for all α. We note $[e_\alpha R, R]=Re_\alpha$ and $[Re_\alpha, R]=e_\alpha R$. Since R is an injective cogenerator in $\mathfrak{M}_R^+$, $r_{e_\alpha R}(l_{Re_\alpha}(\mathfrak{r}))=\mathfrak{r}$ for any R-submodule $\mathfrak{r}$ in $e_\alpha R$ and $l_{Re_\alpha}(r_{e_\alpha R}(\mathfrak{l}))=\mathfrak{l}$ for a finitely generated left R-submodule $\mathfrak{l}$ of Re_α. Hence, Re_α is noetherian by the assumption and artinian from Proposition 2 and the above. Moreover, the above facts imply, from Theorem 2 and the remark, that ${}_R\mathfrak{M}^+$ is QF.

Corollary. *Let R be as above. We assume that R is a cogenerator in $\mathfrak{M}_R^+$ and ${}_R\mathfrak{M}^+$, and $\mathfrak{M}_R^+$ is locally noetherian. Then the following are equivalent.*

1) *R is injective in ${}_R\mathfrak{M}^+$.*
2) *${}_R\mathfrak{M}^+$ is locally noetherian.*
3) *$\mathfrak{M}_R^+$ is locally artinian.*

In those cases $\mathfrak{M}_R^+$ and ${}_R\mathfrak{M}^+$ are QF, (cf. [17], p. 406).

Proof. We first show that $\mathfrak{M}_R^+$ is QF from the assumption. We quote here the idea of Kasch [17]. Let E be an injective hull of R in $\mathfrak{M}_R^+$. We put $\mathfrak{r}=\cup \operatorname{Im} f$, $f\in[E, R]$, then $\mathfrak{r}$ is a two-sided ideal of R. If $\mathfrak{r}\neq R$, there exists $s\neq 0$ in $[R, R]$ as left R-modules such that $\mathfrak{r}^s=0$, since R is a cogenerator in ${}_R\mathfrak{M}^+$. We take an idempotent e_α in R such that $e_\alpha{}^s\neq 0$. Then for any $f\in[E, R]$, $0=(f(e_\alpha))^s=(f(e_\alpha)e_\alpha)^s=f(e_\alpha)e_\alpha{}^s=f(e_\alpha{}^s)$. On the other hand, R is a cogenerator in $\mathfrak{M}_R^+$. Hence, we have shown $\mathfrak{r}=R$, which implies that R is a retract of $\sum_{[E,R]}\oplus E$. Since R is locally noetherian, R is injective in $\mathfrak{M}_R^+$. Therefore, $\mathfrak{M}_R^+$ is QF. Similarly, we can prove that ${}_R\mathfrak{M}^+$ is QF if ${}_R\mathfrak{M}^+$ is locally noetherian. Hence, 2) implies 1) and 3). The remaning parts are clear from Theorem 3.

3. Property II

In this section, we shall study a relation between the property II and a QF-category. Faith and Walker [6] showed that a ring T with identity is QF if and only if II is satisfied. However, the following examples show that the above fact is not true for Grothendieck categories.

EXAMPLE 3. In Example 2 we replace relations $f_ie_j=\delta_{i,j+1}f_i$ and $e_if_i=f_i$ by $e_jf_i=\delta_{j-1,i}f_i$ and $f_ie_i=f_i$, respectively. Then $R=\sum\oplus e_iR=\sum\oplus Re_i$ is perfect and locally artinian and noetherian. We can show that e_iR for $i\geqslant 2$ are injective. Let E be an injective object in $\mathfrak{M}_R^+$. Then E contains non-zero homomorphic image of some e_iR. Hence, E contains e_iR or $e_{i+1}R$ as an isomorphic image. Therefore, $E\approx\sum_{i\geqslant 2}\oplus e_iR^{(\alpha)}$, since R is locally noetherian. Thus,

R satisfies II and $\sum_{i\geq 2}\oplus e_i R$ is an injective cogenerator in $\mathfrak{M}_R^+$. On the other hand, e_1R is not injective and hence, $\mathfrak{M}_R^+$ is not QF.

In Example 2 an injective hull $E(e_1R/e_1N)$ of e_1R/e_1N is not projective and hence, $\mathfrak{M}_R^+$ does not satisfy II. On the other hand,

EXAMPLE 4. Let R be a vector space over a field K with basis $\{e_i, f_i\}_{-\infty}^{\infty}$ and define the multiplication in R in Example 2. Then $\mathfrak{M}_R^+$ and $_R\mathfrak{M}^+$ are QF and R is a cogenerator in $\mathfrak{M}_R^+$ and $_R\mathfrak{M}^+$.

Let $\mathfrak{A}$ be a Grothendieck category with a generating set $\{P_\alpha\}_I$ of small projectives. We assume $\mathfrak{A}$ satisfies II. Then considering the induced ring from $\mathfrak{A}$, we can show from [6], Theorem 1.1 that $\mathfrak{A}$ is locally noetherian. Thus, we have from the argument in Example 3

Proposition 3. *Let $\mathfrak{A}$ be as above. Then $\mathfrak{A}$ satisfies* II *if and only if $\mathfrak{A}$ is locally noetherian and every indecomposable injective object is projective.*

Corollary 1. *Let $\mathfrak{A}$ be as above. If $\mathfrak{A}$ satisfies* II, *$P=\sum_I\oplus P_\alpha$ is a cogenerator in $\mathfrak{M}_R^+$. Conversely, if $\mathfrak{A}$ is locally noetherian and artinian and P is a cogenerator, then $\mathfrak{A}$ satisfies* II.

Proof. We assume $\mathfrak{A}$ satisfies II. Then for any minimal object S_α in $\mathfrak{M}_R^+$, $E_\alpha=E(S_\alpha)$ is projective indecomposable. Hence, E_α is isomorphic to a retract of some P_β by [21], Lemma 2. Since P_α's are finitely generated, $\sum\oplus E_\alpha$ is a cogenerator. Therefore, P is a cogenerator. Conversely, we assume $\mathfrak{A}$ is locally noetherian and artinian. Every indecomposable injetive E is the injective hull of its socle. If P is a cogenerator, E is a retract of P. Hence, E is projective. Therefore, $\mathfrak{A}$ satisfies II by Proposition 3.

Corollary 2. *Let $\mathfrak{A}$ be as above. We assume $\mathfrak{A}$ is QF and semi-artinian. Then $P=\sum_I\oplus P_\alpha$ is a cogenerator if and only if $\mathfrak{A}$ satisfies* II.

Proof. If P is a cogenerator, $\mathfrak{A}$ is locally noetherian, and hence, locally artinian by the assumption.

The following lemma is essentially due to Faith and Walker [6]. However, we shall give the proof as an application of [9], Theorem 1.

Lemma 4 ([6]). *Let R be the induced ring from a category and let $\{E_\alpha\}_L$ be a set of projective, injective and indecomposable objects in $\mathfrak{M}_R^+$. Then every coproducts P of any family of E_α's are injective if and only if* E(P) *is projective for all P.*

Proof. "Only if" part is clear. We denote the cardinal number of a set K' by $|K'|$. Let $\zeta=|R|$ and K a countably infinite set. We put $M=\sum_{i\in K}\oplus E_i$;

$E_i \in \{E_i\}_L$, (E_i may be equal to E_j). Since E_i is projective, $E_\alpha \approx f_\alpha R$ for some primitive idempotent f_α in R by [11], Corollary to Lemma 2. Let J be a set of $|J| = \max(\zeta, \aleph_0) = \xi$ and put $M^\xi = \sum_{\alpha\in J} \oplus M^{(\alpha)}$; $M^{(\alpha)} \approx M$. Then $M^\xi = \sum_{\beta\in K} \sum_{\delta\in J_\beta} \oplus (f_\beta R)^{(\delta)}$; $(f_\beta R)^{(\delta)} \approx f_\beta R$ and $|J_\beta| = \xi$. Let $E = E(M^\xi)$. Since E is projective by the assumption, E is a retract of a form $\sum_{\varepsilon\in T} \oplus e_\delta R$. Hence, $E \approx \sum \oplus g_\varepsilon R$ and $e_\varepsilon R = g_\varepsilon R \oplus g_\varepsilon' R$ by [21], Lemma 2. Now, we consider those injective modules in the category $\mathfrak{C}$ of injective modules modulo the radical of $\mathfrak{C}$ defined in [9], §1. Then $\sum_K \sum_{J_\beta} \oplus \overline{(f_\beta R)}^{(\delta)} = \sum_T \oplus \overline{g_\varepsilon R}$, where $\overline{f_\beta R}$ and $\overline{g_\varepsilon R}$ mean the residue classes of $f_\beta R$ and $g_\varepsilon R$, respectively. Since $\overline{f_\beta R}$ is minimal in $\mathfrak{C}$, $\overline{g_\varepsilon R} \approx \sum_{K\in\beta} \sum_{J_\beta'(\varepsilon)} \oplus \overline{f_\beta R}^{(\delta)}$, $J_\beta'(\varepsilon) \subseteq J_\beta$, which means that $g_\varepsilon R = E(\sum_K \sum_{J_\beta'} \oplus (f_\beta R)^{(\delta)})$. Hence, every element φ in $[\sum_{J_\beta'} \oplus (f_\beta R)^{(\delta)}, \sum_{J_\beta'} \oplus (f_\beta R)^{(\delta)}]$ is extended to φ' in $[g_\varepsilon R, g_\varepsilon R] = g_\varepsilon R g_\varepsilon$. On the other hand, $[\sum_{J_\beta'} \oplus (f_\beta R)^{(\delta)}, \sum_{J_\beta'} \oplus (f_\beta R)^{(\delta)}] = \prod_{J_\beta'} [(f_\beta R)^{(\delta)}, \sum_{J_\beta'} \oplus (f_\beta R)^{(\delta)}] \supset \prod_{J_\beta'} f_\beta R f_\beta$ and $\zeta \geqslant |g_\varepsilon R g_\varepsilon| \geqslant 2^{|J'_\beta(\varepsilon)|}$. Hence, $|J_\beta'(\varepsilon)| < \zeta \leqslant \xi$. Next, we take an index β in K and consider the subset $T_\beta = \{\varepsilon \mid \in T, J_\beta'(\varepsilon) \neq \phi\}$. Since $J_\beta = \bigcup_{\varepsilon\in T_\beta} J_\beta'(\varepsilon)$, $|T_\beta| = |J_\beta| = \xi \geqslant \aleph_0$. Hence, for each β we can find an index $\varepsilon(\beta)$ in T_β such that $\varepsilon(\beta) \neq \varepsilon(\beta')$ if $\beta \neq \beta'$. Therefore, $\sum_{\alpha\in K} \oplus f_\alpha R$ is a retract of E and hence, M is injective. Thus, we have proved the lemma by virtue of the proof of Theorem 2, (see [5]).

Proposition 4. *Let $\mathfrak{A}$ be the Grothendieck category with generating set $\{P_\alpha\}_I$ of small projectives. We assume $\mathfrak{A}$ satisfies* II. *Then the representative class $\{S_\gamma\}_K$ of the minimal objects is a set. Furthermore,* $E(\sum_K \oplus S_\gamma)/(J(E(\sum_K \oplus S_\gamma)) \approx \sum_K \oplus S_\gamma$ *if and only if $\mathfrak{A}$ is QF and every projective contains the non-zero socle, where $J(\ )$ means the Jacobson radical.*

Proof. "Only if". We take the induced ring R from $\mathfrak{A}$. Let S_α be an minimal object (cf. [11], Proposition 2) and $E_\alpha = E(S_\alpha)$. Then $E_\alpha \approx f_\alpha R$ by the assumption. Hence, $\{S_\alpha\}_K$ is a set. It is clear that $\bigcup_K E_\alpha = \sum_K \oplus E_\alpha$ and $\sum \oplus E_\alpha$ is injective by Lemma 4. Therefore, $E(\sum \oplus S_\alpha) = \sum \oplus E_\alpha$. We assume any $S_\alpha \approx E_\alpha'/J(E_\alpha') = f_\alpha' R/f_\alpha' J(R)$. Let P be projective. Then P contains a maximal subobject P_0 by [11], Proposition 2. $P/P_0 \approx E_\alpha/J(E_\alpha)$ for some α by the assumption. Since E_α is perfect, E_α is a retract of P. We consider the set of submodules in R which are coproducts of some E_α's. Using the Zorn's lemma and the above fact, we know $R = \sum \oplus E_\alpha$. Hence, $\mathfrak{M}_R^+$ is QF and every projective contains a minimal module. "If". We assume the above properties, then $E = E(\sum_K \oplus S_\alpha) \approx \sum_K \oplus e_\alpha R$ and every indecomposable projective P is isomorphic to $e_\alpha R$ for some $\alpha \in K$. Hence, $E/J(E) \approx \sum \oplus S_\alpha$.

We shall apply the above to a ring with identity.

Theorem 4 ([6]). *Let S be a ring with identity. Then S is a QF-ring if and only if* II *is satisfied in* $\mathfrak{M}_R$.

Proof. "Only if" part is clear from Corollary 1 to Proposition 3. We assume II. Then $\sum_K \oplus E_\alpha$ is a direct summand of S as the proof of Proposition 4. Hence, K is finite. Therefore, $E/J(E)\approx\sum_K \oplus S_\alpha$.

Finally, we shall consider the category of covariant additive functors $(\mathfrak{C}, Ab)$, where $\mathfrak{C}$ is a small abelian category.

Proposition 5. *Let $\mathfrak{C}$ be a small abelian category. Then the following are equivalent.*

1) $(\mathfrak{C}, Ab)$ *is semi-simple (completely reducible).*
2) $(\mathfrak{C}, Ab)$ *is QF.*
3) $(\mathfrak{C}, Ab)$ *satisfies* II.

In such a case, every object in $\mathfrak{C}$ is a finite coproduct of minimal objects.

Proof. Put $\mathfrak{A}=(\mathfrak{C}, Ab)$ and $H^C=[C, -]$ for $C\in\mathfrak{C}$, then $\{H^C\}_{C\in\mathfrak{C}}$ is a generating set of small projectives of $\mathfrak{A}$. We assume $\mathfrak{A}$ is QF. Then $\mathfrak{A}$ is perfect By Theorem 1. Hence, every object C in $\mathfrak{C}$ is a finite coproduct of completely indecomposable objects $\{C_\alpha\}$ by [11], Proposition 5. Furthemore, since H^C is injective in $\mathfrak{A}$, C is projective in $\mathfrak{C}$ by [15], p. 100, Proposition 2.3. Hence, every C_α is minimal and $\mathfrak{A}$ is semi-simple by [20], Proposition 5. Next, we assume $\mathfrak{A}$ satisfies II. Then $\mathfrak{A}$ is locally noetherian by Proposition 3. Hence, $\mathfrak{C}$ is artinain. Let C be minimal in $\mathfrak{C}$. Then we can easily see that H^C is minimal in $\mathfrak{A}$, (cf. [20]). Let $C\supset C_1$ be objects in $\mathfrak{C}$ and C_1 minimal. Then $0\to H^{C/C_1}\to H^C\to H^{C_1}\to 0$ is exact, since H^{C_1} is minimal. Hence, C_1 is a retract of C, since H^{C_1} is projective. Therefore, $\mathfrak{C}$ is semi-simple and artinian, which implies $\mathfrak{A}$ is semi-simple from [20], Proposition 5.

We note that every perfect Grothendieck category $\mathfrak{A}$ is equivalent to $(\mathfrak{C}^0, Ab)$ by [11], Theorem 4, where $\mathfrak{C}$ is a small amenable preadditive category. Hence, if $\mathfrak{A}$ is non semi-simple QF, $\mathfrak{C}$ is not abelian.

4. Projective and injective objects

From the definition of a QF-category, every projectives are injective and so we shall study, in this section, projective, injective objects in the Grothendieck categroy $\mathfrak{A}$ with generating set $\{G_\alpha\}_I$ of small objects. Which is a supplement of [10].

As a dual of weakly distinguished objects [9], we define a weakly co-distinguished object. If an object P in $\mathfrak{A}$ has a property $[P, P_1/P_2]\neq 0$ for any subobjects $P_1 \supset P_2$ of P such that P_1/P_2 is minimal, then P is called *weakly co-distinguished*. Since $\mathfrak{A}$ has $\{G_\alpha\}$, if P is projective, then P is weakly co-distinguished if and only if $[P, P_1] \supsetneqq [P, P_2]$ for any subobjects $P_1 \supsetneqq P_2$ of P.

Put $S=[P, P]$. For any subset T of S $r_S(T)=\{s| \in S,\ Ts=0\}$, $l_S(T)=\{s| \in S,\ sT=0\}$ and $TP=\bigcup_{f \in T} \mathrm{Im}\, f$.

Lemma 5. *Let P be projective and $S=[P, P]$. For any left ideal $\mathfrak{l}$ and right ideal $\mathfrak{r}$ of S, $r_S(\mathfrak{l})=[P, r_P(\mathfrak{l})]$ and $l_S(\mathfrak{r}P)=l_S(\mathfrak{r}P)$.*

Proof. It is clear that $r_S(\mathfrak{l})P \subseteq r_P(\mathfrak{l})$ and $r_S(\mathfrak{l}) \subseteq [P, r_P(\mathfrak{l})]$. Let f be in $[P, r_P(\mathfrak{l})]$. Then $\mathfrak{l}f(P) \subseteq \mathfrak{l}r_P(\mathfrak{l})=0$. Hence $f \in r_S(\mathfrak{l})$. The last statement is clear.

Proposition 6. *Let P be projective and weakly co-distinguished in $\mathfrak{A}$ and $S=[P, P]$. Then*

1) *$r_P(\mathfrak{l})=r_S(\mathfrak{l})P$ for any left ideal $\mathfrak{l}$ in S.*
2) *$P_0=[P, P_0]P$ for any suboject P_0 of P.*

Furthermore, we assume P is injective and weakly distinguished, then

3) *$\mathfrak{l}_S(r_S(l))=l_S(r_P(\mathfrak{l}))=\mathfrak{l}$ for any finitely generated left ideal $\mathfrak{l}$ of S.*
4) *$r_S(l_S(\mathfrak{r})=\mathfrak{r}$ for any finitely generated right ideal $\mathfrak{r}$ of S.*
5) *$r_P(l_S(\mathfrak{r}))=\mathfrak{r}P$ for any right ideal $\mathfrak{r}$ of S.*

Proof. We assume that P is projective and co-distinguished. We have from Lemma 5 that $r_S(\mathfrak{l}) \subseteq [P, r_S(\mathfrak{l})P] \subseteq [P, r_P(\mathfrak{l})]=r_S(\mathfrak{l})$. Hence, $r_P(\mathfrak{l})=r_S(\mathfrak{l})P$. Similarly, we have 2). We further assume P is injective. Then $l_S(r_S(\mathfrak{l}))=l_S(r_S(\mathfrak{l})P)=l_S(r_P(\mathfrak{l}))$ by Lemma 5 and 1). If $\mathfrak{l}$ is finitely generated, $l_S(r_P(\mathfrak{l}))=\mathfrak{l}$, (Theorem 2). Finally, we further assume P is injective and distinguished. $r_P(l_S(\mathfrak{r}))=r_P(l_S(\mathfrak{r}P))=\mathfrak{r}P$ for any right ideal $\mathfrak{r}$ by Lemma 5 and [10]. Hence, if $\mathfrak{r}$ is finitely generated, $\mathfrak{r}=[P, \mathfrak{r}P]=[P, r_P(l_S(\mathfrak{r}))]=r_S(l_S(\mathfrak{r}))$ by Lemma 5 and [8], Lemma 2.6.

Corollary. *Let P and S be as above. Then P is artinian if and only if S is right artinian. Furthermore, if P is injective, the following are equivalent.*

1) *P is artinian.*
2) *P is noetherian.*
3) *S is right noetherian, (artinian).*

If P is projective, injective, weakly distinguished and co-distinguished, then the following are equivalent.

1)~3).

4) *S is left noetherian, (artinian).*
5) *S is a QF-ring.* (cf. [10], Theorem 2, [16], Satz and [19], §3).

Proof. The first statment is clear from Proposition 6, 2) and [11], Corollary 2 to Lemma 2. We assume P is injective. 1)→3). Since S is right artinian from the above, S is noetherian. 3)$\leftrightarrow$2). It is evident from Proposition 6, 2) and [8], Proposition 2.7. 2)$\rightarrow$1). S is semi-primary by [10], Theorem 1 and hence, S is right artinian. Therefore, P is artinian from the first statement. Finally, we assume further that P is weakly distinguished. 1)$\rightarrow$4). It is clear from Proposition 6, 3). Furthermore, S is left artinian, since S is semi-primary. 4)$\rightarrow$1). P is artinian, since P is injective and weakly distinguished. 1)$\leftrightarrow$5). It is clear from the proof of Theorem 2 and Proposition 6, 3) and 4).

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References

[1] J. Beck: *Σ-injective modules*, J. Algebra **21** (1972), 232–249.

[2] A. Caillear: *Une caractérisation des modules Σ-injectifs*, C.R. Acad. Sci. Paris Ser. A. **269** (1969), 997–999.

[3] S.U. Chase: *Direct products of modules*, Trans. Amer. Math. Soc. **97** (1960), 457–473.

[4] S. Eilenberg and T. Nakayama: *On the dimension of modules and algebras* II, Nagoya Math. J. **9** (1955), 1–16.

[5] C. Faith: *Rings with ascending condition on annihilators*, Nagoya Math. J. **27** (1966), 179–191.

[6] ——— and E.A. Walker: *Direct-sum representations of injective modules*, J. Algebra **5** (1967), 203–221.

[7] A. Grothendieck: *Sur quelques points d'algèbre homologique*, Tohoku Math. J. **9** (1957), 119–221.

[8] M. Harada: *On semi-simple abelian categories*, Osaka J. Math. **7** (1970), 89–95.

[9] ——— and Y. Sai: *On categories of indecomposable modules* I, Osaka J. Math. **8** (1971), 309–321.

[10] ——— and T. Ishii: *On endomorphism rings of noetherian quasi-injective modules*, Osaka J. Math. **9** (1972), 217–223.

[11] M. Harada: *On perfect categories* I, Osaka J. Math. **10** (1973), 329–341.

[12] ———: *On perfect categories* II, Osaka J. Math. **10** (1973), 343–355.

[13] ———: *On perfect categories* III, Osaka J. Math. **10** (1973), 357–367.

[14] M. Ikeda and T. Nakayama: *On some characteristic properties of quasi-Frobenius and regular rings*, Proc. Amer. Math. Soc. **5** (1954), 15–19.

[15] B. Mitchell: Theory of Categories, Academic Press, New York and London, 1965.

[16] T. Onodera: *Ein Zatz über koendlich erzeugte RZ-modules*, Tohoku Math. J. **23** (1971), 691–695.

[17] ———: *Über Kogeneratoren*, Arch. Math. **19** (1968), 402–410.

[18] B. Osofsky: *A generalization of quasi-Frobenius rings*, J. Algebra **3** (1966), 37–386.

[19] A. Rosenberg and D. Zelinsky: *Annihilators*, Portugal. Math. **20** (1961), 53–65.

[20] Y. Sai: *On regular categories*, Osaka J. Math. **7** (1970), 301–306.

[21] R.B. Warfield: *Decomposition of injective modules*, Pacific J. Math. **31** (1969), 263–276.