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Osaka J. Math.
10 (1973), 329-341

PERFECT CATEGORIES I

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(Received March 15, 1972)

Let R be a ring with identity. We assume that an R-module M has two decompositions: $M=\sum_{\alpha\in I}\oplus M_\alpha=\sum_{\beta\in J}\oplus N_\beta$, where M_α's and N_β's are completely indecomposable. Then it is well known as the Krull-Remak-Schmidt-Azumaya's theorem that M satisfies the following two conditions:

I. *The decompositions are unique up to isomorphism.*

II′. *For a given finite set* $\{N_{\beta_1}, \cdots, N_{\beta_n}\}$ *we can find a set* $\{M_{\alpha_1}, \cdots, M_{\alpha_n}\}$ *such that* $M=N_{\beta_1}\oplus\cdots\oplus N_{\beta_n}\oplus\sum_{\alpha\neq(\alpha_i)}\oplus M_\alpha$ *and* $N_{\beta_i}\approx M_{\alpha_i}$ *for* $i=1, 2, \cdots, n$ (*or* $M=M_{\alpha_1}\oplus\cdots\oplus M_{\alpha_n}\oplus\sum_{\beta\neq(\beta_i)}\oplus N_\beta$).

Those facts were generalized in a Grothendieck category $\mathfrak{A}$ by P. Gabriel, [5]. Recently, the author and Y. Sai have treated

II. *The condition* II′ *is true for any infinite subset* $\{N_{\beta_i}\}$,

in a case of modules in [7], and shown that Condition II is satisfied for any M in the induced full subcategory $\mathfrak{B}$ from $\{M_\alpha\}$ in the category $\mathfrak{M}_R$ of R-modules if and only if $\{M_\alpha\}$ is an elementwise T-nilpotent system with respect to a certain ideal $\mathfrak{C}$ of $\mathfrak{B}$. Furthermore, the author and H. Kanbara have shown in [10] and [12] that Condition II is satisfied for a given M if and only if $\{M_\alpha\}$ is an elementwise semi-T-nilpotent system with respect to $\mathfrak{C}\cap \mathrm{Hom}_R(M, M)$.

Conditions I and II′ are categorical and hence, we can easily generalize the arguments in modules to those in $\mathfrak{A}$ (see [5] and [7]). However, the definition of the elementwise T-nilpotency is not categorical. Therefore, we treat, in this paper, a Grothendieck category with a generating set of small objects, e.g. $\mathfrak{M}_R$, locally noteherian categories and functor categories of small additive categories to the category Ab of abelian groups.

We shall show in the section two that almost all of essential properties in [7], [8], [9], [10], [11] and [12] are valid in such a category.

In the final section, making use of such generalized properties, we define perfect (resp. semi-prefect) Grothendieck categories $\mathfrak{A}$ and give a characterization of them with respect to a generating set of $\mathfrak{A}$. This characterization gives us a generalization of [2]. Theorem P for $(\mathfrak{C}, Ab)$, where $\mathfrak{C}$ is an amenable additive small category. Especially, if $\mathfrak{C}$ is a full additive subcategory with finite coproducts of finitely generated abelian groups, we show that $(\mathfrak{C}, Ab)$ is perfect if and only if the complete isomorphic class of indecomposable p-torsion groups in $\mathfrak{C}$ is finite for every prime p.

1. Perliminary results

Let $\mathfrak{A}$ be a Grothendieck category, namely a complete, co-complete C_3-abelian category (see [14], Chap. III). We call an object A in $\mathfrak{A}$ *samll* if $[A, \sum\oplus -]\approx\sum\oplus[A, -]$ and call $\mathfrak{A}$ *quasi-small* if every object A in $\mathfrak{A}$ is a union of some small subobjects A^{α} in $A: A=\bigcup_{\alpha} A^{\alpha}$.

If $\mathfrak{A}$ has a generating set of small objects, then $\mathfrak{A}$ is quasi-small. For example, the category $\mathfrak{M}_R$ of modules over a ring R is quasi-small and more generally the functor category $(\mathfrak{C}, Ab)$ and its full subcategory $\mathrm{L}(\mathfrak{C}, Ab)$ of left exact functors are quasi-small, where $\mathfrak{C}$ is a small additive category and Ab is the category of abelian groups, (cf. [13], p. 109, Theorem 5.3 and p. 99, Proposition 2.3). It is clear that if $\mathfrak{A}$ is locally noetherian (see [4], p. 356), then $\mathfrak{A}$ is quasi-small.

By $\mathrm{J}(A)$ we denote the Jacobason radical for any object A in $\mathfrak{A}$, i.e. $\mathrm{J}(A)=\cap N$, where N runs through all maximal subobjects in A and $\mathrm{J}(A)=A$ if A does not contain any maximal subobjects. A is called *finitely generated* if $A=\bigcup_{\alpha\in I} A_{\alpha}$ for some subobjects A_{α} of A, then $A=\bigcup_{\beta\in J} A_{\beta}$ for a finite subset J of I.

Let N be a subobject in M. N is called *samll in M* if $N+T=M$ implies $T=M$ for any subobject T in M. Following to [13], we define a semi-perfect (resp. perfect) objest P in $\mathfrak{A}$. P is called *semi-perfect* (resp. *perfect*) if P is projective and every factor object of P has a projectove cover (resp. any coproduct of copies of P is semi-perfect).

From the proof of Lemma in [16], we have

Lemma 1. *Let P be a projective object in an abelian category $\mathfrak{C}$. Then* $\mathrm{J}([P, P])=\{f\,|\in[P, P],\ \mathrm{Im} f$ *is samll in* $P\}$.

Proposition 1. *Let P be a projective object in the Grothendieck category $\mathfrak{A}$. Then the following statements are equivalent.*

1) *$S_P=[P, P]$ is a local ring; $S_P/\mathrm{J}(S_P)$ is a division ring.*
2) *Every proper subobject in P is small in P.*
3) *P is semi-perfect and directly indecomposable.*

(cf. [8], Theorem 5).

Proof. 1)$\to$2). Since S_P is local, $\mathrm{J}(S_P)$ consists of all non-isomorphisms. Let N be a proper subobject of P and assume $P=T+N$. Since $P/T\overset{\varphi}{\approx}N/N\cap T$, we have a diagram:

$$\begin{array}{ccccccc} 0 \to & T\cap N & \to & N & \xrightarrow{\nu'} & N/N\cap T & \to 0 \\ & & & & \nwarrow & \uparrow \varphi & \\ & & & & & P/T & \\ & & & \alpha & & \uparrow \nu & \\ & & & & & P & , \end{array}$$

where ν and ν are the canonical epimorphisms. Since P is projective, we obtain $\alpha \in [P, N] \subseteq S_P$ such that $\nu'\alpha = \varphi\nu$. Since $N \neq P$, $\alpha \in \mathrm{J}(S_P)$. Hence, $N = \mathrm{Im}\,\alpha + T \cap N$ and $P = \mathrm{Im}\,\alpha + T$. Therefore, $P = T$ by Lemma 1.

2)→1). Let f be not isomorphic. If $\mathrm{Im} f = P$, $P = P_0 + \mathrm{Ker} f$. Since $\mathrm{Ker} f$ is proper, $\mathrm{Ker} f$ is small in P, which is a contradiction. Hence, $\mathrm{Im} f \neq P$. Let g be another non-isomorphism. Since $\mathrm{Im} f$ and $\mathrm{Im}\, g$ are samll in P, $P \neq \mathrm{Im} f + \mathrm{Im}\, g \supseteq \mathrm{Im}(f+g)$. Hence, S_P is a local ring.
2)→3). It is clear from the definition.
3)→2). Let T be a proper subobject of P and $P' \to P/T$—0 a projective cover of P/T. Since P is indecomposable, $P \approx P'$. Hence, T is small in P.

For the rest of this section, we always assume that the abelian category $\mathfrak{A}$ is quasi-small in the sense given in the beginning of this section.

We shall generalize the notions of summability and elementiwse T-nilpotent systems in $\mathfrak{M}_R$ to a case of quasi-small categories, (cf. [7] and [8]).

A set of morphisms $\{f_\beta\}_{\beta \in K}$ of an object L to an object Q is called *summable* if for any small subobject L^n in L $f_\beta | L^n = 0$ for almost all $\beta \in K$. Let $M = \sum_I \oplus M_\alpha$ and $N = \sum_J \oplus N_\beta$ be two coproducts in $\mathfrak{A}$, and let i_α, p_β be an injection M_α to M and a projection of N to N_β, respectively. Let f be any element in $[M, N]$ and put $f_{\beta\alpha} = p_\beta f i_\alpha$. If M_α^n is a small subobject of M_α, $f_{\beta\alpha} | M_\alpha^n = 0$ for almost all β. Therefore, the $\{f_{\beta\alpha}\}_\beta$ is a set of summable morphisms of M_α to N. Conversely, let $\{f_{\beta\alpha}\}_{\beta \in J}$ be a set of summable morphisms of M_α to N and $M_\alpha = \cup M_\alpha^n$, where M_α^n's are small subobjects in M_α. Since a finite union of small subobjects is again small, we assume $\{M_\alpha^n\}$ forms a directed family and $M_\alpha = \varinjlim M_\alpha^n$. Furthermore, $\sum_{\beta \in I} f_{\beta\alpha} | M_\alpha^n$ gives an element in $[M_\alpha, N]$. Hence, we have a unique element f in $[M, N]$ such that $f i_\alpha^n = \sum f_{\beta\alpha} | M_\alpha^n$. Thus, we have

Lemma 2. *Let $M_i = \sum_{\alpha_i \in I_i} \oplus M_{i\alpha_i}$ be objects in the quasi-small category $\mathfrak{A}$ for $i = 1, 2$ and 3. Then $[M_1, M_2]$ is isomorphic to the set of row summable matrices with entries $a_{\alpha_j \alpha_i}$. Furthermore, the composition $[M_2, M_3]\,[M_1, M_2]$ corresponds to the product of martices, where $a_{\alpha_j \alpha_i} \in [M_{i\alpha_i}, M_{j\alpha_j}]$.*

Corollary 1. *Let P be projective and directly indecomposable object in $\mathfrak{A}$ with a set of small generators. If $S_P = [P, P]$ is a local ring, then P is semi-perfect and $\mathrm{J}(P)$ is a unique maximal subobject of P. Hence, P is finitely generated.*

Proof. Let $Q_1 \subset Q_2 \subset \cdots \subset Q_n \subset \cdots$ be a series of proper subobjects in P. If $P = \bigcup Q_j$, we have a diagram

$$\begin{array}{ccccc} \sum \oplus Q_j & \xrightarrow{\nu} & P & \rightarrow & 0 \qquad \text{(exact)} \\ & \nwarrow & \| 1_P & & \\ & f & P & & \end{array}$$

, where ν is given naturally by inclusions. We obtain $f \in [P, \sum \oplus Q_j]$ such that $\nu f = 1_P$ and put $f = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_\alpha \\ \vdots \end{pmatrix}$ and $\nu = (i_1, i_2, \cdots, i_\alpha, \cdots)$. Then $1_P = \sum i_\alpha f_\alpha$. However, any of f_α's is not isomorphic, which is a contradiction (cf. [1]). Hence, we have a maximal subobject by the Zorn's lemma. Therefore, J(P) is a unique maximal, subobject of P by Proposition 1.

Corollary 2 ([6], Theorem 2.8.) *Let P be projective and artinian, then P is finitely generated, and S_P is right artinian.*

Proof. Since S_P is a semi-primary ring by [5], Proposition 2.7, it is clear from the above corollary. S_P is right artinian from [6], Lemma 2.6.

2. Coproducts of completely indecomposable objects

We studied Krull-Remak-Schmidt-Azumaya's theorm for a direct decomposition of a module as completely indecomposable modules in [7], [8], [10] and [12]. We shall generalize many results in a case of modules to a case of Grothendieck abelian categories $\mathfrak{A}$ with a set of small generators.

An object M in $\mathfrak{A}$ is called *completely indecomposable* if $S_M = [M, M]$ is a local ring. The following lemma was given in [7], p. 343, Remark 4 without proof. We shall give here its proof for the sake of completeness.

Lemma 3. *Let $M = \sum_{i=1}^{\infty} \oplus M_i$ and M_i's be completely indecomposable objects in a C_3-abelian category $\mathfrak{C}$. Let $\{f_i\}_{i=1}^{n}$ be a set of morphisms $f_i \in [M_i, M_{i+1}]$. Put $M_i' = \mathrm{Im}(1_{M_i} + f_i)$. Then $M_t \cap (M_{i_1}' + M_{i_2}' + \cdots + M_{i_s}') \subseteq \mathrm{Ker}(f_n f_{n-1} \cdots f_t)$ for $1 \leq t \leq n$ and $(i_1, i_2, \cdots, i_s) \subseteq (1, 2, \cdots, n)$ and $M_t \cap (M_i + \sum_{j=1}^{n} M_j') \subseteq \mathrm{Im}(f_{t-1} \cdots f_1) + \mathrm{Ker}(f_n \cdots f_t)$ for $i \leq t \leq n$.*

Proof. We take $\{M_i\}_{i=1}^{n+1}$ and we construct a small full subcategory $\mathfrak{C}_0$ such that $\mathfrak{C}_0$ contains all M_i' and kernels and images in $\mathfrak{C}_0$ are those in $\mathfrak{C}$, (see [14], p. 101, Lemma 2.7). Then there exists an exact covariant imbedding functor of $\mathfrak{C}_0$ to Ab by [14], p. 101, Theorem 2.6. Hence, we may assume that all of M_i are abelian groups. In this case the lemma is clear.

We shall make use of the same condition I. II and III given in [7], p. 331–332, (see the introduction). Condition I is satisfied for any two decompositions as coproducts of completely indecomposable objects in an arbitrary Grothendieck category (see [5] or [8], Theorem 7′). We are now interested in Condition II.

From now on we assume that a Grothendieck category $\mathfrak{A}$ has a generating set of small objects, namely quasi-small in the sense of §1.

First, we shall generalize the notions of elementwise semi-T-nilpotent system defined in [7] and [8].

Let $\mathfrak{C}$ be an ideal in $\mathfrak{A}$. We take a set of objects $\{M_\alpha\}$ and consider morphisms $f_{\alpha_i}\colon M_{\alpha_i}\to M_{\alpha_{i+1}}$, which belong to $\mathfrak{C}$. If for any small subobject $M^n_{\alpha_1}$ of M_{α_1} there exists m such that $f_{\alpha_m}f_{\alpha_{m-1}}\cdots f_{\alpha_1}|M^n_{\alpha_1}=0$, we call $\{f_{\alpha_i}\}$ a *locally right T-nilpotent system* (with respect to $\mathfrak{C}$). If for any subset $\{M_\alpha\}$ and any set $\{f_{\alpha_i}\}$, $\{f_{\alpha_i}\}$ is locally rihgt T-nilpotent system, we call $\{M_\alpha\}$ is a *locally right T-nilpotent system.* If $\alpha_i\neq\alpha_j$ for $i\neq j$ in the above, we call $\{f_{\alpha_i}\}$ and $\{M_\alpha\}$ *locally right semi-T-nilpotent systems.* Similarly, if we replace f_{α_i} by $g_{\alpha_i}\colon M_{\alpha_{i+1}}\to M_{\alpha_i}$ and $g_{\alpha_1}g_{\alpha_2}\cdots g_{\alpha_m}=0$ for some m, we call $\{g_\alpha\}$ left T-nilpotent.

If we replace *elementwise* (*semi-*) *T-nilpotent system* by *locally right* (*semi-*) *T-nilpotent systems* in the arguments in [7], [8], [9], [10] and [12], we know that many results in them are valid in $\mathfrak{A}$ without changing proofs. For instance, in order to prove the same result of [7], Lemma 9 for $\mathfrak{A}$, we can replace the relations 2) and 3) in [7], p. 336 by Lemma 3 and elements x by small subobjects, and we use the same argument, taking a projection of M to M_n if necessary.

Let $\{M_\nu\}$ be a set of completely indecomposable objects and $\mathfrak{B}$ be the induced full additive category from $\{M_\alpha\}$: objects of $\mathfrak{B}$ consist of all coproducts of some M_α (and their all isomorphic images). We can express all morphisms in $\mathfrak{B}$ by row summable matrices $(a_{\beta\alpha})$ by Lemma 2. We define an ideal $\mathfrak{C}$ of $\mathfrak{B}$ as follows: $\mathfrak{C}$ consists of all morphisms $(a_{\beta\alpha})$ such that $a_{\gamma\delta}\colon M_\delta\to M_\gamma$ is not isomorphic for all γ, δ. Then we have from Theorem 9 in [7].

Theorem 1. *Let $\mathfrak{A}$ be a Grothendieck category with a generating set of small objects, and $\mathfrak{B}$ the induced full subadditive category from a set of completely indecomposable objects M_α. Then the following statements are equivalent.*

1) *For any two decompositions $M=\sum_I\oplus Q_\alpha=\sum_J\oplus N_\beta$ of any object M in $\mathfrak{B}$, Condition* II *in* [7] *is satisfied, where Q_α, N_β are indecomposable.*

2) *The ideal $\mathfrak{C}$ in $\mathfrak{B}$ defined above is the Jacobson radical of $\mathfrak{B}$.*

3) *$\{M_\alpha\}$ is a locally right T-nilpotent system.*

Similarly from [12], Theorem or [10], Lemma 5 we have

Theorem 2. *Let $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ be as above. Then the following statements are equivalent.*

1) *For given two decompositions $M=\sum_I\oplus Q_\alpha=\sum_J\oplus N_\beta$ of a given object M*

in $\mathfrak{B}$, *Condition* II *is satisfied, where* Q_α, N_β *are indecomposable.*

2) $\mathfrak{C}\cap S_M=\mathrm{J}(S_M)$, *where* $S_M=[M, M]$

3) $\{Q_\alpha\}_I$ *is a locally right semi-T-nilpotent system with respect to* $\mathfrak{C}$.

REMARK. Using Lemmas 2 and 3, we can obtain theorems concerned with exchange properties in $\mathfrak{A}$ in [6] and [9] if we replace *semi-T-nilpotent* by *locally right semi-T-nilpotent*.

3. Perfect categories

H. Bass defined a perfect or semi-perfect ring in [2]. Recently, M. Weidenfeld and G. Weidenfeld have generalized them to a functor category ($\mathfrak{C}$, *Ab*) of an additive category $\mathfrak{C}$ in [17].

We shall define a perfect or semi-perfect Grothendieck category $\mathfrak{A}$ and study some properties of $\mathfrak{A}$, which are analogous to ones in [2], as an application of §2.

Let $\mathfrak{A}$ be a Grothendieck category. $\mathfrak{A}$ is called *perfect* (resp. *semi-perfect*) if every (resp. finitely generated) object A in $\mathfrak{A}$ has a projective cover (cf. [2]).

Let $\mathfrak{A}'$ be the spectral Grothedieck category given in [7], p. 331, Example 2. Then every object in $\mathfrak{A}'$ has a trivial projetive cover and hence, $\mathfrak{A}'$ is perfect. However, $\mathfrak{A}'$ has completely different properties from ones in $\mathfrak{M}_R$, where R is a right perfect ring.

We are interested, in this section, in perfect categories with similar properties of perfect rings. Hence, in order to exclude such a special perfect category we assume that $\mathfrak{A}$ is quasi-small, namely $\mathfrak{A}$ has a generating set of small objects.

As seen in [2] and [13], the fact $P\neq \mathrm{J}(P)$ for a projective P in $\mathfrak{A}$ is very important to study perfect categories. In the spectral category $\mathfrak{A}'$ above, this fact is not true. On the other hand, that fact was shown in $\mathfrak{M}_R$ and noted in ($\mathfrak{C}$, *Ab*) by [2] and [17], respectively.

We first generalize them as follows:

Proposition 2. *Let* $\mathfrak{A}$ *be a Grothendieck category and* A *an object in* $\mathfrak{A}$. *If* A *is a retract of a coproduct of either*

a) *porjective objects* P *such that* $\mathrm{J}(P)$ *is small in* P, *or*

b) *noetherian objects,*

then $A\neq \mathrm{J}(A)$.

We need two lemmas for the proof, the first of which is well known.

Lemma 4. *Let* P *be a small and projective object in* $\mathfrak{A}$. *Then* P *is finitely generated and* $\mathrm{J}(P)$ *is small in* P.

See [3], p. 105.

Lemma 5. *Let* $\{A_i\}_I$ *be a family of objects in* $\mathfrak{A}$ *such that* $[A_i, \mathrm{J}(A_i)]$ *is* *contained in* $J([A_i, A_i])$ *for all* $i \in I$. *Put* $A = \sum_{\alpha \in I} \oplus A_i$. *Then for* $f \in [A, A]$ *with* $\mathrm{Ker}(1-f) \neq 0$, $\mathrm{Im} f \neq J(\mathrm{Im} f)$.

Proof. Put $B = \mathrm{Im} f$ and assume $B = J(B)$. Since $J(B) \subset J(A)$, $f \in [A, J(A)]$. $\mathrm{Ker}(1-f) \neq 0$ from the assumption and hence, $\mathrm{Ker}(1-f) \cap \sum_{i=1}^{n} \oplus A_{\alpha_1} \neq 0$ for some finite indeces $\alpha_i \in I$. Let e_1 be the projection of A to A_{α_i}. Since $f \in [A, J(A)]$, $e_1 f e_1 | A_{\alpha_1} \in [A_{\alpha_1}, J(A_{\alpha_1})] \subset J(S_{A_{\alpha_1}})$. Hence, $e_1(1-f)e_1 | A_{\alpha_1} = (e_1 - e_1 f e_1) | A_{\alpha_1}$ is automorphic. Therefore, $A = (1-f)(A_{\alpha_1}) \oplus \mathrm{Ker}\, e_1 = (1-f)(A_{\alpha_1}) \oplus \sum_{\beta \neq \alpha_1} \oplus A_\beta$ and $A_{\alpha_1} \overset{1-f}{\approx} (1-f)(A_{\alpha_1})$. Let e_2 be the projection of A to A_{α_2} in the above decompositon. Then we obtain $A = (1-f)(A_{\alpha_1}) \oplus (1-f)(A_{\alpha_2}) \oplus \sum_{\beta \neq \alpha_1, \alpha_2} \oplus A_\beta$ and $A_{\alpha_2} \overset{1-f}{\approx} (1-f)(A_{\alpha_2})$. Repearting this argument, we know that $(1-f) | \sum_{i=1}^{n} \oplus A_{\alpha_i}$ is isomorphic, which is a contraadiction, (this argument is due to [1]).

Proof of Proposition 2. It is clear for the case a) from Lemmas 4 and 5 and [10], Proposition 1. Let A be a noetherian object. Then $A \neq J(A)$ and $J(A)$ is small in A. Hence, $1-f$ is epimorphic for any f in $[A, J(A)]$. Therefore, $1-f$ is unit, since A is noetherian. Thus, $[A, J(A)] \subset J(S_A)$.

Corollary 1 ([2] and [17]). *Let* $\mathfrak{A}$ *be a Grothendieck category which is one of the following types:*

a) $\mathfrak{M}_R$ *for some ring R,*

b) $(\mathfrak{C}, Ab)$, *where* $\mathfrak{C}$ *is a small additive category,*

c) *Locally noetherian.*

Then $P \neq J(P)$ *for every non-zero projective obejcet P.*

Corollary 2. *Let* $\mathfrak{C}$ *be an artinian abelian category and* $L(\mathfrak{C}, Ab)$ *the left exact functor category. Then* $Q \neq J(Q)$ *for every retract Q of any coproduct of generators* $\{H^A\}_{A \in \mathfrak{C}}$, *where* $H^A(-) = [A, -]$.

Proof. $L(\mathfrak{C}, Ab)$ is locally noetherian by [4], Proposition 7 in p. 356.

For the study of perfect categories, we recall an induced category. Let $\{M_\alpha\}_I$ be a given set of some objects in a Grothendieck category $\mathfrak{A}$. By $\mathfrak{C}_f$ we denote the full subadditive category of $\mathfrak{A}$, whose objects consist of all finite coproducts of M_α' which is isomorphic to some M_β in $\{M_\alpha\}_I$. We call $\mathfrak{C}_f$ the *finitely induced additive category from* $\{M_\alpha\}$, (see [7]). If all M_α are completely indecomposable, $\mathfrak{C}_f$ is amenable (see [3], p. 119) by [7], Theorem 7′.

Let A be an object in $\mathfrak{A}$. By $S(A)$ we denote the socle of A, namely $S(A)$ = the union of all minimal subobjects in A.

Following to [15], we call $\mathfrak{A}$ *semi-artinian* if $S(A) \neq 0$ for all non-zero object A in $\mathfrak{A}$.

If $\mathfrak{A}$ is a Grothendieck category with a generating set of small projective, then $\mathfrak{A}$ is equivalent to $(\mathfrak{C}, Ab)$ by Freyd's theorem (see [14], p. 109, Theorem 5.2), where $\mathfrak{C}$ is a small additive category. In this case, $\mathfrak{A}$ is also equivalent to a subcategory of modules by [4]. We give here categorical proofs in the following for some properties in $\mathfrak{A}$, however we note that we can prove them ring-theoretical (see Remark below).

First, we generalize [15], Proposition 3.2.

Proposition 3. ([15]). *Let $\mathfrak{A}$ be a Grothendieck category with a generating set $\{P_\alpha\}$ of small projective. Then $\mathfrak{A}$ is semi-artinian if and only if* 1) *$\{P_\alpha\}$ is a left T-nilpotent system with respect to* $\mathrm{J}(\mathfrak{A})$ *and* 2) $\mathrm{S}(A)\neq 0$ *for every non-zero quotient object A of $P_\alpha/J(P_\alpha)$ for all α.*

Proof. If $\mathfrak{A}$ is semi-artinian, 2) is clear. The following agrument is similar to one in [2], p. 470. Let $\{f_i\}$ be a set in $\mathrm{J}(\mathfrak{A})$ and $f_i\colon P_{i+1}\to P_i$. We define inductively a series of subobjects K_α of $P_{\alpha 1}$ as follows: $K_0=0$, $K_1=\mathrm{S}(P_1)$, $K_2/K_1=S(P_1)$, $\cdots$. If α is a limit, $K_\alpha=\bigcup_{\beta<\alpha} K_\beta$. Since $\mathfrak{A}$ is a Grothendieck category, $P_1=K_\gamma$ for some γ. Put $I_i=\mathrm{Im}\, f_1 f_2\cdots f_i$. Then I_i is finitely generated, since so is P_{i+1}. Let α_i be the least number such that $K_{\alpha_i}\supset I_i$. If α_i is a limit, then $I_i=\bigcup_{\beta<\alpha_i}(K_\beta\cap I_i)$ and hence, $I_i\subset K_\beta$ for some $\beta<\alpha_i$. Therefore, we can express $\alpha_i=\delta_i+1$. Since $K_{\alpha_i}/K_{\delta_i}$ is semi-simple, $\mathrm{J}(K_{\alpha_i}/K_{\delta_i})=0$ and $\mathrm{Im}\, f_{i+1}\subset \mathrm{J}(P_i)$ by Lemma 1. Hence, $\mathrm{Im}\, f_1 f_2\cdots f_{i+1}=\mathrm{Im}\,((f_1 f_2\cdots f_i)f_{i+1})\subset K_{\delta_i}$. Therefore, $\alpha_i>\alpha_{i+1}$ which means that $\{f_i\}$ is a left T-nilpotent. Conversely, we assume 1) and 2). We show that for any non-zero object A, there exists P_1 and $f\in[P_1, A]$ such that $f(\mathrm{J}(P_1))=0$ and $f\neq 0$. If it were not true, we would have some P_1 and $f\in[P_1, A]$ such that $f(\mathrm{J}(P_1))\neq 0$. If we consider an exact sequence,

$$\begin{array}{ccccc} \mathrm{J}(P_1) & \xrightarrow{f} & f(\mathrm{J}(P_1)) & \longrightarrow & 0 \\ & \nwarrow{\scriptstyle f_1} & \uparrow{\scriptstyle f_1'} & & \\ & & P_2 & & \end{array}$$

we have some P_2, $f_1'\in[P_2, f(\mathrm{J}(P_1)]$ and $f_1\in[P_2, \mathrm{J}(P_1)]$ such that $f_1'=ff_1$. Since $f_1'(\mathrm{J}(P_2))\neq 0$, we can find P_3 and $f_2\in[P_3, \mathrm{J}(P_2)]$ such that $f_2'=ff_1f_2\in[P_2, A]$ and $f_2'(\mathrm{J}(P_2))\neq 0$. Repeating this argument we have $f_n'=ff_1\cdots f_n\neq 0$ and $f_i\in[P_{i+1}, \mathrm{J}(P_i)]\subset \mathrm{J}(\mathfrak{A})\cap[P_{i+1}, P_i]$ for all n by Lemma 1, which contradicts to 1). Hence, $\mathfrak{A}$ is semi-artinian from 2).

In order to characterize some perfect Grothendieck categories, we give some notes here. For a project object P such that $P\neq \mathrm{J}(P)$ we obtain from [13], Theorem 5.2 that $P=\sum\oplus P_\alpha$ is semi-perfect if and only if P_α's are semi-perfect of $\mathrm{J}(P_\alpha)\neq P_\alpha$ and $\mathrm{J}(P)$ is small in P. Further if P is semi-perfect, $P=\sum\oplus Q_\alpha$

by [13], Corollary 4.4, where Q_α's are completely indecomposable. Similarly from Lemma 5 and [10], Proposition 1 and Corollary 1 to Theorem 3 we obtain

Lemma 6. *Let $\mathfrak{A}$ be a quasi-samll Grothendieck category and $\{P_\alpha\}_I$ a family of semi-perfect objects in $\mathfrak{A}$. Then $P=\sum_I \oplus P_\alpha$ is semi-perfect (resp. perfect) and $P \neq \mathrm{J}(P)$ if and only if $\{P_\alpha\}_I$ is a locally right semi-T-nilpotent (resp. T-nilpotent) system with respect to $\mathrm{J}([P, P])$ and $P_\alpha \neq \mathrm{J}(P_\alpha)$ for all α.*

Theorem 3. *An abelian category $\mathfrak{A}$ is a Grothendieck category with a generating set of finitely generated objects and is semi-perfect if and only if $\mathfrak{A}$ is equivalent to $(\mathfrak{C}_f^0, Ab)$, where $\mathfrak{C}_f$ is the finitely induced sub-additive category from $\{P_\alpha\}_I$, where P_α's are completely indecomposable objects in $\mathfrak{A}$.*

Proof. Let $\{G_\alpha\}$ be a generating set of finitely generated objects. If $\mathfrak{A}$ is semi-perfect, we have a projective cover P_α of G_α; $0\to K_\alpha \to P_\alpha \xrightarrow{f} G_\alpha \to 0$ is exact and K_α is small in P_α. Furthermore, P_α contains a finitely generated subobject P' such that $f(P')=G_\alpha$. Hence, $P_\alpha=K+P'$ implies that P_α is also finitely generated. Therefore, $\mathfrak{A}$ has a generating set of projective small P_α. We have $P \neq \mathrm{J}(P)$ for every projective object P by Proposition 2. Thus $P_\alpha = \sum_{i=1}^{n_\alpha} \oplus P_{\alpha_i}$ by [13], Corollary 4.4, where P_{α_i}'s are completely indecomposable. Let $\mathfrak{C}_f$ be the induced subadditive category from $\{P_{\alpha_i}\}$. Then $\mathfrak{A}$ is equivalent to $(\mathfrak{C}^0, Ab)$ by Freyd's Theorem. Conversely, if $\mathfrak{A} \approx (\mathfrak{C}^0, Ab)$, $\{H_C(-)=[-, C]\}$ is a generating set of finitely generated projective objects by Lemma 4. Further $\mathfrak{A}$ is semi-perfect by Proposition 1 and [14], Corollary 5.3.

If a ring R is right artinian, then $\mathfrak{M}_R$ is right (semi-) perfect. Similarly, we have

Proposition 4. *Let $\mathfrak{A}$ be a Grothendieck category with a generating set $\{P_\alpha\}_I$ of projective objects with finite length. Then $\mathfrak{A}$ is semi-perfect. $\mathfrak{A}$ is perfect if and only if $\sum_I \oplus P_\alpha$ is semi-perfect,* (cf. Remark 2 below)

Proof. We may assume that $\mathfrak{A}$ has a generating set of completely indecomposable and small projective objects P_α. Then P_α is semi-perfect by Proposition 1 and hence, $\mathfrak{A}$ is semi-perfect. If $\sum_I \oplus P_\alpha$ is semi-perfect, then $\sum \oplus P_\alpha$ is perfect by Lemma 6 and [6], Proposition 2.4.

Analogously to Theorem 3, we have

Theorem 4. *An abelian category $\mathfrak{A}$ is a Grothendieck category with a generating set of finitely generated objects and is perfect if and only if $\mathfrak{A}$ is equivalent to $(\mathfrak{C}_f^0, Ab)$, where $\mathfrak{C}_f$ is the finitely induced additive category from a set of some completely indecomposable objects P_α such that $\{P_\alpha\}$ is a right T-nilpotent system* *with respect to* $\mathrm{J}(\mathfrak{C}_f)$.

Proof. If $\mathfrak{A}$ is a perfect Grothendieck category as above, then $\mathfrak{A}\approx(\mathfrak{C}_f^0, Ab)$ by Theorem 3. It is clear from Lemma 6 that $\{P_\alpha\}$ is a right T-nilpotent system with respect to $\mathrm{J}(\mathfrak{C}_f)$, since P_α is small. Conversely, if $\mathfrak{A}\approx(\mathfrak{C}_f^0, Ab)$, $\mathfrak{A}$ is a perfect category as in the theorem by Lemmas 4 and 6.

We have immediately from Corollary to Lemma 2, Proposition 3 and Theorems 3 and 4

Corollary 1. *Let* $\mathfrak{A}$ *be a Grothendieck category with a generating set of finitely generated. Then* $\mathfrak{A}$ *is semi-perfect if and only if* $\mathfrak{A}$ *has a generating set* $\{P_\alpha\}$ *of completely indecomposable projective objects. In this case* $\{P_\alpha\}$ *is right (resp. left) T-nilpotent if and only if* $\mathfrak{A}$ *is perfect (resp. semi-artinian).*

Let $\mathfrak{A}$ be a Grothendieck category as in the above. Then the induced category from $\{P_{\alpha'}/\mathrm{J}(P_{\alpha'})\}_J$ is equivalent to $\sum_J \oplus \mathfrak{M}_{\Delta_{\alpha'}}$, where $\Delta_{\alpha'}=[P_{\alpha'}/\mathrm{J}(P_{\alpha'}), P_{\alpha'}/\mathrm{J}(P_{\alpha'})]$, where $\{P_{\alpha'}/\mathrm{J}(P_{\alpha'})\}$ is a complete isomorphic representative of $\{P_\alpha/\mathrm{J}(P_\alpha)\}$. Hence, we have

Corollary 2. *A (semi-) perfect Grothendieck category with a generating set of finitely generated is equivalent to* $\mathfrak{M}_R$ *with* R *(semi-) perfect if and only* J *is finite.*

From Theorems 3 and 4, we may restrict ourselves to a case of functor categories $(\mathfrak{C}, Ab)$, if we are interested in perfect Grothendieck categories. First, we note

Proposition 5 ([17]). *Let* $\mathfrak{C}$ *be an amenable additive and small category. Then* $(\mathfrak{C}, Ab)$ *is semi-perfect if and only if every object in* $\mathfrak{C}$ *is finite coproduct of completely indecomposable objects.*

Proof. It is clear from Theorem 3 and [3], p. 119.

For a ring R, ${}_R\mathfrak{M}$ (resp. $\mathfrak{M}_R$) is naturally equivalent to (R, Ab) (resp. (R^0, Ab)). Hence, an analogy of [2], Theorem 2.1 is

Corollary. *Let* $\mathfrak{C}$ *be as above. Then* $(\mathfrak{C}, Ab)$ *is semi-perfect if and only if* $(\mathfrak{C}^0, Ab)$ *is semi-perfect.*

Our next aim is to generalize Theorem P of [2] to a case of $(\mathfrak{C}_f, Ab)$. First we shall recall the idea given in [4], Chapter II. Put $R=\sum_{X,Y\in\mathfrak{C}_f}\oplus[X, Y]$ and we can make R a ring by the compositions of morphisms in $\mathfrak{C}$. If we denote the indentity morphism of X by I_X, I_X is idemoptent and $I_XI_Y=I_YI_X=0$ if $X\neq Y$. Hence, $R=\sum_{X\in\mathfrak{C}}\oplus RI_X=\sum_{X\in\mathfrak{C}}\oplus I_XR$. In general, R does not contain the identity. We know by [4], Proposition 2 in p. 347 that the covariant functor category $(\mathfrak{C}, Ab)$ is equivalent to the full subcategory of ${}_R\mathfrak{M}$ whose objects consist of all left R-modules A such that $RA=A$. Similarly, we know the contravariant functor category $(\mathfrak{C}^0, Ab)$ is equivalent to the full sbucategory of $\mathfrak{M}_R$ with $AR=A$.

Lemma 7. *Let $\mathfrak{C}_f$ and $R=\sum\oplus[X, Y]$ be as above. Then* $J(R)=\sum\oplus([X, Y]\cap J(\mathfrak{C}_f))$.

Proof. Let x be in $J(R)$. Then there exists a finite number of objects X_i such that $x=(\sum I_{X_i})x(\sum I_{X_i})\in(\sum I_{X_i})J(R)(\sum I_{X_i})=J((\sum I_{X_i})R(\sum I_{X_i}))$. On the other hand $(\sum I_{X_i})R(\sum I_{X_i})\approx[\sum\oplus X_i, \sum\oplus X_i]$. Hence, $x\in\sum([X, Y]\cap J(\mathfrak{C}_f))$ by [7]., Lemma 8. The converse is clear from the above argument.

We can prove the following theorem by the same method given in [2], Part 1 even though R does not contain the identity (see Remark 1 below). However, we shall give here the proof rather directly (without homological method).

Theorem 5 (cf. [2]. Theorem P). *Let $\mathfrak{A}$ be an arbitrary Grothendieck category, $\{M_\alpha\}_I$ a set of completely indecomposable objects in $\mathfrak{A}$ and $\mathfrak{C}_f$ the finitly induced additive subcategory from $\{M_\alpha\}$. Put $R=\sum_{\mathfrak{C}_f}\oplus[X, Y]$ as above. Then the following conditions are equivalent.*

1) *$(\mathfrak{C}_f, Ab)$, is perfect.*
2) *$\{M_\alpha\}$ is a left T-nilpotent system with respect to $J(\mathfrak{C}_f)$.*
3) *$J(R)$ is left T-nilpotent.*
4) *R satisfies the descending chain condition on principal right ideals in $J(R)$.*
5) *Every object in $(\mathfrak{C}_f^0, Ab)$ contains minimal subobjects.*

We have the similar result for $(\mathfrak{C}_f^0\ Ab)$.

Proof. 1)$\leftrightarrow$2) is nothing but Lemma 6.
2)$\rightarrow$3). Let x_n be in $J(R)$. Then $x_n=\sum x_{n\,j(n)}$, $x_{n\,j(n)}\in[X_{j(n)}, Y_{j(n)}]\cap J(\mathfrak{C}_f)$. by Lemma 7, where we may assume that X, Y are isomorphic to ones in $\{M_\alpha\}$. Hence, $\{x_n\}$ is left T-nilpotent by König Graph Theorem.
3)$\rightarrow$4)$\rightarrow$2) is clear.
2)$\leftrightarrow$5) is given by Proposition 3.

REMARK 1. We can prove Theorem 5 by making use of idea in [2], Part 1. For instance, let $\{a_i\}$ be a sequence of elements in R. There exist indempotents I_i such that $I_ia_i=a_i$, $a_{i-1}I_i=a_{i-1}$. Then we denote by $[F, \{a_n\}, G]$

1) $F=\sum_{i=1}^{\infty}\oplus RI_ix_i$, 2) The subgroup G of F generated by $\{I_ix_i-a_iI_{i+1}x_{i+1}\}$, where x_i is a base. Then this $[F, \{a_i\}, G]$ takes the place of $[F, \{a_n\}, G]$ given in [2], p. 468, even though R does not contain the identity. From those

arguments we can shown that we may take out the assumption "*in* J(R)" in 4), (cf. [17], Proposition in p. 1571).

REMARK 2. Let $\{R_i\}_I$ be a set of perfect rigns. Then $\mathfrak{M}_{R_i}$ is perfect and $\prod_I \mathfrak{M}_{R_i}$ is also perfect, however $\prod_I R_i$ is not a perfect ring if I is infinite.

If a ring R is right artinian, then $\mathfrak{M}_R$ has a generator R of finite length and $\mathfrak{M}_R$ is perfect. However, in gneral categories with a generating set of projective and finite length need not be perfect. For instance, let K a be field and I the set of natural numbers. We define an abelian category $[I, \mathfrak{M}_K]$ of commutative diagrams as follows; the objects of $[I, \mathfrak{M}_K]$ consist of all form $(A_1, A_2, \cdots, A_j, \cdots)$ with arrow $d_{kj}\colon A_j \to A_k$ such that $d_{kj}=0$ for $k>j$, where $A_i \in \mathfrak{M}_K$. Then $[I, \mathfrak{M}_K]$ is an abelian category with a generating set of projective objects $(K, K, \cdots, K, 0, \cdots)=U_i$ of finite length (see [11], Proposition 2.1 and [14], p. 227). We have natural monomorphisms $f_i\colon U_i \to U_{i+1}$. Hence, $[I, \mathfrak{M}_K]$ is not perfect, however $[I, \mathfrak{M}_K]$ is semi-artinian by Proposition 3.

Finally, we shall give the following corollary as an example.

Corollary. *Let $\mathfrak{C}$ be a full additive amenable subcategory with finite coproduct in the category of finitely generated torsion abelian groups. Then the following statements are equivalent.*

1) *($\mathfrak{C}$, Ab) is perfect.*

2) *($\mathfrak{C}^0$, Ab) is perfect.*

3) *Every object in ($\mathfrak{C}$, Ab) contains minimal subobjects.*

4) *Every object in ($\mathfrak{C}^0$, Ab) contains minimal subobjects.*

5) *The completely isomorphic representative class of indecomposable p-torsion objects in $\mathfrak{C}$ is finite for all p.*

6) *($\mathfrak{C}_f$, Ab) is equivalent to $\prod \mathfrak{M}_{R_\alpha}$, where R_α's are right artinian rings.*

Proof. The indecomposable objects are left (or right) T-nilpotent with respect to J($\mathfrak{C}$) if and only if 5) is satisfied.

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